Valuation Models for Default-Risky Securities: An Overview

SAIKAT NANDI
The author is a senior economist in the financial section of the Atlanta Fed's research department.
He thanks Lucy Ackert, Jerry Dwyer, Will Roberds, and Larry Wall for helpful comments.

VALUING FINANCIAL SECURITIES OFTEN ASSUMES THAT THE CONTRACTUAL OBLIGATIONS OF THE SECURITY ARE GOING TO BE HONORED. HOWEVER, FREQUENTLY A PARTY TO A CONTRACT WILL DEFAULT ON ITS OBLIGATIONS. AN ISSUER OF A CORPORATE BOND MAY BE UNABLE TO MAKE ITS PROMISED COUPON AND PRINCIPAL PAYMENTS, AND A PARTY TO A DERIVATIVES CONTRACT SUCH AS AN INTEREST RATE SWAP MAY DEFAULT ON THE PERIODIC PAYMENTS OF THE SWAP CONTRACT.

Because the contractual features of defaultable securities are usually complex and it may be difficult to find comparable securities for which to observe prices, valuation based on simple rules of thumb is often infeasible. For example, one may have to value an interest rate swap subject to termination if one of the parties has its credit downgraded, and there are no comparable swaps that one can look up for a reference price (see the glossary for a definition of a swap and of other terms used throughout this discussion). Hence it becomes necessary to resort to formal models that can value a defaultable security on the basis of expected future cash flows, taking into account the contractual features of the defaultable security and the uncertainties surrounding the future cash flows. Many financial institutions hold large amounts of default-risky securities of various degrees of complexity in their portfolios, and it is important that these institutions have a reliable estimate of the resulting credit exposure. Estimating the credit exposure often involves knowing the possible values of the defaultable security at various times in the future. Therefore, understanding the different valuation models of default-risky securities and the strengths and drawbacks of various modeling approaches (see Table 1) is also important for implementing prudent risk-management policies to manage credit exposures.

This article discusses some of the models for valuing financial instruments that are subject to default risk, the implications of these models, and the difficulties that might be encountered in implementing them. The focus is on the valuation of default-risky bonds and swaps, although the general principles of valuation can be applied to other related instruments.[1] The first section explains the classic Merton (1974) model for valuing a default-risky bond. Subsequently, the text discusses some of the more recent models for valuing such bonds. These models differ in terms of how predictable the degree of default is and whether the firm's value is needed as an input in the valuation formula. Next, some of the valuation models for default-risky swaps are considered, including ones in which both parties to a swap can default. The discussion concludes with a review of the strengths and drawbacks of different valuation models and some thoughts for future research.

Merton's Model of Default-Risky Debt

One of the first models for valuing defaultable bonds was developed by Robert Merton (1974) using the principles of option pricing developed by Black and Scholes (1973) and Merton (1973). To understand the model, consider a firm that has equity and a single zero-coupon bond in its capital structure. Let the face value of the bond be $1,000, so that $1,000 has been promised to the bondholder at a future date when the bond matures. If the firm's value (the value of its assets) when the bond matures is less than $1,000, the firm cannot completely pay the bondholder even by liquidating all its assets. As a result, the firm will default on the debt.

How is a default-risky bond valued, given the knowledge that the firm may default at the maturity of the debt and that the claims of the bondholders are senior to those of the equityholders? Note that the payment to the bondholder at the maturity of the debt is the smaller of two quantities: the face value of the bond or the market value of the firm. If the firm's market value at maturity is greater than the face value of the bond, then the bondholder gets back the face value of the bond. However, if the value of the firm is less than the face value of the bond, the equityholders get nothing and the bondholder gets back the market value of the firm. The payoff from the default-risky bond at maturity resembles the payoff from an option, where the underlying asset is the value of the firm. Specifically, the payoff to the bondholder at maturity is the face value of the bond minus a put option on the firm's value with an exercise price equal to the face value of the bond.[2]

Using insights from option pricing theory, Merton (1974) derived an explicit valuation formula for default-risky bonds that default at maturity. The valuation formula requires knowing the following inputs: the value of the firm, the face amount of the debt, the volatility of the firm's value, the yield on a default-free bond that matures at the same time, and the time to maturity of the bond.

The yield on a defaultable bond should exceed the yield on an otherwise identical default-free bond because risk-averse investors must be compensated for the default risk in the form of additional yield or they will not hold a defaultable bond. The differential yield between a defaultable and a default-free bond (identical along all other dimensions) is known as the credit spread of the defaultable bond. In Merton's model, the credit spread increases as the leverage of the firm rises. The firm's leverage is measured by the ratio of the present value of the face amount of debt (discounted by the risk-free rate) to the value of the firm. This increase in the credit spread is natural because increased leverage heightens the probability that the firm may default. Higher default probability is reflected in an increase in the credit spread. Similarly, a rise in the volatility of the firm's value increases the probability that the firm may default, thus expanding the credit spread.

An important aspect of any model for valuing defaultable bonds is the term structure of credit spreads. As the term structure of interest rates in the Treasury market shows how the yields of zero-coupon bonds vary with their maturities, the term structure of credit spreads depicts the relationship between the credit spreads of defaultable bonds and their respective maturities. The term structure of credit spreads not only is important from a valuation perspective but also has some relevance for banking regulations related to allocating capital for credit-risky instruments. At the time of this writing, such regulations do not always recognize the term structure of credit spreads in defaultable bonds and other instruments. In other words, for allocating capital to cover potential defaults and credit downgrades, a one-year default-risky bond is treated the same as a ten-year default-risky bond although the two bonds may have quite different default and downgrade probabilities.

Many financial institutions hold large amounts of default-risky securities of various degrees of complexity in their portfolios, and it is important that these institutions have a reliable estimate of the resulting credit exposure.

In Merton's model, the term structure of credit spreads depends on the current credit quality of the issuer. Credit quality is measured by the ratio of debt to the firm's value; higher levels of debt lower credit quality by increasing the probability of default. In particular, the term structure of credit spreads is upward-sloping for high-credit-quality issuers, downward-sloping for low-credit-quality issuers, and hump-shaped for intermediate-quality issuers. Why do these patterns hold true? The value of the default-risky bond depends on the probability of default, which in turn depends on the value of the firm. If a firm is currently enjoying high credit quality, the impact on the bond value of further improvement in the credit quality through further increases in the firm value is limited because the payoff from the bond at maturity is capped at its face value. On the other hand, the firm's

1. *An example would be instruments that help hedge default/credit risk, such as credit derivatives.*
2. *Payoff from the bond at maturity is* $\min(V, B) = B - \max(B - V, 0)$, *where* min( ) *and* max( ) *give the minimum and maximum, respectively, of two quantities,* V *is the value of the firm, and* B *is the face value of the bond.*

Glossary

Absolute Priority: The strict seniority order in which claims to the firm's assets are paid in the event of bankruptcy/default.

Call Option: An option that gives its owner the right (but not the obligation) to buy the underlying asset at a fixed price (called the strike or the exercise). This right can be exercised at some fixed date in the future (European option) when the option matures or at any time until the option matures (American option).

Full Two-Way Payments Scheme: A swap settlement scheme (upon default) in which the party for whom the swap is an asset, even though the party may be the defaulter, has a claim equal to the current value of the swap from the counterparty.

LIBOR: An acronym for the London Interbank Offer Rate, the interest rate on dollar-denominated deposits outside the United States, deposited by one bank with another bank.

Poisson Process: A type of statistical process often used to describe the random arrival (through time) of customers in a queue, such as the arrival of telephone calls in a switchboard. The number of arrivals until a certain point of time has a Poisson distribution, and the interarrival times (that are statistically independent) are exponentially distributed. Thus, the Poisson process jumps by a certain amount at each arrival time.

Put Option: An option that gives its owner the right (but not the obligation) to sell the underlying asset at a fixed price (called the strike or the exercise). This right can be exercised at some fixed date in the future (European option) when the option matures or at any time until the option matures (American option).

Swap: A contract involving periodic exchange of payments between two parties according to some prespecified terms. In an interest rate swap, one party pays a fixed rate (called the swap rate) and receives a floating rate, usually tied to the LIBOR, on a notional principal.

Swap Rate: The fixed rate paid in a fixed-for-floating swap.

Term Structure of Credit Spreads: A relationship that shows how the differential yields between default-risky and default-free zero-coupon bonds that are otherwise similar vary with maturity.

Transition Matrix: A matrix in which a typical element corresponds to the probability of transition (of the debt of a firm) to a different credit rating over a period of time, given that the debt has a particular credit rating as of today.

credit quality may deteriorate with the passage of time, thus increasing the risk of default. In other words, the upside potential is limited and the downside risk is substantial as time elapses. As a result, credit spreads widen as maturity increases for high-quality bonds, resulting in an upward-sloping term structure of credit spreads. On the other hand, for a firm that currently has low credit quality, the downside risk is limited and the upside potential is substantial as time elapses. Therefore, one would expect credit spreads to decrease with maturity, yielding a downward-sloping term structure of credit spreads.

The empirical evidence on term structure of credit spreads and credit quality is mixed. Sarig and Warga (1989) and Fons (1994) document evidence supporting the relation between the term structure of credit spreads and credit quality mentioned above. However, Helwege and Turner (1997) examine bonds of different maturity issued by the same firm and find that the term structure of credit spreads of some low-quality firms is upward-sloping.

Despite its simplicity and intuitive appeal, Merton's model has many limitations. First, in the model the firm defaults only at maturity of the debt, a scenario that is at odds with reality. Second, for the model to be used in valuing default-risky debts of a firm with more than one class of debt in its capital structure, the priority/seniority structures of various debts have to be specified. Also, this framework assumes that the absolute-priority rules are actually adhered to upon default in that debts are paid off in the order of their seniority. However, empirical evidence in Franks and Torous (1989, 1994) and from other researchers indicates that the absolute-priority rule is often violated. Yet another problem with the Merton model is that the value of the firm, an input to the valuation formula, is very difficult to ascertain. Unlike the stock price in the Black-Scholes-Merton formula for valuing equity options, the current market value of a firm is not easily observable. One can argue that, ideally, the firm's market value is equal to the market value of its

TABLE 1
Strengths and Drawbacks of Various Models for Default-Risky Bonds and Swaps

Model	Advantages	Drawbacks
Merton (1974)	Simple to implement.	(a) Requires inputs related to the value of the firm. (b) Default occurs only at the maturity of the debt. (c) Information in the history of defaults and credit-rating changes cannot be used.
Longstaff and Schwartz (1995)	(a) Simple to implement. (b) Allows for stochastic term structure and correlation between defaults and interest rates.	(a) Requires inputs related to the value of the firm. (b) Information in the history of defaults and credit-rating changes cannot be used.
Jarrow, Lando, and Turnbull (1997)	(a) Simple to implement. (b) Can exactly match the existing prices of default-risky bonds to infer risk-neutral probabilities of defaults and credit-rating changes. (c) Uses the information in the history of defaults and credit-rating changes.	(a) Correlation not allowed between default probabilities and the level of interest rates. (b) Credit spreads change only when credit ratings change.
Lando (1998)	(a) Allows correlation between default probabilities and interest rates. (b) Allows many existing term-structure models to be easily embedded in the valuation framework.	(a) Historical probabilities of defaults and credit-rating changes are used under the assumption that the risk premiums due to defaults and rating changes are zero.
Duffie and Singleton (1997a, b)	(a) Allows correlation between default probabilities and the level of interest rates. (b) Recovery ratio can be random and depend on the predefault value of the security. (c) Any default-free term structure model can be accommodated, and existing valuation results for default-free term structure models can be readily used.	(a) Information in the history of credit-rating changes and defaults cannot be used.
Duffie and Huang (1996) (swaps)	(a) Has all the advantages of Duffie and Singleton. (b) Asymmetry in credit qualities is easily accommodated. (c) ISDA guidelines for settlement upon swap default can be incorporated.	(a) Information in the history of credit-rating changes and defaults cannot be used. (b) Can be computationally burdensome to implement for some swaps, such as cross-currency swaps, if domestic and foreign interest rates are taken to be random.

equity plus the market value of the debt; however, the market values of the various classes of debt a firm issues are difficult to ascertain because many corporate debts are very thinly traded, and those arranged with a private lender are not traded at all. Further, values of many firms may embody intangible brand-name components that may simply be unobservable except perhaps during mergers or acquisitions.

The drawbacks of the Merton (1974) model have led researchers to develop other models for valuing defaultable debt. One class of models relaxes some of the questionable assumptions of the Merton model but still requires parameters related to the firm's value as inputs in the valuation formula. This approach is often called the structural approach.[3] The other approach to valuing defaultable debt, called the reduced-form approach, does not require any parameters related to the value of the firm.

Shortcomings of structural models make it necessary to look at other classes of models for valuing defaultable securities that are not predicated on the value of the firm and that take into account credit-rating changes.

Structural Models

The structural models for valuing defaultable debt relax one of the unrealistic assumptions of the Merton model, namely, that default can occur only at the maturity of the debt when the firm's assets are no longer sufficient to cover debt obligations. Instead it is assumed that default may occur any time between the issuance and maturity of the debt and that default is triggered when the value of the firm (that is, its assets) reaches a lower threshold level.[4] It is also often assumed that debtholders, upon default, get back a fraction of the face value of the debt, sometimes called the recovery ratio, and that the recovery ratio is known a priori. While this assumption is somewhat unrealistic, it circumvents the difficult issue of explicitly specifying the seniority structure of debt, a drawback of the Merton (1974) model. Some authors—for example, Longstaff and Schwartz (1995)—argue that, by looking at the history of defaults and the recovery ratios for various classes of debt of comparable firms, one can form a reliable estimate of the recovery ratio.

The structural models may be difficult to implement in terms of actually valuing defaultable debt. This difficulty lies in the fact that some of the inputs to the valuation formula require parameters related to the value of the firm and, as noted before, the value of the firm is difficult to quantify. However, one may argue that the model's parameters could always be backed out or inferred from the market prices of some traded bonds of the firm in question.[5] For example, if a valuation model has n unknown parameters, then observing the prices of the firm's n bonds (say, of n different maturities) at a given time will yield the values of the n parameters such that the model prices of the defaultable bonds equal their market prices.[6]

Inferring the model parameters by matching the model's bond prices with the market bond prices is conceptually similar to the widely used practice of inferring a volatility (an unobservable quantity) from the observed market price of an option using the popular Black-Scholes (1973) model for equity options. In turn, the inferred/implied parameters can be used to price other bonds of the firm. However, there are some differences. First of all, one can easily relate to the volatility inferred from an option using the Black-Scholes model because the implied volatility is simply the expected average volatility that is supposed to prevail until the option expires. Observing past stock prices provides a good estimate of historical volatility, so it is not difficult to judge whether the inferred volatility estimate is reasonable. It is difficult, though, to ascertain whether the implied parameters related to the firm's value are reasonable. As a result, using these implied parameters to price other classes of bonds issued by the firm with a sufficient degree of confidence is not easy.

In addition, there have been cases in which an otherwise solvent firm has sought the protection of the bankruptcy court for previously unanticipated future legal liabilities (see Franks and Torous 1989). The relevance of structural models for these cases is questionable given that the defaults are not directly related to the current firm values but are "sudden surprises" (Duffie and Lando 1998). Another drawback of the structural models is that they cannot incorporate credit-rating changes that occur quite frequently for default-risky corporate debts. Many corporate bonds undergo credit downgrades by credit-rating agencies before they actually default, and bond prices react to these rating changes either in anticipation or when they occur. Thus, any valuation model should take into account the uncertainty associated with credit-rating changes as well as the uncertainty surrounding default. The shortcomings of structural models make it necessary to look at other classes of models for valuing defaultable securities that are not predicated on the value of the firm and that take into account credit-rating changes.

Reduced-Form Models

Unlike structural models, reduced-form models do not condition default explicitly on the value of the firm, and parameters related to the firm's value need not be estimated to implement the model. Also, reduced-form models fundamentally differ from typical structural models in the degree of predictability of the default. In fact, they are more general than the structural

models because they can easily accommodate defaults that are sudden surprises.

A typical reduced-form model assumes that an exogenous random variable drives default and that the probability of default over any time interval is nonzero. Many reduced-form models further assume that the probability of default could vary through time, possibly with variations in the level of interest rates.[7] Actual default occurs when the random variable undergoes a discrete shift in its level. The time at which the discrete shift will occur cannot be foretold on the basis of information available today. In other words, the time at which default might occur is a random variable. However, even in a structural model such as that of Longstaff and Schwartz (1995), the default time is not known in advance because the value of the firm is a random variable (the exact time the value of the firm will touch the lower threshold that results in default is not necessarily predictable on the basis of current information). Yet there are technical conditions that make a crucial distinction between the properties of the default time in most structural models and those in reduced-form models. Roughly speaking, the default time in reduced-form models is much more unpredictable than in structural models, where the time of bankruptcy can be foretold just before it occurs so that, as Duffie and Lando have remarked, in most structural models "bankruptcy occurs not with a bang but with a whimper" (1998).[8]

It may appear that reduced-form models are somewhat ad hoc in that default is modeled only implicitly through the discrete shift of an exogenous variable. However, the value of a firm used in structural models can only be imprecisely observed, and one of the motivations for building reduced-form models is to circumvent using parameters related to an imprecisely observed quantity. Further, by allowing the default probability to vary through time and to depend on the level of interest rates, a reduced-form model can be made rich enough to accommodate many stylized features of defaults, as in Lando (1998) and Duffie and Singleton (1997a).

The following example shows how one can value a defaultable zero-coupon bond in a simple reduced-form model, specifically the model of Jarrow, Lando, and Turnbull (1997). Suppose that the value of a default-free zero-coupon bond that will mature at time T and pay one dollar at maturity is known at time t. Denote the value of this bond by $p(t, T)$. If $v_i(t, T)$ denotes the value of a defaultable zero-coupon bond of a firm that currently has credit rating i (for example, AAA) at time t, will mature at time T, and has a promised payoff of \$1 at maturity, then Jarrow, Lando, and Turnbull show that

$$v_i(t, T) = p(t, T)[\phi + (1 - \phi)q_i(t, T)], \qquad (1)$$

where ϕ is the recovery ratio—the fraction of the face value (\$1) that is recovered at time T after default—and $q_i(t, T)$ denotes the probability of a default occurring after T given that the debt has credit rating i as of time t. The valuation formula indicates that the higher the probability of default not occurring before maturity, the higher the value of the defaultable bond is and therefore the lower the credit spread is. To arrive at the valuation formula, Jarrow, Lando, and Turnbull (1997) assume that default is independent of the level of interest rates.

The assumption of independence is not critical to reduced-form modeling, however. Lando (1998) relaxes the independence assumption and extends Jarrow, Lando, and Turnbull's model so that the default probability can depend on the level of interest rates. Since interest rates vary through time, Lando does not restrict changes in credit spreads in terms of when they can occur. Lando's model is discussed in Box 1. (Box 1 also illustrates the valuation principle of Duffie and Singleton 1997a for defaultable debt, which is related to but somewhat different from the principles of Jarrow, Lando, and Turnbull 1997 and Lando 1998 in that credit-rating changes are not taken into account.)

At this stage, it might seem that valuing a defaultable bond is relatively straightforward in a typical reduced-form model: one needs to know the recovery ratio, the price of an identical maturity default-free bond, and the probability of default. Recovery ratios can be estimated by looking at past recovery ratios of similar bonds, and

3. Technically, Merton's model is also a structural model.

4. One of the earlier studies based on this framework is Black and Cox (1976). More recent ones include Longstaff and Schwartz (1995) and Nielsen, Saa-Requejo, and Santa-Clara (1993).

5. The same argument can be made about the Merton model.

6. The parameters are found by solving a system of n *simultaneous equations (that are generally nonlinear) in* n *unknowns using a numerical routine for finding out the roots of equations.*

7. It is assumed that defaults are governed by what are known as Poisson processes. In a model without credit-rating changes, default occurs when the Poisson counter changes for the first time, for example, from zero to one. The intensity of the Poisson process, which determines the probability of default over a small time interval, could depend on the level of interest rates.

8. Most structural models based on the value of a firm assume that, as a mathematical function, the value of the firm is continuous (in time). As a result, the time of the bankruptcy can actually be predicted just before it happens and hence there are no sudden surprises (Duffie and Lando 1998). One can, however, model the value of a firm as a function consisting of a continuous part and a jump part—the so-called jump diffusion process, as in Zhou (1997)—so that the default can be a sudden surprise.

BOX 1

Reduced-Form Models of Duffie and Singleton (1997a) and Lando (1998)

It is well known that the price (at time t) of a default-free zero-coupon bond (which is traded continuously) that matures at time T (and pays out $\$Y$) is given by

$$E_t[\exp(-\int_t^T r_u du)Y],$$

where r_t is the interest rate on a loan that matures at the next instant and $E_t(\ )$ is the conditional expectation under the risk-neutral distribution of r_t. Duffie and Singleton (1997a) assume that defaults are governed by a Poisson process with intensity λ_t. The probability of default over a small time interval could be thought of as proportional to λ_t. In particular, the probability of default could be time-varying and depend on the level of interest rates. Denote λ_u^* to be the intensity corresponding to the risk-neutral default probability. Under the assumption that upon default one recovers a fraction—say, ϕ—of the predefault value of the bond, Duffie and Singleton show that the price of a defaultable zero-coupon bond that matures at time T (and is supposed to pay $\$Y$) is

$$E_t\left(\exp\{-\int_t^T [r_u + \lambda_u^*(1-\phi)]du\}Y\right).$$

Thus, the valuation formula for risky bonds is the same as for risk-free bonds, except for an adjusted short rate given by $r_t + \lambda_t^*(1-\phi)$ that is used for discounting purposes. In other words, with the possibility of default, a default risk premium has to be added to the interest rate for discounting. The required risk premium increases with an increase in the probability (risk-neutral) of default and increases in the amount of value lost upon default.

Lando's (1998) model also gives rise to a valuation formula somewhat similar to that of Duffie and Singleton (1997a) under a similar set of assumptions. Additionally, the model can incorporate credit-rating changes, specifically the probabilities of a firm transitioning to a different credit rating or default as computed from historical data. However, the risk-neutral probabilities of default and credit-rating changes are taken to be the same as the ones computed from historical data, a somewhat questionable assumption.

the price of an identical maturity default-free bond (that is, a Treasury bond/bill) can be observed in the market or, alternatively, can be estimated from observed prices of zero-coupon bonds of various maturities.[9] The problem lies in determining the default probability—not the historical probability of default but, roughly speaking, an artificial probability called the risk-neutral probability of default. The risk-neutral probability of default can be thought of as an adjusted probability that takes into account investors' compensation for default risk (see Box 2 for the principles behind risk-neutral valuation).

How does one determine the risk premium attributed to default for various bonds and thereby estimate the risk-neutral probabilities of default needed for valuation? All that can be estimated are the historical probabilities of credit-rating changes and defaults from historical data, probabilities that could be computed using data available from credit-rating agencies such as Moody's and Standard and Poor's. Of course, as with structural models, one can always infer the model parameters from a cross section of bond prices observed in the market and then use these inferred parameters to price similar bonds. However, as discussed earlier, a default-risky bond can undergo several credit-rating changes before it actually defaults, and the changes are priced by the market. The information from past credit-rating changes and defaults is useful and should not simply be ignored. Furthermore, there are instruments such as credit-sensitive notes and certain types of swaps (with credit triggers) whose payoffs explicitly depend on particular credit events occurring. A more complete framework for modeling defaultable instruments, therefore, has to take into account credit-rating changes.

Jarrow, Lando, and Turnbull (1997) have shown how one can use the probabilities of credit-rating changes and defaults computed from historical data to price defaultable bonds. The migration/transition probabilities of credit-rating changes and defaults can be estimated from the

information available in credit reports, such as *Moody's Special Report* (1992) or Standard and Poor's *Creditreview* (1993). Although the historical transition probabilities may not reflect the most current information because the probabilities may not be updated frequently, the information is useful. Jarrow, Lando, and Turnbull (1997) also show how one can use historical probabilities to estimate the risk-neutral probabilities (of defaults and credit-rating changes) at various future dates from a cross section of defaultable and default-free bonds of various maturities by exactly matching the observed market prices of defaultable and default-free bonds. These risk-neutral default probabilities can, in turn, be used to value other defaultable financial instruments of the firm, such as a swap. (See the appendix for Jarrow, Lando and Turnbull's procedure.)

There are other reduced-form models, such as those of Jarrow and Turnbull (1995) and Madan and Unal (1995). However, both of these models assume that defaults are not correlated with interest rates and cannot incorporate information in credit-rating changes.

Shortcomings of the Models

The structural and reduced-form models for valuing default-risky debt that have been discussed so far cannot readily incorporate financial restructuring that often occurs upon default, such as renegotiating of the terms of the debt contract by extending the maturity or lowering/postponing the promised payments, exchanging the debt for other forms of securities, or some combination of the above. Similarly, the institutional features of a reorganization under court supervision, such as Chapter 11 bankruptcy, cannot be incorporated in any of these valuation models without making them intractable. Debt restructurings anticipated by the market will be priced into the value of a defaultable bond in ways that none of these models captures. Another issue relevant to pricing default-risky securities is that, unlike Treasury securities, many defaultable securities are thinly traded. A liquidity premium may therefore be incorporated into the prices of these securities, another factor that is outside the realm of the valuation models discussed.

Empirical evidence regarding the validity of these models is rather limited. Duffee (1996) finds that reduced-form models based on the Duffie and Singleton (1997a) framework have difficulty explaining the observed term structure of credit spreads across firms of different credit qualities. Such problems could arise from incorrect statistical specifications of default probabilities and interest rates or from the model's inability to incorporate some of the features of default/bankruptcy mentioned in the previous paragraph.

Default-Risky Swaps

So far the discussion has described the different valuation models for default-risky bonds. Another class of instruments that is very heavily traded and could be subject to defaults is swaps—obligations created when parties swap streams of payments. Swaps come in various forms. The focus here is on the plain vanilla interest rate swap, which involves swapping interest payments at a fixed rate for payments at a floating rate, and simple currency swaps, in which principal and interest payments in one currency are swapped for payments in another currency. However, the general valuation principles discussed can be adapted to other types of swaps. Only reduced-form models for valuing default-risky swaps are discussed because models based on the value of the firm are usually difficult to implement.[10]

When a swap is initiated, by definition it has a market value of zero for both parties. However, with the passage of time, as interest rates and exchange rates change, a swap can become an asset to one party and a liability to the other. As a result, the expected loss profile from a swap is somewhat different from that of a straight loan such as a bond. For example, if default by the counterparty occurs when the swap is an asset (that is, has positive value) to a party, then the default represents a real loss as opposed to its occurring when the swap is a liability (has negative value).

A defaultable swap has option-like features embedded in it. In fact, valuation for defaultable swaps based on options theory has been advocated by Bollier and Sorensen (1994). This approach is somewhat ad hoc because the value of a defaultable swap is not derived using the fundamental principles of valuation, namely, by calculating the expectation (under the risk-neutral distribution) of the discounted future dividends/cash flows, including the cash flows that would accrue upon default. Instead, the value of a defaultable swap is arrived at by making some adjustments to the default-free value. Nevertheless, the approach is intuitive as it clearly illustrates the option-like features embedded in a defaultable swap.

To understand these adjustments, consider a swap in which only one of the parties—say, party B—can default at the next date while party A is default-free. Also assume that A does not recover anything upon default. If B defaults, then A's loss is either the current value of the swap (if the swap is an asset to A) or zero (if the swap is

9. See Bliss (1997) and Waggoner (1997) for various methods of computing the price of a zero-coupon bond of a given maturity from the observed prices of various Treasury bills, notes, and bonds.

10. For valuation of defaultable interest-rate swaps that is predicated on the value of the firm, see Cooper and Mello (1991) and Abken (1993).

Risk-Neutral Valuation

This box illustrates the basic principles behind risk-neutral valuation, which is often used to value risky assets such as derivatives and bonds. Consider a hypothetical economy with two assets, one risky and the other risk-free. Suppose the risky asset will pay \$3 and \$6, respectively, in the two states of nature, one of which will be realized tomorrow; the risk-free asset will pay \$1 irrespective of which state occurs. To find out the current values of these two assets, one needs to know how much the future dollars in the two states are worth as of today. Suppose one dollar in state 1 is worth p_1 and one dollar in state 2 is worth p_2 as of today. In fact, we may view the current price of an elementary security that pays off \$1 in state 1 (and nothing in state 2) as p_1 and the current price of an elementary security that pays off \$1 in state 2 (and nothing in state 1) as p_2. These elementary securities are often known as Arrow-Debreu securities, named after two famous economists, Kenneth Arrow and Gerard Debreu. In general, p_1 and p_2 may be different. If an additional dollar in state 2 is more valuable to the investor than in state 1, then the security that pays off in state 2 will be worth more to the investor than the security that pays off in state 1. The reason an additional dollar may be more valuable in state 2 than in state 1 could be a general market downturn in state 2.

A well-functioning economy should not admit free lunches or arbitrage in that there should not exist any portfolio of the two assets that does not cost anything today but pays off a positive quantity with certainty tomorrow.[1] It turns out that in the absence of arbitrage, a set of simple equations links p_1, p_2, and the future payoffs of the two assets to the present values of these assets. In general, one does not know p_1 and p_2. However, given the current values of the two assets, p_1 and p_2 can be inferred and, in turn, can be used to price a derivative asset such as an option on the risky asset. Assume that the current prices of the risky and risk-free asset are \$4.2 and \$0.9, respectively. Given the payoff structure, absence of arbitrage implies that the following system of equations has to hold:

$$3p_1 + 6p_2 = 4.2$$

$$p_1 + p_2 = 0.9.$$

One can solve the two equations to find $p_1 = 0.4$ and $p_2 = 0.5$. Thus, the market price of \$1 received in state 1 is 40 cents and the market price of \$1 received in state 2 is 50 cents. Also note that the above system of equations basically says that the two securities are linear combinations of the elementary Arrow-Debreu securities and that the prices of these Arrow-Debreu securities are positive. For example, the risky asset can be viewed as comprising three Arrow-Debreu securities that pay off in state 1 and six Arrow-Debreu securities that pay off in state 2.[2] Thus the pricing rule is linear and positive. A different valuation equation can be constructed from an algebraic rearrangement of the above set of equations to yield what is known as the risk-neutral valuation relationship. Under the risk-neutral valuation, the actual probabilities are adjusted so that the mean return on every asset (risky and risk-free) becomes the risk-free rate. The above system of equations can be rewritten as

$$(p_1 + p_2)[3p_1/(p_1 + p_2) + 6p_2/(p_1 + p_2)] = 4.2$$

$$(p_1 + p_2)[p_1/(p_1 + p_2) + p_2/(p_1 + p_2)] = 0.9.$$

Define two variables, q_1 and q_2, such that $q_1 = p_1/(p_1 + p_2)$ and $q_2 = p_2/(p_1 + p_2)$. These variables can be viewed as probabilities because each of them is nonnegative and $q_1 + q_2 = 1$. Also note that the gross return of the risk-free asset is $R = 1/(p_1 + p_2)$ because the current price of the risk-free security is $p_1 + p_2$ and it pays \$1 with certainty tomorrow. In terms of q_1, q_2, and R, the above system of equations can be written as

$$(1/R)(3q_1 + 6q_2) = 4.2$$

$$(1/R)(q_1 + q_2) = 0.9.$$

The equations above indicate that as of today (time t), the price of a security is $V_t = (1/R)E^q(V_{t+1})$, where $E^q(\ )$ denotes the statistical expectation over the probabilities q_1 and q_2, and V_{t+1} is tomorrow's (that is, $t + 1$) payoff from the asset. This expression is known as valuing the asset under risk-neutral probabilities. Note that under the artificial probabilities, q_1 and q_2, the expected price of the asset in the future is the current price multiplied by the risk-free return, R. In other words, the expected/mean return on every asset (risky and risk-free) is the risk-free rate. If investors are risk-neutral, the expected appreciation on every asset has to equal the risk-free rate. Hence, q_1 and q_2 are called the risk-neutral probabilities and are, in general, different from the

actual probabilities of the occurrences of the two states. Calculating the value of a risky asset as being the discounted expectation under the risk-neutral probabilities/distribution is the very fundamental principle of valuation and is valid even in more general settings with multiple states and multiple trading periods.

One can value all the derivative assets, such as call and put options, using either q_1 and q_2 or p_1 and p_2. Often it is the case that valuing a primary asset, such as a bond, or a derivative asset, such as an option, is much easier (algebraically or computationally) using the risk-neutral probabilities. Researchers and practitioners therefore often use the risk-neutral approach to value derivative assets and bonds, including defaultable assets. However, the actual probabilities of default (that can be computed from historical data) have to be adjusted to arrive at the risk-neutral default probabilities used for valuation.

1. Also, there should not exist any portfolio of the two securities such that an investor (who owns the portfolio) gets a net cash inflow as of today and does not have to pay anything tomorrow.

2. The prices of Arrow-Debreu securities are unique because there are two possible states of nature and two securities to span these two states.

a liability to A). Thus the exposure of A to B's default is the maximum of the following two quantities: the value of the swap to A or zero. In other words, the exposure of A to B's default resembles the payoff from a European call option on the value of the swap with a strike price of zero.[11] Party A has implicitly written an option to B and should be compensated for the same. The compensation is equal to the probability that B will default at the next date times the value of the above option. Since default can also take place at other future dates, this add-on option approach suggests that the amortized value of these compensations over the remaining life of a swap should be taken into account to arrive at the default-adjusted value of the swap to A.

In general, the value of the option will depend on the current shape of the term structure. For example, if the swap in question is an interest rate swap and A is the fixed rate payer, then in an upward-sloping term structure environment, the option has value to A. It has value because the floating rate payments, and hence the value of the swap to A, are expected to increase. Therefore, the possibility of default by B will result mostly in a lower fixed rate (the swap rate) being paid by A. If A can also default, then, following the same arguments as before, it is easy to see that A has bought an option from B and written another option to B. If the valuation allows for default possibilities at various other dates during the life of the swap, then, as for swaps in which only one party can default, each of these options has to be valued separately to get the default-adjusted value of the swap or, equivalently, the swap rate.

Although intuitive, the add-on option approach does not explicitly take into account some of the settlement issues upon default per the guidelines of the International Swaps and Derivatives Association (ISDA) that are usually part of a swap agreement. For example, in a full two-way payments scheme between two parties who are engaged in only one swap, the party for whom the swap is an asset, even though it may be the defaulter, has a claim equal to the current value of the swap from the counterparty. Often it could be the case that only a fraction of the swap value can be recovered. In particular, Duffie and Huang (1996) show that if parties to a swap are not symmetric in terms of their credit characteristics, settlement issues do matter, and accounting for them explicitly could yield swap values quite different from those achieved by using the add-on option approach. Before turning to Duffie and Huang's model, the discussion examines a more complete model for pricing defaultable swaps in which the parties are symmetric in terms of their credit characteristics.

Swaps with Symmetric Credit Risk

As Box 1 illustrates, computing the price of a defaultable bond is similar to computing a default-free bond price. The difference lies in the factor used for discounting. In pricing a default-free bond, cash flows from the bond are discounted at the risk-free interest rates whereas for a defaultable bond the relevant discounting factor is the sum of two terms: the risk-free interest rate and the product of two factors—one proportional to the risk-neutral default probability and another

11. Formally, the exposure of A is $\max(V_t, 0)$*, where* V_t *is the value of the swap at time* t *to A without default.*

that equals the fraction of predefault value of the bond lost upon default. In other words, discounting is done with respect to an interest rate adjusted for default probability and the fraction of value lost upon default. Since valuing a swap amounts to computing the expected value of the stream of cash flows at an appropriate discount rate, it is reasonable to expect that the above methodology for valuing defaultable bonds can be extended to value defaultable swaps.

Note that the price of a default-risky instrument differs from an equivalent (in terms of promised cash flows and maturity) default-free instrument through the extra term used in discounting that captures the probability of default and loss in value upon default. Thus, this extra term could be thought of as representing the credit quality of a firm/institution that can default on its obligations. In fact, Duffie and Singleton (1997b) show that in a swap involving both parties of symmetric credit quality (that is, the extra term is the same for both parties), valuation of defaultable swaps, taking into account the settlement payments upon default, can be very similar to that of defaultable bonds discussed in Box 1. In other words, the value of the swap is simply the statistical expectation (under the risk-neutral distribution) of the future cash flows/dividends from the swap, discounted by an interest rate adjusted to reflect the risk-neutral default probability and the fraction of the predefault swap value lost upon default. Duffie and Singleton's model does implicitly take into account the previously discussed options embedded in a defaultable swap. However, it is more complete than the add-on option approach in that it values the swap using the fundamental valuation principles, instead of some adjustments to the default-free value of the swap. It is to be noted that the model requires risk-neutral probability of default in the valuation formula and therefore cannot readily be implemented solely on the basis of historical data. Instead, the extra term used in discounting (and other parameters) has to be inferred from the observed market prices of swaps. Also, because the model does not take into account credit-rating changes, the model cannot readily price swaps with embedded credit triggers that result in termination of the swap if a particular credit event occurs.

By allowing the default probability to vary through time and to depend on the level of interest rates, a reduced-form model can accommodate many stylized features of defaults.

To arrive at a swap value in Duffie and Singleton (1997b) for an interest rate swap, one needs to assume the type of statistical process that the adjusted interest rate follows through time. In their empirical work, the authors assume that the default-adjusted interest rate follows what is known as the square root process (see Cox, Ingersoll, and Ross 1985).[12] The model also permits other statistical specifications for the default-adjusted interest rate. Duffie and Singleton find that their model fits the interest rate swap data reasonably well. However, they do not describe how well their estimated model predicts future swap rates. Although a particular model may fit the data well during the course of statistical estimation in that the pricing errors could be quite small, it is not necessarily true that the model would be equally good at predicting future market values of the securities based on the parameters estimated.[13]

Swaps with Asymmetric Credit Risk

Duffie and Huang (1996) develop a model for valuing defaultable swaps in which the credit qualities of a swap's two parties can be asymmetric. The model also incorporates features of settlement payments upon default per some of the ISDA guidelines.

Consider a plain vanilla interest rate swap in which A is the floating rate payer (paying the LIBOR), B is the fixed rate payer, and settlements upon default will allow for full two-way payments. For A, predefault dividends are simply the net payments (fixed minus floating) that would accrue according to the swap agreement whereas the postdefault dividend is the payment made after default per the full two-way payments scheme. Thus, the postdefault dividend may be the value of the swap (or a fraction thereof) at the time of default or zero. The authors calculate the value of a defaultable swap to be the statistical expectation (under the risk-neutral distribution) of the discounted dividends that would be paid until default.[14] However, the discounting rate used is quite different from that of Duffie and Singleton because of the asymmetric credit quality.

According to Duffie and Huang's valuation procedure, if the swap's value is positive for a party, then it is the default characteristics of the other party that matter for arriving at the default-adjusted interest rate used for discounting. For example, if the value of the swap to A is positive, then the extra term added in the discounting factor is the product of a factor proportional to the risk-neutral default probability of B and a factor that equals the fraction of the swap value lost upon default through transacting with B. Since the swap's value can switch between positive and negative values for either of the parties, the discounting rate switches between the default characteristics of A and B. The switching discount rate makes the valuation model nonlinear in that multiplying the promised dividends/cash flows (to A) by a factor does not change the value of the swap to A by the same factor. This characteristic is in contrast to the valuation of a typical

financial instrument without default in which, for example, doubling/halving the promised cash flows would result in the value of the instrument being doubled/halved.

One of Duffie and Huang's (1996) primary insights is that the degree of asymmetry in the default characteristics of the two parties in a plain vanilla interest rate swap (with no exchange of principals) is not very important in determining swap rates. For example, replacing the fixed rate payer with a riskier counterparty whose corporate bond yield is 100 basis points higher changes the swap rate (the fixed rate paid in the swap) by approximately a basis point. In contrast, the authors claim that the add-on option approach can substantially overstate the impact of asymmetry in credit qualities despite the netting of fixed and floating payments that in general make a swap much less credit sensitive than a straight loan, such as a bond. Consequently, the estimate of expected default losses in switching from a higher-credit-quality to a lower-credit-quality counterparty can be substantially higher under the add-on option approach, perhaps resulting in unnecessary capital being set aside to cover default losses during the course of managing credit risk. Duffie and Huang (1996) also show that in a currency swap involving fixed-for-fixed payments in which an exchange of principals takes place, the impact of the asymmetry in the credit risk is somewhat higher. The asymmetry matters more because, besides the periodic exchange of interest payments, principal payments are exchanged at the end of the swap, resulting in an additional exposure. The exposure increases if the volatility of the exchange rates is higher. A similar result for currency swaps has also been found by Hull and White (1995) but under more restrictive conditions.

Conclusion

Although valuation models for defaultable securities date back to Merton (1974) and researchers have improved considerably on the basic Merton framework, problems remain. Table 1 highlights some of the advantages and disadvantages of using the valuation models discussed in this article.

One class of models, sometimes referred to as structural models, requires the use of an imprecisely observed quantity (or quantities), such as a firm's value or variables related to it, in the valuation formula. In contrast, another class of models known as reduced-form models does not need firm-value-related variables and so holds more promise. It is often the case that the valuation formulas of reduced-form models are very similar to those used for valuing the corresponding default-free securities. The only difference is that the discounting factor is adjusted upward, taking into account the probability of default and the fraction of value lost upon default. Therefore, many of the existing results for valuing default-free securities such as default-free bonds can be readily extended to price default-risky securities. This advantage is significant as some of the models for valuing default-free securities are analytically and computationally tractable. Some of the reduced-form models can also incorporate the historical probabilities of credit-rating changes and defaults. These probabilities not only expand the information set used in valuation but also can be crucial for pricing instruments whose payoffs are explicitly linked to credit events, such as credit upgrades or downgrades.

However, because a limited amount of work has been done so far in validating the empirical efficacy of various reduced-form models, caution is warranted in using these models for pricing and hedging defaultable securities. Also, it is often the case that if one allows for realistic features, such as correlation between the interest rate level and default probabilities, historical probabilities of credit-rating changes and defaults can be used in a tractable fashion only under the questionable assumption that the risk premiums due to defaults and credit-rating changes are zero. Many of the institutional features of bankruptcy and defaults, such as renegotiation between the debtor and creditors and rescheduling of debts, cannot be readily incorporated in any of the valuation models discussed in this article as otherwise the models would be rendered intractable. It is hoped that the next generation of valuation models will be able to incorporate at least some institutional features and be able to use the historical probabilities of defaults and credit-rating changes without making unnecessarily strong assumptions.

12. This type of interest rate has the advantage that negative interest rates are precluded.

13. In technical jargon, good in-sample pricing by a model does not always lead to good out-of-sample pricing. This characteristic is especially true if a model is not parsimonious in the number of parameters that need to be estimated.

14. Duffie and Huang (1996) show that in the presence of settlement payments, one needs to take into account a feedback effect from the value of the swap itself to the stream of dividends that needs to be valued to arrive at the swap value. The add-on option approach ignores this type of feedback effect.

Estimating the Risk-Neutral Probabilities of Default and Credit-Rating Change

The purpose of this appendix is to show how one can estimate the risk-neutral probabilities of default and credit-rating changes from a cross section of default-free and defaultable bond prices of various maturities using the methodology of Jarrow, Lando, and Turnbull (1997). The authors suggest an algorithm such that one can exactly match the observed prices of default-free and defaultable bonds and at the same time use the historical probabilities of migration to other credit ratings, including default. Alternatively, if one assumes that the defaultable bond prices are observed with error, their procedure yields estimates of risk-neutral default probabilities by using the information in the history of credit-rating changes. The estimated risk-neutral default probabilities can then be used to price other defaultable securities of a firm, such as a plain vanilla swap the firm is party to or a swap with embedded credit triggers.

The risk-neutral probabilities of default or credit-rating changes are computed by multiplying the historical probabilities by a factor that can be interpreted as a default risk premium.[1] Table A presents a hypothetical transition matrix of credit-rating changes and defaults. There are three possible rating categories—AAA, BBB, and D. Assume that D represents default by a firm on its debt. The matrix has nine entries, and a typical entry shows the probability of moving from one rating category to another in one period. For example, the entry of 0.1 for row 2, column 3 indicates that the probability is 0.1 that a firm currently rated BBB will default in one period. Similarly, an entry of 0.06 for row 1, column 2 indicates that the probability of a currently AAA-rated debt becoming BBB-rated is 0.06 in the next time period. Note that all entries in the last row of the transition matrix are 0 except the last, which is 1. The probability of a debt migrating to a different credit-rating category once it enters default is 0, and therefore the probability of remaining in the default state once the debt enters default is 1.

Let $v_i(0, 1)$ and $p(0, 1)$ denote the prices of a defaultable and a default-free bond, respectively, at time 0 (that is, the current time); all bonds mature at time 1 and are supposed to pay $1 at maturity. Assume that the fraction of face value to be recovered upon default from the defaultable bond is ϕ. Given this information and the assumptions the authors make, two risk premiums have to be determined, namely, moving from AAA and BBB to other credit categories. With these risk premiums, the risk-neutral probabilities of credit-rating changes, including default, can be found by multiplying the historical probabilities and the risk premiums. Given this example, Jarrow, Lando, and Turnbull (1997) show that the risk premium attributed to a credit-rating category i, denoted by Π_i, is given by

TABLE A
Transition Matrix of Defaults and Rating Changes

	AAA	BBB	D
AAA	0.9	0.06	0.04
BBB	0.05	0.85	0.1
D	0	0	1

$$\Pi_i = [p(0, 1) - v_i(0, 1)]/[p(0, 1)(1 - \phi)q(\mathrm{i}, 3)],$$

where $q(\mathrm{i}, 3)$ is the entry in the row i and column 3 of the transition matrix and represents the probability of moving from rating category i to default. In the example, Π_i needs to be calculated only for the AAA and BBB firms. Once Π_i is computed the risk-neutral default probability for a firm currently rated AAA will be given as $\Pi_{\mathrm{AAA}} \times 0.04$ (the entry in row 1, column 3 of the transition matrix). Similarly, the risk-neutral default probability for a BBB-rated firm is $\Pi_{\mathrm{BBB}} \times 0.1$ (the entry in row 2, column 3 of the transition matrix). The risk-neutral probabilities calculated in the example are due to credit-rating changes and defaults occurring between time 0 and 1. For risk-neutral probabilities of credit-rating changes and defaults at other times in the future—say, between time 1 and time 2—one has to use defaultable and default-free bonds that mature at time 2. For details in calculating the risk-neutral probabilities at other times in the future, see Jarrow, Lando, and Turnbull (1997).

1. *Jarrow, Lando, and Turnbull (1997) make the somewhat questionable assumption that the risk premium in moving from credit rating* i *to credit rating* j *is the same as moving from credit rating* i *to credit rating* k.

REFERENCES

ABKEN, PETER. 1993. "Valuation of Default-Risky Interest-Rate Swaps." *Advances in Futures and Options Research* 6:93–116.

BLACK, FISCHER, AND JOHN COX. 1976. "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions." *Journal of Finance* 31:351–67.

BLACK, FISCHER, AND MYRON SCHOLES. 1973. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81:637–54.

BLISS, ROBERT. 1997. "Testing Term Structure Estimation Methods." *Advances in Futures and Options Research* 9:197–231.

BOLLIER, THIERRY F., AND ERIC H. SORENSEN. 1994. "Pricing Swap Default Risk." *Financial Analysts Journal* 50 (May/June): 23–33.

COOPER, IAN, AND ANTONIO MELLO. 1991. "The Default Risk of Swaps." *Journal of Finance* 46:597–620.

COX, JOHN, JONATHAN INGERSOLL, AND STEVEN ROSS. 1985. "A Theory of the Term Structure of Interest Rates." *Econometrica* 53:385–407.

DUFFEE, GREGORY. 1996. "Estimating the Price of Default Risk." Board of Governors of the Federal Reserve System Working Paper No. 96-29.

DUFFIE, DARRELL, AND MING HUANG. 1996. "Swap Rates and Credit Quality." *Journal of Finance* 51:921–49.

DUFFIE, DARRELL, AND DAVID LANDO. 1998. "Term Structure of Credit Spreads with Incomplete Accounting Information." Stanford University Working Paper.

DUFFIE, DARRELL, AND KENNETH SINGLETON. 1997a. "Modeling Term Structures of Defaultable Bonds." Stanford University Working Paper.

———. 1997b. "An Econometric Model of the Term Structure of Interest-Rate Swap Yields." *Journal of Finance* 52:1287–321.

FONS, JEROME S. 1994. "Using Default Rates to Model the Term Structure of Credit Risk." *Financial Analysts Journal* 50 (September/October): 25–32.

FRANKS, JULIAN, AND WALTER TOROUS. 1989. "An Empirical Investigation of U.S. Firms in Reorganization." *Journal of Finance* 44:747–69.

———. 1994. "A Comparison of Financial Recontracting in Distressed Exchanges and Chapter 11 Reorganizations." *Journal of Financial Economics* 35:349–70.

HELWEGE, JEAN, AND CHRISTOPHER TURNER. 1997. "The Slope of the Credit Yield Curve for Speculative-Grade Issuers." Federal Reserve Bank of New York Working Paper No. 97-25.

HULL, JOHN, AND ALAN WHITE. 1995. "The Impact of Default Risk on the Prices of Options and Other Derivative Securities." *Journal of Banking and Finance* 19:299–322.

JARROW, ROBERT, DAVID LANDO, AND STUART TURNBULL. 1997. "A Markov Model for the Term Structure of Credit Spreads." *Review of Financial Studies* 10:481–523.

JARROW, ROBERT, AND STUART TURNBULL. 1995. "Pricing Options on Financial Securities Subject to Default Risk." *Journal of Finance* 50:53–86.

LANDO, DAVID. 1998. "On Cox Processes and Credit Risky Securities." *Review of Derivatives Research*, forthcoming.

LONGSTAFF, FRANCIS, AND EDUARDO SCHWARTZ. 1995. "A Simple Approach to Valuing Risky Fixed and Floating Debt." *Journal of Finance* 50:789–819.

MADAN, DILIP, AND HALUK UNAL. 1995. "Pricing the Risks of Default." University of Maryland Working Paper.

MERTON, ROBERT. 1973. "The Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4:141–83.

———. 1974. "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates." *Journal of Finance* 29:449–70.

MOODY'S INVESTORS SERVICE. 1992. "Corporate Bond Defaults and Default Rates." *Moody's Special Report*, December.

NIELSEN, LARS, JESUS SAA-REQUEJO, AND PEDRO SANTA-CLARA. 1993. "Default Risk and Interest Rate Risk: The Term Structure of Default Spreads." INSEAD Working Paper.

SARIG, ODEG, AND ARTHUR WARGA. 1989. "Some Empirical Estimates of the Risk Structure of Interest Rates." *Journal of Finance* 44:1351–60.

STANDARD AND POOR'S. 1993. "Corporate Default, Rating Transition Study Updated." *Creditreview* (supplement to *Creditweek*), January 25.

WAGGONER, DANIEL F. 1997. "Spline Methods for Extracting Interest Rate Curves from Coupon Bond Prices." Federal Reserve Bank of Atlanta Working Paper No. 97-10, November.

ZHOU, CHUNSHENG. 1997. "A Jump-Diffusion Approach to Modeling Credit Risk and Valuing Defaultable Securities." Board of Governors of the Federal Reserve System Working Paper No. 97-15.