An Intertemporal Model of Consumption and Portfolio Allocation

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Abstract: We develop an infinite time horizon, continuous time model of portfolio choice and consumption allocation for an investor seeking to maximize the expected utility of his life-time consumption. In this model, the investor is endowed with capital that can be invested in long-lived capital assets and has, in addition, a stochastic stream of cash flows that could be interpreted as either a wage income stream or a stochastic endowment flow. We obtain a complete and original solution to the consumption-portfolio choice problem for the negative exponential and quadratic utility functions and special case solutions for the general power and log utility functions. The results obtained in this paper have significant implications for the theory of asset prices, the theory of mutual funds, optimal portfolio strategies of investors, and so forth. The results of the model can also be easily extended to one with a finite time horizon.

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SECTION I: INTRODUCTION

An important topic in the area of investments relates to the optimal consumption-portfolio allocation problem. The objective is to first characterize investor behavior and, second, develop an equilibrium valuation rule for different investment alternatives facing a set of investors.

The single-period capital asset pricing model (CAPM) is the simplest of all the models of asset valuation and portfolio choice. Intertemporal models on lines similar to the CAPM have also been developed, of which the consumption-based "beta" model is an important one.

The basic assumption underlying all these models, whether with a heterogenous set of investors or a homogenous group, is that the markets in assets are complete and hence there exists a market-based valuation that is in some respects an aggregate of individual investors' private valuations. This assumption leads to simple valuation equations and optimal portfolios that satisfy what may be termed as the mutual fund conditions.

Investors facing an opportunity set of several investment alternatives will choose to allocate their wealth among a common set of unique portfolios or mutual funds. Different investors will allocate different amounts of their investment capital to these portfolios. As a consequence, portfolio management can be decentralized and the mutual funds can generate benefits from scalar economies.

This approach has attracted a significant research effort together with attempts to test the applicability of the "mutual fund" theorems to the "real world." Unfortunately, the real world experience is not very supportive of the modelling approach hitherto undertaken. Investors do not, in general, hold "diversified" portfolios or even several mutual funds at a time.

Several explanation are offered for this observed behavior, chief among which are those relating to transaction costs, information asymmetries, bounded rationality, etc. A closer examination reveals, however, that the reasons offered are not very strong. Costs of investing in mutual funds are continuously being driven down, as well as the information about mutual funds being upgraded to enable investors to make rational choices.

The primary shortcoming in all the models has been the almost complete emphasis on the "market" set of assets and attendant neglect of those investment opportunities that are not available to everyone, or those investment alternatives that are individual-specific and claims on which cannot be issued. A further complicating feature is the pre-eminence of revenue streams from the labor market which are the primary source of wealth aggregation for a great majority of agents in the economy.

Another complication arises from the fact that the markets are not necessarily complete, and this, with the existence of "private" assets and wage or endowment shocks, can result in the breakdown of the simple models of asset valuation and portfolio allocation.

We cite three examples to clarify our above description of the state of affairs in the area of portfolio choice. Take the case of employees at Delta Airlines, who form an investment pool (their pension plan). Given that their investment capital for the pool comes from their wages generated by the airline's operation, diversification of risk would demand that their holding of Delta (or even other airline stock) be different from an otherwise identical group which is not employed by the airline.

Another example relates to the farming sector. Given the seasonal nature of their activity, farmers generate off-farm income. In addition, the investment in the farm itself is in the nature of a non-market investment, as the farmer cannot issue financial claims on the farm activity. Farmers as a group will therefore hold portfolios which are different from those of non-farm workers. Among farmers themselves, different groups with different endowments, raising different types of crops or livestock, will again hold different types of portfolios.

A third example is of investors with different investment horizons and life-spans. Typically, younger individuals with a more recent entry into the wage market have smaller capital endowments, smaller levels of human capital/abilities, attendant with lower wage rates and/or more volatile income streams. On the other hand, they have a greater time span of productive activities. Older agents are endowed with greater wealth stocks as a consequence of accumulation. They have accumulated human capital resulting in higher wage levels and/or more stable earnings. Their investment horizon is shorter as a consequence of their being older. Thus, these two groups, even if they were faced with the same set of "market" assets and had identical tolerances for risks, will choose different optimal portfolios.

As the above examples suggest, the existence of "non-market" assets, private endowment or wage streams, and different investment horizons in an economy with incomplete markets, lead to the breakdown of conventional theories and modelling prescriptions.

There is, therefore, a need to incorporate some of these features into the models of portfolio choice and asset valuation. Of these, the portofolio allocation problem is much more tractable and provides investment rules that are valuable for the study of the mutual fund industry, not to speak of possible regulatory prescriptions for the operation of pension plans, etc.

In the next section, we give a brief overview of the literature, with a description of two models developed in recent years. We also discuss our modelling approach and contrast it with the extant models. In section III, we develop our model and derive the optimality results and some simple comparative statics. In section IV, we describe the testable propositions arising out of our modelling approach and suggest some of the ways to derive empirical implications.

SECTION II: A VERY BRIEF SURVEY OF LITERATURE

In the interest of brevity and clarity of focus, we discuss below two of the approaches that in recent years have focused on the problem and contrast our model with them.

The three important papers in this area are those by Bodie, Merton, and Samuelson (1992), Svensson (1988), and Svensson and Werner (1993). Each of these papers considers the consumption-portfolio allocation problem with a stochastic wage or non-market asset cash-flow stream. Bodie, et al., consider the allocation problem for a finite- lived investor who has a wage stream that is spanned by the market asset set or whose unspanned component is treated as pure "noise" and is not valued by the investor. In their model, the wage stream follows a geometric Brownian motion.

Svensson and Svensson and Werner, on the other hand, treat the stochastic cash-flow stream to be produced by a non-market or private asset. This stochastic revenue stream follows an arithmetic Brownian motion, and they point to this by stating that under certain circumstances this stochastic flow has the flavor of a liability rather than an asset. Using this approach enables them to get a closed-form solution to the exponential utility case. In their model, the investor is an infinite-lived individual, unlike the finite-lived investor considered by Bodie and others. Svensson and Werner try to find a market equivalent value for the stochastic stream, and in their model they augment the actual wealth of the investor by this capitalized value. In their model, the investor uses the capitalized component of wealth to buy the claim to the cash-flow stream.

With such an approach, they obtain an advantage in that the derived utility function is a function of the augmented wealth alone, and hence the Hamilton-Jacobi-Bellman equation (HJBE) and the optimal choices take simpler functional forms that are amenable to solution tecniques. This advantage comes with a price, however. The "private" asset in which the capitalized component of wealth is invested has return dynamics whose characteristics are unknown and, strictly speaking, have to be endogenously determined. To characterize the private asset return dynamics, they presume it to be a function of the wealth and the cash-flow dynamics and use Ito's lemma to derive the valuation equation. This leads two types of problems: one, the valuation problem cannot be solved without the specification of appropriate boundary conditions, and two, this valuation is treated as independent of investor characteristics, which is not yet established. In other words, it is *a priori* not clear why different investors would have the same capitalized value for the stochastic cash-flow stream.

As an aside, we are able to show that our approach subsumes their approach and has the added benefit that we do not assume a market valuation. In fact, we can disaggregate their approach and show it to have a form like ours when no restrictions are imposed. In our model, the derived utility function is a function of wealth and the stochastic wage income which plays the same role as the stochastic cash-flow stream of Svensson, et al., and the wage stream of Bodie and others. This 'explicit' dependence of the derived utility function on the wage income, v, is a very crucial part of our model and distinguishes it from those developed by the above referred authors. In addition, there are other features of their models that we modify and generalize further, as discussed below.

Svensson considers only the case where there is no state variable and in essence models a setup where the market asset returns and the non-market asset returns are driven by a common set of Wiener processes, albeit with a general correlation structure. Svensson's non-market asset can be considered to play a role identical to that for the wage income stream in our model, as his asset does not require any capital outlay, but has a stochastic revenue stream. However, and this is critical, Svensson ignores the fact that the 'non-market' asset's revenue stream is not spanned by the market asset revenue streams, and therefore this revenue stream itself should be treated as a state variable, with an impact on the dynamics of the problem. In fact, in his general setup, Svensson does allow for this possibility, but when solving for an explicit case, he opines that if the 'non-market' asset's revenue stream has constant drift and diffusion parameters, then it is equivalent to a case where there is no state variable. This approach is at odds with a statement early on, where he notes that the income stream from the non-market source is akin to the state variable.

As a consequence of this simplification, his model results are very straightforward, for the market and non-market asset streams are driven by a common vector of Wiener processes, and the HJBE does not contain any terms with respect to a state variable. In fact Svensson's results can also be obtained in a case where the non-market asset revenue stream is an arithmetic Brownian motion. When we incorporate explicitly the assumption that the derived utility of wealth function depends upon the non-market asset revenue dynamics, the resulting first order conditions, the HJBE, and the optimal consumption and asset allocation relationships are radically different.

Bodie, et al., do not have a private set of assets in their model, but they do consider the problem with a wage income stream in addition to the general set of market assets. In their model, the investor has labor endowment in addition to wealth endowment, and his objective is to obtain optimal allocations of wealth, consumption, leisure, and labor efforts. In their general setup, they state that the wage income stream is not necessarily spanned by the revenue streams from the market assets. However, in their explicit derivations, they consider only the case, where they assume essentially that (a) spanning obtains, or equivalently, (b) the investor behaves as if the component for the wage stream that is not spanned by (nor is orthogonal to) the market revenue streams, can be treated as noise and is therefore without value or price.

Their focus is on the labor-leisure trade-off and the impact of the wage dynamics on the investor's optimal consumption-market asset portfolio allocation decisions. They consider this impact in two cases: one where the investor has no flexibility in terms of the labor allocation, and the second, where he does have the flexibility in terms of his leisure-labor allocations.

In terms of our model, their model with rigid or non-flexible labor allocation is the closest comparable one. There is one important consequence of their 'projection' approach to the treatment of the wage income stream. This approach, which originated with Merton's work (1971), gives a very simple and tractable solution approach, but there is a price to pay. The price lies in the fact that we account only for those components that lie in the span of the market set and essentially ignore the orthogonal components that any rational investor would not choose to ignore. The results are therefore biased, and this bias can be significant.

The results of Bodie, et al., therefore 'mis-estimate' the impact of the wage stream on the optimal consumption-portfolio choices of the investor. Further, as their focus is on the quantification of the impact of these wage sources on the portfolio holding patterns of different investors with different wage dynamics, their treatment of the wage dynamics, as stated above, has a critical impact on the model prescriptions. The point that we are attempting to make is that the assumption or treatment as noise of the orthogonal components is not a very trivial or even necessarily benign one.

Our model, in contrast, straddles both the above models of Svensson and Bodie, et al., (BMS). While it is less general than the BMS model as far as its treatment of the labor endowment is considered (the flexible labor supply case), our model is, nevertheless, more general in its correlation structure and its explicit treatment of the impact of wage dynamics on the derived utility function. It is a richer model, and we are able to derive the optimal set of conditions or rules for an investor's welfare, consumption choice, and asset allocation. In addition, we also derive closed-form solutions for specific utility functions.

We initially derive the results for the infinite time horizon case. In the case of the 'exponential' utility function, we derive results which, when we make the assumption akin to that made by Svensson, lead to representation which is identical to his.

Given the closed-form solutions, we are able to examine the sensitivities of consumption and portfolio allocations to changes in different factors ranging from different correlation structures to changes in the wage dynamics.

An additional feature of our model is its possible extension for the development of an investor-specific 'private valuation' measure for the private or non-market asset(s). Whereas in the case of the wage income stream, we developed a valuation or capitalized equivalent to determine the impact on the decision making exercise, here the approach will be radically different. In the first case, there was no wealth investment in the 'wage asset,' whereas here these private assets call for wealth allocations in order to generate the revenue streams for the investor. Therefore, the approach relies on finding that increment to the current wealth endowment which, when invested together with the current wealth in a smaller set of capital investment projects, will nevertheless result in the same level of welfare as in the original setup. In order to undertake this exercise, it is clear that we need to employ specific utility functions. As we have already developed closed-form solutions for the derived utility of wealth and also characterized the optimal consumption-investment allocations, we have a closed form expression for the investor's welfare. Therefore, it is a straightforward approach to estimating the marginal, 'private' value for any non-market asset.

As stated earlier, we have developed the model initially for the infinite-time horizon case using specific utility functions The extension of these results to the finite-time horizon problem would render the results sensitive to the investor's time frame, and hence is richer in its predictions or prescriptions. For example, two otherwise identical investors with different time horizons will have different consumption and portfolio strategies, and hence in the context of mutual funds, there are clientele effects that any fund promoter must pay attention to. Alternatively, investors with same time frames but different wage stream parameters will also behave differently. This has implications for pension fund investments, among others, as employees in different industries have different wage dynamics, and even if they are otherwise homogeneous in their preferences, nevertheless, their optimal portfolio choices will be different.

Finally, this model redresses one of the primary shortcomings of the extant models of optimal portfolio choice and allocations. It takes a partial equilibrium approach, with primary focus on an individual investor, and develops the appropriate measures for optimal allocations. In part, the model answers questions as to why a typical investor's portfolio is very different from the 'well-diversified' one expected from the other models. It is an extension of the BMS approach in this respect, which has the added feature that the impact of 'non-systematic or idiosyncratic' component of the wage stream is not ignored but explicitly incorporated in the decision making process. Empirical tests of this model are easy to develop, as we have the necessary metrics, and also one can categorize different labor groups by industry or activity into different wage streams and estimate their correlation structures with a market asset set such as the S&P 500 or the DJIA. An examination of the composition or characteristics of the investors in different types of mutual funds is also an approach to check for the viability of this model. In section III, we describe the model setup and derive the major results.

SECTION III: A CONTINUOUS TIME MODEL OF CONSUMPTION AND PORTFOLIO ALLOCATION

In this section, we develop an infinite-horizon, continuous time model of consumption choice and portfolio allocation. We focus our attention on an investor who is an expected utility of consumption maximizer. To render the problem tractable to the technique of dynamic programming, we assume that the utility function of the investor is amenable to the dynamic programming approach. The investor's opportunity set consists of long-lived capital assets, requiring investments of wealth. The capital assets consist of both "market" and "non-market" assets, athough in this paper we do not identify them separately.

The investor also receives a continuous stream of cash flow or revenue, which we term "wage income stream," although one could view the stream as an endowment stream as well. This is a single good economy where the single-consumption good also serves as the numeraire. We define below all the terms and also elaborate on the assumptions before deriving the fundamental equation to be solved, namely the Hamilton-Jacobi-Bellman equation (HJBE). There are two alternative treatments of the wage stream, resulting in two different HJBEs, and we derive both of them.

Initially, we express the HJBE(s) in general derived utility of wealth and income terms ($I\,[W,v]$), but later on we use the HJBE to solve for specific utility functions. The problem is a complicated one and, with the specification of a given form for the utility function, we can at times derive a closed-form solution only in special cases where the system parameters satisfy certain functional relationships. Nevertheless, we do derive the general result for the class of negative exponential utility functions.

Once the HJBE has been solved, and a closed-form solution obtained for the derived utility function over wealth and wage income stream, the next step is to characterize fully the optimal consumption choice [$c^*(t)$] and the portfolio vector ($W\underline{w}^*$). Once these have been expressed in terms of the system parameters, we are then in a position to examine their dependence on the various variables of interest ranging from the wealth endowment and the wage income level to the correlation structure between the capital assets and the wage stream. This "comparative-statics" exercise will enable us to fully characterize the investor's optimal choice and also lead to prescribing strategies for developing specialized, optimal portfolios for different classes of investors with different sources of income, among other things.

In this section and in this paper, we do not undertake this exercise and limit ourselves to the task of simply characterizing the consumption and portfolio allocation decisions.

We begin below with the definitions and notations employed hereinafter.

Definitions and Notations:

$c(t)$: rate of consumption at time t.

ρ: time preference coefficient for the investor.

$v(t)$: the wage level at time t.

$W(t)$: the wealth level at time t of the investor.

Capital Asset Dynamics:

Capital assets are assets which require investment of wealth or capital to generate returns.

$$\underline{\underline{I}}^{-1}(P)\cdot d\underline{P} = \underline{\overline{Z}}dt + \underline{\underline{S}}_Z \cdot d\underline{B}_Z \cdot$$

where

$$\underline{\underline{I}}(P) \equiv \begin{bmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_n \end{bmatrix}_{nxn}$$

is the diagonal matrix and P_i is the level of investment in the ith asset.

$\underline{\overline{Z}} \equiv \left[\overline{Z}_1, \overline{Z}_2, \cdots \overline{Z}_n\right]^T$ is the instantaneous mean rate of return vector.

$\underline{\underline{\Sigma}} \equiv \underline{\underline{S}}_Z \cdot \underline{\underline{S}}_Z^T$ is the (nxn) matrix representing the instantaneous covariance matrix of returns on the capital assets.

$\underline{\underline{\Sigma}}$ is assumed to be positive definite.

Wage Dynamics:

$v(t)$ represents the wage flow at time t, with the following value dynamics:

$$dv = v \cdot \overline{Z}_v dt + v\sigma_v \cdot dB_v$$

where and σ_v are the instantaneous drift and diffusion parameters for the wage stream.

Brownian Motion Relationships:

$d\underline{B}_Z$: Kx1 vector of incremental Wiener processes affecting the capital asset value return dynamics.

dB_v: The incremental Wiener process affecting the wage stream value dynamics.

The processes $d\underline{B}_Z$ and dB_v are assumed to have a general covariance structure as shown below:

$$E_t\left[\left(\underline{\underline{P}}^{-1}\cdot d\underline{P}\right)\cdot\frac{dv}{v}\right]=\underline{\Sigma}_{Zv}\ dt$$

where $\underline{\Sigma}_{ZV}$ is the instantaneous covariance vector reflecting the relationship between the capital assets and wage stream dynamics.

Risk-Free Asset Dynamics:

$$\frac{dP_f}{P_f}=Rdt.$$

P_f : level of investment at t in the risk-free asset.

R: instantaneous rate of return on the risk-free investment.

Investor Characteristics:

$U[c,t]$: utility function for the investor.

1. Negative exponential utility function:

$$-e^{-\rho t}\cdot\frac{e^{-\delta[c]}}{\delta}$$

2. Power utility function:

$$\frac{[c-\hat{c}]^{\gamma}}{\gamma}\quad \gamma<1,$$

$$\frac{-[\hat{c}-c]^{\gamma}}{\gamma}\quad \gamma>1.$$

3. Log utility function:

$$Ln[c-\hat{c}].$$

Note 1: The utility function $U(c,t)$ is assumed to be separable, i.e.,

$$U[c,t]=f(t)\cdot U(c).$$

Note 2: The utility function is assumed to satisfy the differentiability and continuity requirements necessary for the use of stochastic dynamic programming techniques.

Time Horizon Assumption:

Initially, we assume that the investor is an infinite lived agent. His objective function is therefore set out as:

$$Max\ E_t\left[\int_t^{\infty} U\left[c,s\right]ds\right]$$

subject to budget and wealth dynamic constraints at each decision point.

E_t: Expectation operator at time t.

Starting Point for the Investor's Optimization Problem:

The infinite horizon, Hamilton-Jacobi-Bellman equation (HJBE), is the starting point for the investor's optimization problem.

Definition:

$e^{-\rho t}I\,[W, v]$ = derived utility over wealth and wage stream for the infinite lived investor.

1. $I\,[W, v] = \ -e^{\Delta}\dfrac{e^{-\delta R[f(W,v)]}}{\delta R}$ for negative exponential utility function.

2. $I\,[W, v] \equiv \Delta\cdot\dfrac{\left[f(W,v)\right]^{\gamma}}{\gamma}$ for power utility function.

3. $I\,[W, v] \equiv \Delta\cdot Ln\left[f(W,v)\right]$ for log utility function.

The original HJBE:

$$\begin{aligned}&U(c,t)dt + J_t dt + J_W E_t\left[dW\right]\\&+J_v E_t\left[dv\right]+\tfrac{1}{2}J_{WW}E_t\left[dW^2\right]+\tfrac{1}{2}J_{vv}E_t\left[dv^2\right]\\&+J_{Wv}E_t\left[dW\cdot dv\right]=0\end{aligned} \tag{1}$$

where $J \equiv J\left[W,v,t\right]$ is the derived utility of wealth (and wage) function for the investor.

Note: With $U(c,t)\equiv e^{-\rho t}U(c)$ and $J\left[W,v,t\right]\equiv e^{-\rho t}I\left[W,v\right]$ we have:

$$
\begin{aligned}
&U(c^*)-\rho I+I_W\left[E_t(dW)/dt\right]+I_v E_t\left[(dv)/dt\right]\\
&+\tfrac{1}{2}I_{WW}\left[E_t\left(dW^2\right)/dt\right]+\tfrac{1}{2}I_{vv}\left[E_t\left(dv^2\right)/dt\right]+I_{Wv}\left[E_t(dW\cdot dv)/dt\right]=0
\end{aligned}
\tag{1a}
$$

where $dW \equiv$ incremental change in the investor's wealth level and $dv \equiv$ incremental wage stream flow accruing to the investor.

Definition:

$\underline{w}$ = vector of proportions of wealth allocated to the various capital assets.

$W(t)-c(t)dt \equiv$ wealth allocated at time t to the capital assets.

$\Rightarrow W(t+dt)$, level of wealth at time $(t+dt)$

$$
\begin{aligned}
&=\left[W(t)-c(t)dt\right]+\left[W(t)-c(t)dt\right]\left[\left[R+\underline{w}^T\left(\underline{\overline{Z}}-R\underline{1}\right)\right]dt+\underline{w}^T\underline{\underline{S}}_Z dB_Z\right]+dv\\
&=\left[W(t)-c(t)dt\right]+\left[W(t)-c(t)dt\right]\left[\left[R+\underline{w}^T\left(\underline{\overline{Z}}-R\underline{1}\right)\right]\cdot dt+\underline{w}^T\underline{\underline{S}}_Z d\underline{B}_Z\right]\\
&+v\overline{Z}_v dt+v\sigma_v dB_v
\end{aligned}
\tag{1b}
$$

$$
\begin{aligned}
&dW \equiv W\left[t+dt\right]-W(t)=-c(t)dt\\
&+\left[W(t)-c(t)dt\right]\left[\left[R+\underline{w}^T\left(\underline{\overline{Z}}-R\underline{1}\right)\right]dt+\underline{w}^T\underline{\underline{S}}_Z d\underline{B}_Z\right]\\
&+v\overline{Z}_v dt+v\sigma_u dB_v
\end{aligned}
\tag{1c}
$$

We have:

$$
\begin{aligned}
&E_t\left[dW(t)\right]=\left[-c(t)+W\left[R+\underline{w}^T\left(\underline{\overline{Z}}-R\underline{1}\right)\right]+v\overline{Z}_v\right]dt\\
&E_t\left[dW(t)^2\right]=\left[W^2\underline{w}^T\underline{\underline{\Sigma}}w+2vW\underline{w}^T\underline{\Sigma}_{Zv}+v^2\sigma_v^2\right]dt\\
&E_t\left[dv(t)\right]=v\overline{Z}_v dt\\
&E_t\left[dv(t)^2\right]=v^2\sigma_v^2 dt\\
&E_t\left[dW(t)\cdot dv(t)\right]=\left[vW\cdot\underline{w}^T\underline{\Sigma}_{Zv}+v^2\sigma_v^2\right]dt
\end{aligned}
\tag{1d}
$$

Substituting for these in the HJBE, we obtain the following expression:

$$
\begin{aligned}
&\Rightarrow H[\cdot,\cdot] \\
&U[c^*] - \rho I + I_W\left[-c + WR + W\underline{w}^T(\overline{\underline{Z}} - R\underline{1}) + v\overline{Z}_v\right] \\
&+ I_v v\overline{Z}_v + \tfrac{1}{2} I_{WW}\left[W^2 \underline{w}^T \underline{\underline{\Sigma}} w + 2vW\underline{w}^T \underline{\Sigma}_{Zv} + v_2\sigma_v^2\right] + \tfrac{1}{2} I_{vv} v_2 \sigma_v^2 \\
&+ I_{Wv}\left[vW \cdot \underline{w}^T \Sigma_{Zv} + v_2\sigma_v^2\right] = 0.
\end{aligned}
\tag{1e}
$$

The optimization problem becomes:

$$
\underset{\{c(t), \underline{w}\}.}{\text{Max}} \quad H[\cdot,\cdot] \tag{1}
$$

which gives us the following first order conditions:

$$
1. \qquad U_c(c^*) = I_W \tag{2}
$$

$$
2. \qquad I_W\left[W(\overline{\underline{Z}} - R\underline{1})\right] + I_{WW}\left[W^2 \underline{\underline{\Sigma}}\underline{w}^* + vW\underline{\Sigma}_{Zv}\right] + I_{Wv}\left[vW\underline{\Sigma}_{Zv}\right] = 0. \tag{3}
$$

or equivalently

$$
W\underline{w}^* = -\underline{\underline{\Sigma}}^{-1}\left[\left(\frac{I_W}{I_{WW}}\right)(\overline{\underline{Z}} - R\underline{1}) + \left(1 + \frac{I_{Wv}}{I_{WW}}\right)v\underline{\Sigma}_{Zv}\right]. \tag{4}
$$

Substituting the relationships implied by the second fonc into the HJBE and simplifying, we get the modified HJBE as

$$
\begin{aligned}
&\Rightarrow U(c^*) - \rho I + I_W[-c^* + WR] + \left[(I_W + I_v)\overline{Z}_v - \hat{\theta}_1 \frac{I_W(I_{WW} + I_{Wv})}{I_{WW}}\right]v \\
&+ \left(\tfrac{1}{2}\right)\left[(I_{WW} + 2I_{Wv} + I_{vv})\sigma_v^2 - \frac{(I_{WW} + I_{Wv})^2}{I_{WW}} \cdot \hat{\theta}_2\right]v^2 - \tfrac{1}{2}\frac{I_W^2}{I_{WW}} \cdot \hat{\theta}_0 = 0.
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
\hat{\theta}_0 &\equiv (\overline{\underline{Z}} - R\underline{1})^T \underline{\underline{\Sigma}}^{-1}(\overline{\underline{Z}} - R\underline{1}) \\
\hat{\theta}_1 &\equiv (\overline{\underline{Z}} - R\underline{1})^T \underline{\underline{\Sigma}}^{-1}\underline{\Sigma}_{Zv} \text{ and} \\
\hat{\theta}_2 &\equiv \underline{\Sigma}_{Zv}^T \underline{\underline{\Sigma}}^{-1}\underline{\Sigma}_{Zv}.
\end{aligned}
$$

Note: Henceforth, our starting point is the below given HJBE:

$$U[c^*]-\rho I+I_W[-c+WR]-\frac{1}{2}\frac{I_W^2}{I_{WW}}\cdot\hat{\theta}_0+v\left[(I_W+I_v)\overline{Z}_v-\frac{I_W(I_{WW}+I_{Wv})}{I_{WW}}\cdot\hat{\theta}_1\right] \tag{6}$$

$$+\left(\tfrac{1}{2}\right)v^2\left[(I_{WW}+2I_{Wv}+I_{vv})\sigma_v^2-\frac{\left[I_{WW}+I_{Wv}\right]^2}{I_{WW}}\hat{\theta}_2\right]=0.$$

Using the above equation in conjunction with the appropriate utility function over consumption and the corresponding derived utility function, will generate the differential equations that need to be solved in order to obtain the appropriate functional form(s) for the $f(w,v)$ function. Once we solve for the $f(w,v)$ correctly, the next step is to solve for the optimal consumption rate, $c^*(t)$, and the optimal capital asset investment allocation vector $\underline{w}^*(t)$. We can then undertake a detailed study of the consumption-investment behavior of the investor as a function of the various system parameters!

Aside: note that in the setting up of the investor's optimization problem, we have assumed that in the time interval $[t, t+dt]$, the wage stream accruing to the investor is 'dv'! There can be an alternative assumption; namely, that in the incremental interval dt, the investor's wage stream accrual is actually vdt. This has the flavor of the wage rate being negotiated at t and being in force over the interval length dt.

With such an alternative set up, we would then have:

$$\Rightarrow dW=\left[-c+WR+W\underline{w}^T\left(\underline{\overline{Z}}-R\underline{1}\right)+v\right]dt+W\underline{w}^T\underline{\underline{S}}_Z d\underline{B}_Z, \tag{7}$$

so that the wealth dynamics are quite different now, and the HJBE becomes:

$$U(c^*)-\rho I+I_W\left[-c+v+WR+W\underline{w}^T\left(\underline{\overline{Z}}-R\underline{1}\right)\right]+I_v\cdot v\overline{Z}_v+\tfrac{1}{2}I_{WW}\left[W^2\underline{w}^T\underline{\underline{\Sigma}}_w\right] \tag{8}$$
$$+\tfrac{1}{2}I_{vv}v^2\sigma_v^2+I_{Wv}\left[vW\underline{w}^T\underline{\Sigma}_{Zv}\right]=0$$

with the foncs as:

1. $U_c=I_W$ (9)

2. $I_W\left[W\left(\underline{\overline{Z}}-R\underline{1}\right)\right]+I_{WW}\left[W^2\underline{\underline{\Sigma}}\underline{w}^*\right]+I_{Wv}\left[vW\underline{\Sigma}_{Zv}\right]=0$ (10)

or equivalently:

$$W\underline{w}^* = -\underline{\underline{\Sigma}}^{-1}\left[\left(\frac{I_W}{I_{WW}}\right)(\underline{Z} - R\underline{1}) + \left(\frac{I_{Wv}}{I_{WW}}\right)v\underline{\Sigma}_{Zv}\right]W\underline{w}^*, \tag{11}$$

and the HJBE is transformed to:

$$\begin{aligned} &U[c^*] - \rho I + I_W[-c + v + WR] + I_v v\overline{Z}_v + \tfrac{1}{2}I_{vv}v^2\sigma_v^2 \\ &-\tfrac{1}{2}I_{WW}\left[\left(\frac{I_W}{I_{WW}}\right)(\underline{Z} - R\underline{1}) + \left(\frac{I_{Wv}}{I_{WW}}\right)v\underline{\Sigma}_{Zv}\right]^T \underline{\underline{\Sigma}}^{-1}\left[\left(\frac{I_W}{I_{WW}}\right)(\underline{Z} - R\underline{1}) + \left(\frac{I_{Wv}}{I_{WW}}\right)v\underline{\Sigma}_{Zv}\right] = 0. \end{aligned} \tag{12}$$

Rearranging the terms, we get:

$$\begin{aligned} &U(c^*) - \rho I + I_W[-c + WR] - \tfrac{1}{2}\frac{I_W^2}{I_{WW}}\cdot\hat{\theta}_0 + v\left[\left(I_W + I_v\overline{Z}_v\right) - \left(\frac{I_W \cdot I_{Wv}}{I_{WW}}\right)\cdot\hat{\theta}_1\right] \\ &+\left(\tfrac{1}{2}\right)v^2\left[I_{vv}\sigma_v^2 - \left(\frac{I_{Wv}^2}{I_{WW}}\right)\hat{\theta}_2\right] = 0 \end{aligned} \tag{12a}$$

$\hat{\theta}_0, \hat{\theta}_1$, and $\hat{\theta}_2$ are as defined before.

Again, with the appropriate utility function and assumed derived utility function, we end up with a differential equation that needs to be solved for the equilibrium form of the function $f(w,v)$ and, therefore, we can obtain the equilibrium characterizations as before.

Note: In the second case, the HJBE is much more compact.

The essential characteristic of this approach is that the wage stream contribution is vdt and not dv in the interval $[t, t + dt]$ and hence the simpler relationship.

In the next few pages, we derive the final differential equations that need to be solved for the $f(w, v)$ function for each of the three utility functions, using the first HJBE relationship where the wage stream contribution to $dW(t)$ is dv and not vdt.

The first HJBE:

$$\begin{aligned} &U(c) - \rho I + I_W[-c + WR] - \tfrac{1}{2}\frac{I_W^2}{I_{WW}}\cdot\hat{\theta}_0 + \left[(I_W + I_v)\overline{Z}_v - \left[I_W\left(1 + \frac{I_{Wv}}{I_{WW}}\right)\right]\cdot\hat{\theta}_1\right]v \\ &+\left[(I_{WW} + 2I_{Wv} + I_{vv})\sigma_v^2 - I_{WW}\left(1 + \frac{I_{Wv}}{I_{WW}}\right)^2\hat{\theta}_2\right]\tfrac{1}{2}v^2 = 0. \end{aligned} \tag{6}$$

Before we move on to specific utility functions, we will first attempt an examination of the optimal portfolio choice and try to characterize the behavior of the investor.

***An Examination of the Optimal Portfolio Choice*:**

Irrespective of the specific utility function employed, we have the optimal portfolio of capital assets specified as below. We have from equation (4) above:

$$W\underline{w}^* = -\underline{\underline{\Sigma}}^{-1}\left(\underline{\overline{Z}} - R\underline{1}\right)\left(\frac{I_W}{I_{WW}}\right) - \underline{\underline{\Sigma}}^{-1}\underline{\Sigma}_{Zv}v\left(1 + \frac{I_{Wv}}{I_{WW}}\right). \tag{13a}$$

We now do some simple manipulations of the above expression:

$$W1^T\underline{w}^* = -1^T\underline{\underline{\Sigma}}^{-1}\left(\underline{\overline{Z}} - R\underline{1}\right)\left(\frac{I_W}{I_{WW}}\right) - 1^T\underline{\underline{\Sigma}}^{-1}\underline{\Sigma}_{Zv}v\left(1 + \frac{I_{Wv}}{I_{WW}}\right).$$

Please note that the above expression is a scalar, and represents the total wealth invested in the capital assets.

We define

$$\underline{\hat{w}}^* \equiv \frac{W\underline{w}^*}{W1^T\underline{w}^*}.$$

Please note $1^T \cdot \underline{\hat{w}}^* = 1.$

We therefore obtain $\underline{\hat{w}}^*$ as:

$$\underline{\hat{w}}^* = \frac{\underline{\underline{\Sigma}}^{-1}\left(\underline{\overline{Z}} - R\underline{1}\right)\left(\frac{I_W}{I_{WW}}\right) + \underline{\underline{\Sigma}}^{-1}\underline{\Sigma}_{Zv}v\left(1 + \frac{I_{Wv}}{I_{WW}}\right)}{1\underline{\underline{\Sigma}}^{-1}\left(\underline{\overline{Z}} - R\underline{1}\right)\left(\frac{I_W}{I_{WW}}\right) + 1^T\underline{\underline{\Sigma}}^{-1}\underline{\Sigma}_{Zv}v\left(1 + \frac{I_{Wv}}{I_{WW}}\right)}. \tag{13b}$$

This expression can be further manipulated to give a more compact expression below:

$$\underline{\hat{w}}^* = \frac{\underline{\hat{w}}_c^* + \underline{\delta}_v\left(\frac{I_{WW} + I_{Wv}}{I_W}\right)}{1 + \left(1^T\delta_v\right)v\left(\frac{I_{WW} + I_{Wv}}{I_W}\right)}, \tag{14}$$

where

$$\underline{\hat{w}}_c^* = \frac{\underline{\underline{\Sigma}}^{-1}(\underline{\bar{Z}} - R\underline{1})}{1^T \underline{\underline{\Sigma}}^{-1}(\underline{\bar{Z}} - R\underline{1})} \text{ and } \underline{\delta}_v = \frac{\underline{\underline{\Sigma}}^{-1}\underline{\Sigma}_{Zv}}{1^T \underline{\underline{\Sigma}}^{-1}(\underline{\bar{Z}} - R\underline{1})}.$$

Please note that $\underline{\hat{w}}_c^*$ is the optimal portfolio that would be held by the investor if either he has no wage stream v or if the wage stream dynamics are independent of the dynamics of the capital assets, i.e., $\underline{\Sigma}_{Zv} = 0$.

Please also note that the denominator of the expression for $\underline{\hat{w}}^*$ above is a scalar.

To recap, we write $\underline{\hat{w}}^*$ as:

$$\underline{\hat{w}}^* = \frac{1}{\left[1 + (1^T \underline{\delta}_v) v \left[\frac{I_{WW} + I_{Wv}}{I_W}\right]\right]} \left[\underline{\hat{w}}_c^* + \underline{\delta}_v \cdot v \left[\frac{I_{WW} + I_{Wv}}{I_W}\right]\right] \quad . \tag{14}$$

We can now interpret the portfolio choice as adjustments to the $\underline{\hat{w}}_c^*$ portfolio as a consequence of the existence of the wage revenues. The first adjustment is adding weights

$$\underline{\delta}_v \cdot v \left[\frac{I_{WW} + I_{Wv}}{I_W}\right]$$

and then scaling the resultant portfolio by

$$\left[1 + (\underline{1}^T \underline{\delta}_v) v \left(\frac{I_{WW} + I_{Wv}}{I_W}\right)\right]^{-1}.$$

In the absence of any further information about the derived utility function $I[W,v]$ and its partial derivatives, we cannot draw any more fruitful insight into the optimal portfolio composition. This is true, even more so, for the optimal consumption choice, c^*. We therefore need to revisit the first HJBE and try to solve it explicitly for specific utility functions, after which the comparative static results can be obtained. This we do next.

I. Negative Exponential:

For the negative exponentail utility case, we assume the following forms for the utility function and the derived utility function.

$$U(c) \equiv \frac{-e^{\delta c}}{\delta} \ ;$$

$$I(W,v) \equiv -e^{\Delta}\frac{e^{-\delta Rf}}{\delta R}$$

Thus, we have the HJBE from equation (6) transformed to:

$$\begin{aligned}
&-\frac{f_W}{\delta}+\frac{\rho}{\delta R}+\left[\frac{\Delta}{\delta}-\frac{Ln(f_W)}{\delta}-Rf+RW\right]f_W-\tfrac{1}{2}\frac{f_W^2}{\left[f_{WW}-(\delta R)f_W^2\right]}\hat{\theta}_0 \\
&+\left[(f_W+f_v)\overline{Z}_v-\frac{f_W\left[(f_{WW}+f_{Wv})-(\delta R)f_W(f_W+f_v)\right]\hat{\theta}_1}{\left[f_{WW}-(\delta R)f_W^2\right]}\right]v \\
&+\left[\begin{array}{l}\left[(f_{WW}+2f_{Wv}+f_{vv})-\delta R\left(f_W+f_v^2\right)\right]\sigma_v^2 \\ -\dfrac{\left[(f_{WW}+f_{Wv})-\delta Rf_W(f_W+f_v)\right]^2}{\left[f_{WW}-(\delta R)f_W^2\right]}\hat{\theta}_2\end{array}\right](\tfrac{1}{2})v^2=0.
\end{aligned} \tag{15}$$

And the optimal choices as:

$$c^* = -\frac{\Delta}{\delta}-\frac{Ln(f_W)}{\delta}+Rf(w,v), \text{ and} \tag{16}$$

$$W\underline{w}^* = -\underline{\underline{\Sigma}}^{-1}\left[\frac{f_W\cdot(\overline{\underline{Z}}-R\underline{1})}{f_{WW}-(\delta R)f_W^2}+\frac{\left[(f_{WW}+f_{Wv})-(\delta R)f_W(f_W+f_v)\right]}{\left[f_{WW}-(\delta R)f_W^2\right]}v\underline{\Sigma}_{Zv}\right] \tag{17}$$

We next obtain the solution(s) to the HJBE above. Now, we have upon having separability of the function $f(W, v)$ in W and v as $f(W, v) = W + f(v)$, i.e.,

with $U(c)=\dfrac{-e^{-\delta c}}{\delta}$ and $I[W,v]\equiv\dfrac{-e^{\Delta}e^{-\delta R[W+f(v)]}}{\delta R}$, we obtain upon simplification:

$$\begin{aligned}
&\left[\frac{\Delta}{\delta}-\frac{1}{\delta}+\frac{\rho}{\delta R}+\frac{1}{2(\delta R)}\hat{\theta}_0\right] \\
&-Rf(v)+\hat{\theta}_1 v(1+f_v)-\theta_2 v^2(1+f_v)^2+\theta_3 v^2 f_{vv}=0
\end{aligned} \tag{18}$$

where

$\theta_1 \equiv \overline{Z}_v - \hat{\theta}_1 = \overline{Z}_v - \left(\underline{Z} - R\underline{1}\right)^T \underline{\underline{\Sigma}}^{-1} \underline{\Sigma}_{Zv}$;

$\theta_2 \equiv \frac{1}{2}(\delta R)\left[\sigma_v^2 - \hat{\theta}_2\right] = \frac{1}{2}(\delta R)\left[\sigma_v^2 - \underline{\Sigma}_{Zv}^T \underline{\underline{\Sigma}}^{-1} \underline{\Sigma}_{Zv}\right]$; and

$\theta_3 \equiv \frac{1}{2}\sigma_v^2$.

By assuming that $(i)\dfrac{\Delta}{\delta} - \dfrac{1}{\delta} + \dfrac{\rho}{\delta R} + \dfrac{1}{2\delta R}\hat{\theta}_0 = 0$ and $(ii) Rf(v) - \theta_1 v(1 + f_v) + \theta_2 v^2 (1 + f_v)^2 - \theta_3 v^2 f_{vv} = 0$, we can solve separately for Δ and for $f(v)$.

This gives us:

$$\frac{\Delta}{\delta} = \frac{1}{\delta R}\left[R - \rho - \tfrac{1}{2}\hat{\theta}_0\right] \text{ or } \Delta = \frac{1}{R}\left[R - \rho - \tfrac{1}{2}\hat{\theta}_0\right]. \tag{19}$$

We now turn our attention to $f(v)$. We have:

$$Rf(v) - \theta_1 v(1 + f_v) + \theta_2 v^2 (1 + f_v)^2 - \theta_3 v^2 f_{vv} = 0. \tag{20}$$

We now do some transformations on the above expression. First, define:

$$g(v) = f(v) + v.$$

This gives us:

$$Rg(v) - Rv - \theta_1 v g_v + \theta_2 v^2 g_v^2 - \theta_3 v^2 g_{vv} = 0.$$

Next define $h(Lnv) = g(v)$, and define $Lnv = x$; these, upon subsitution, give us:

$$Rh(x) - \mathrm{Re}^x - \overline{\theta}_1 h_x + \overline{\theta}_2 h_x^2 - \overline{\theta}_3 h_{xx} = 0 \tag{22}$$

where $\overline{\theta}_1 = \theta_1 - \theta_3$; $\overline{\theta}_2 = \theta_2$; and $\overline{\theta}_3 = \theta_3$.

Now if $R - \dfrac{\overline{\theta}_1}{2} - \dfrac{\overline{\theta}_3}{4} = 0,$ we have a very simple solution to the above equation.

Let $h(x) = ae^{\frac{x}{2}}$

$h_x = \dfrac{a}{2}e^{\frac{x}{2}}; h_x^2 = \dfrac{a}{4}e^x;\ h_{xx} = \dfrac{a}{4}e^{\frac{x}{2}}.$

Substituting, we get:

$$Rae^{\frac{x}{2}} - \mathrm{Re}^{x} - \overline{\theta}_1 \frac{a}{2} e^{\frac{x}{2}} + \overline{\theta}_2 \frac{a^2}{4} e^{x} - \overline{\theta}_3 \frac{a}{4} e^{\frac{x}{2}} = 0$$

$$\text{or} \left[R - \frac{\overline{\theta}_1}{2} - \frac{\overline{\theta}_3}{4} \right] ae^{\frac{x}{2}} + \left[\overline{\theta}_2 \frac{a^2}{4} - R \right] e^{x} = 0. \tag{23}$$

If stated as above, we do have $R - \frac{\overline{\theta}_1}{2} - \frac{\overline{\theta}_3}{4} = 0$, then we have $a = 2\sqrt{\frac{R}{\theta_2}}$, assuming the positive root, so that,

$$h(x) = 2\sqrt{\tfrac{R}{\theta_2}}\, e^{\frac{x}{2}}, \text{ and}$$

$$f(v) = g(v) - v = \sqrt{\tfrac{4Rv}{\theta_2}} - v. \tag{24}$$

Thus we have:

$$I[W, v] = \frac{-e^{\Delta} e^{-\delta R[W + f(v)]}}{\delta R}$$

where

$$\Delta = \frac{1}{R}\left[R - \rho - \tfrac{1}{2}\hat{\theta}_0\right]$$

$$\hat{\theta}_0 = \left(\underline{\overline{Z}} - R\underline{1}\right)^T \underline{\underline{\Sigma}}^{-1} \left(\underline{\overline{Z}} - R\underline{1}\right) \tag{25}$$

$$f(v) = \sqrt{\tfrac{4Rv}{\theta_2}} - v$$

$$\overline{\theta}_2 = \theta_2 = \tfrac{1}{2}(\delta R)\left[\sigma_v^2 - \underline{\Sigma}_{Zv}^T \underline{\underline{\Sigma}}^{-1} \underline{\Sigma}_{Zv}\right],$$

provided, and this is important,

$$R - \frac{\overline{\theta}_1}{2} - \frac{\overline{\theta}_3}{4} = 0; \text{ i.e., } R = \frac{\overline{\theta}_1}{2} + \frac{\overline{\theta}_3}{4}$$

$$= \tfrac{1}{2}\left(\theta_1 - \theta_3\right) + \frac{\theta_3}{4} = \tfrac{1}{2}^{\theta_2} - \tfrac{1}{4}^{\theta_3} \tag{26}$$

$$\text{or } 4R = \theta_1 - 2\theta_3; \text{ i.e., } 4R = \left[\overline{Z}_v - \left(\underline{\overline{Z}} - R\underline{1}\right)^T \underline{\underline{\Sigma}}^{-1} \underline{\Sigma}_{Zv}\right] - \sigma_v^2.$$

On the other hand, if $R - \frac{\overline{\theta}_1}{2} - \frac{\overline{\theta}_3}{4} \neq 0$, then we have a much more complicated problem. Going back to the ODE per equation (22), we have:

$$Rh(x) - Re^x + \overline{\theta}_2 h_x^2 - \overline{\theta}_1 h_x - \overline{\theta}_3 h_{xx} = 0, \text{ where}$$

$$\overline{\theta}_1 = \theta_1 - \theta_3 = \left[\overline{Z}_v - \left(\underline{\overline{Z}} - R\underline{1}\right)^T \underline{\underline{\Sigma}}^{-1} \underline{\Sigma}_{Zv}\right] - \tfrac{1}{2}\sigma_v^2$$

$$\overline{\theta}_2 = \theta_2 = \tfrac{1}{2}(\delta R)\left[\sigma_v^2 - \underline{\Sigma}_{Zv}^T \underline{\underline{\Sigma}}^{-1} \underline{\Sigma}_{Zv}\right]$$

$$\overline{\theta}_3 = \theta_3 = \tfrac{1}{2}\sigma_v^2.$$

Rewriting the expression, and employing the series form expression for e^x, equation (22) becomes:

$$Rh(x) - R\sum_{i=0}^{\infty}\left(\tfrac{x^i}{i!}\right) + \overline{\theta}_2 h_x^2 - \overline{\theta}_1 h_x - \overline{\theta}_3 h_{xx} = 0 \ . \tag{27}$$

Assume:

$$h(x) = a_0 + \frac{a_1}{1}x^1 + \frac{a_2}{2}x^2 + \frac{a_3}{3}x^3 + \ldots + \frac{a_2}{2}x^2 \ldots \equiv a_0 + \sum_{i=1}^{\infty}\frac{a_i}{i}x^i . \tag{28a}$$

This gives us:

$$h_x = \sum_{i=1}^{\infty} a_i x^{i-1} , \tag{28b}$$

$$h_{xx} = \sum_{i=1}^{\infty} a_i (i-1) x^{i-1} \tag{28c}$$

$$h_x^2 = \left[\sum_{i=1}^{\infty} a_i x^{i-1}\right]^2 = 2\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} a_i a_{i+k} x^{2i+k-2} + \sum_{i=1}^{\infty} a_i^2 x^{2i-2}. \tag{28d}$$

Substituting, we get:

$$
\begin{aligned}
&R\left[a_0 + a_1 x + \frac{a_2}{2} x^2 + \frac{a_3}{3} x^3 + \ldots + \frac{a_n}{x} x^n + \ldots\right] \\
&-R\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots\right] \\
&+\bar{\theta}_2\left[a_1^2 + a_2^2 x^2 + a_3^2 x^4 + a_4^2 x^6 + \ldots + a_n x^{2n-2} \ldots\right] \\
&+2\bar{\theta}_2\left[a_1 a_2 x + a_2 a_3 x^3 + a_3 a_4 x^5 + a_4 a_5 x^7 + \ldots + a_n a_{n+1} x^{2n-1} + \ldots\right] \\
&+2\bar{\theta}_2\left[a_1 a_3 x^2 + a_2 a_4 x^4 + a_3 a_5 x^6 + a_4 a_6 x^8 + \ldots + a_n a_{n+2} x^{2n} + \ldots\right] \\
&+2\bar{\theta}_2\left[a_1 a_4 x^3 + a_2 a_5 x^5 + a_3 a_6 x^7 + a_4 a_7 x^9 + \ldots + a_n a_{n+3} x^{2n+1} + \ldots\right] \\
&+\ldots \\
&-\bar{\theta}_1 a_1 - \bar{\theta}_1 a_2 x - \bar{\theta}_1 a_3 x^2 - \ldots - \bar{\theta}_1 a_n x^{n-1} \ldots \\
&-\bar{\theta}_3 a_2 - 2\bar{\theta}_3 a_3 x - 3\bar{\theta}_3 a_4 x^2 \ldots (n-1)\bar{\theta}_1 a_n x^{n-2} \ldots = 0.
\end{aligned}
\tag{29}
$$

Rearranging the terms, we obtain the following expressions:

$$
\begin{aligned}
&\Rightarrow\left[Ra_0 - R + \bar{\theta}_2 a_1^2 - \bar{\theta}_1 a_1 - \bar{\theta}_3 a_2\right]x^0 \\
&+\left[Ra_1 - R + 2\bar{\theta}_2 a_1 a_2 - \bar{\theta}_1 a_2 - 2\bar{\theta}_3 a_3\right]x^1 \\
&+\left[R\frac{a_2}{2} - \frac{R}{2!} + \bar{\theta}_2 a_2^2 + 2\bar{\theta}_2 a_1 a_3 - \bar{\theta}_1 a_3 - 3\bar{\theta}_3 a_4\right]x^2 \\
&+\left[R\frac{a_3}{3} - \frac{R}{3!} + 2\bar{\theta}_2 a_1 a_4 + 2\bar{\theta}_2 a_2 a_3 - \bar{\theta}_1 a_4 - 4\bar{\theta}_3 a_5\right]x^3 \\
&+\left[R\frac{a_4}{4} - \frac{R}{4!} + 2\bar{\theta}_2 a_1 a_5 + 2\bar{\theta}_2 a_2 a_4 + \bar{\theta}_2 a_3^2 - \bar{\theta}_1 a_5 - 5\bar{\theta}_3 a_6\right]x^4 \\
&+\left[R\frac{a_5}{5} - \frac{R}{5!} + 2\bar{\theta}_2 a_1 a_6 + 2\bar{\theta}_2 a_2 a_5 + 2\bar{\theta}_2 a_3 a_4 - \bar{\theta}_1 a_6 - 6\bar{\theta}_3 a_7\right]x^5 \\
&+\left[R\frac{a_6}{6} - \frac{R}{6!} + 2\bar{\theta}_2 a_1 a_7 + 2\bar{\theta}_2 a_2 a_6 + 2\bar{\theta}_2 a_3 a_5 + \bar{\theta}_2 a_4^2 - \bar{\theta}_1 a_7 - 7\bar{\theta}_3 a_8\right]x^6 \\
&+\left[R\frac{a_7}{7} - \frac{R}{7!} + 2\bar{\theta}_2 a_1 a_8 + 2\bar{\theta}_2 a_2 a_7 + 2\bar{\theta}_2 a_3 a_6 + 2\bar{\theta}_2 a_4 a_5 - \bar{\theta}_1 a_8 - 8\bar{\theta}_3 a_9\right]x^7
\end{aligned}
\tag{30}
$$

$+....$

$$+\begin{bmatrix} R\dfrac{a_{2n}}{(2n)}-\dfrac{R}{(2n)!}+2\bar{\theta}_2\,a_1a_{2n+1}+2\bar{\theta}_2a_2a_{2n}+2\bar{\theta}_2a_3a_{2n-1} \\ +....+2\bar{\theta}_2\,a_na_{n+2}+\bar{\theta}_2\,a_{n+1}^2-\bar{\theta}_1\,a_{2n+1}-(2n+1)\bar{\theta}_3\,a_{2n+2} \end{bmatrix}x^{2n}$$

$$+\begin{bmatrix} R\dfrac{a_{2n+1}}{(2n+1)}-\dfrac{R}{(2n+1)!}+2\bar{\theta}_2\,a_1a_{2n+2}+2\bar{\theta}_2a_2a_{2n+1}+2\bar{\theta}_2a_3a_{2n} \\ +....+2\bar{\theta}_2\,a_{n+1}a_{n+2}-\bar{\theta}_1\,a_{2n+2}-(2n+2)\bar{\theta}_3\,a_{2n+3} \end{bmatrix}x^{2n+1}$$

$+....=0.$

The above expressions consist of an infinite series of powers of x,
$b_0x^0+b_1x^1+b_2x^2+b_3x^3+....+b_nx^n+....=0.$

By setting each of the coeffecients, $b_0,b_1,b_2,....b_n$, equal to zero, we can solve for the coefficients $a_0,a_1,a_2,....a_n,....a_{2n},a_{2n+1}$, etc.

Thus, we have:

$$\left[Ra_0-R+\left(\bar{\theta}_2-a_1-\bar{\theta}_1\right)a_1-\bar{\theta}_3a_2\right]=0.$$
$$\left[Ra_1-R+2\bar{\theta}_2a_1a_2-\bar{\theta}_1a_2-2\bar{\theta}_3a_3\right]=0.$$
$$\left[R\frac{a_2}{2}-\frac{R}{2!}+2\bar{\theta}_2a_1a_3+\bar{\theta}_2a_2^2-\bar{\theta}_1a_3-3\bar{\theta}_3\,a_4\right]=0.$$
$$\left[R\frac{a_3}{3}-\frac{R}{3!}+2\bar{\theta}_2\,a_1a_4+2\bar{\theta}_2a_2a_3-\bar{\theta}_1\,a_4-4\bar{\theta}_3\,a_5\right]=0.$$

.

.

.

$$\begin{bmatrix} R\dfrac{a_{2n}}{(2n)}-\dfrac{R}{2n!}+2\bar{\theta}_2a_1a_{2n+1}+2\bar{\theta}_2\,a_2a_{2n}+2\bar{\theta}_2a_3a_{2n-1} \\ +...+2\bar{\theta}_2\,a_na_{n+2}+\bar{\theta}_2a_{n+1}^2-\bar{\theta}_1\,a_{2n+1}-(2n+1)\bar{\theta}_3\,a_{2n+2} \end{bmatrix}=0.$$
$$\begin{bmatrix} R\dfrac{a_{2n+1}}{(2n+1)}-\dfrac{R}{2n+1!}+2\bar{\theta}_2a_1a_{2n+2}+2\bar{\theta}_2\,a_2a_{2n+1}+2\bar{\theta}_2a_3a_{2n} \\ +...+2\bar{\theta}_2\,a_{n+1}a_{n+2}-\bar{\theta}_1\,a_{2n+2}-(2n+2)\bar{\theta}_3\,a_{2n+3} \end{bmatrix}=0. \qquad (31)$$

and so on.

This is a complex set of relationships that can only be solved sequentially in the following manner. We have:

$$a_{2n+3} = \frac{1}{(2n+2)\bar{\theta}_3}\begin{bmatrix} R\dfrac{a_{2n+1}}{(2n+1)} - \dfrac{R}{2n+1!} + 2\bar{\theta}_2 a_1 a_{2n+2} + 2\bar{\theta}_2\, a_2 a_{2n+1} \\ +\ldots+ 2\bar{\theta}_2\, a_{n+1} a_{n+2} - \bar{\theta}_1\, a_{2n+2} \end{bmatrix} \tag{32}$$

$$a_{2n+2} = \frac{1}{(2n+1)\bar{\theta}_3}\begin{bmatrix} R\dfrac{a_{2n}}{(2n)} - \dfrac{R}{2n!} + 2\bar{\theta}_2 a_1 a_{2n+1} + 2\bar{\theta}_2\, a_2 a_{2n} \\ +\ldots+ 2\bar{\theta}_2\, a_n a_{n+2} + \bar{\theta}_2\, a_{n+1}^2 - \bar{\theta}_1\, a_{2n+1} \end{bmatrix} \tag{33}$$

for $n \geq 1$.

And we have:

$a_3 = \frac{1}{2\bar{\theta}_3}\left[Ra_1 - R + 2\bar{\theta}_2\, a_1 a_2 - \bar{\theta}_1\, a_2\right]$ and $a_2 = \frac{1}{\bar{\theta}_3}\left[Ra_0 - R + \bar{\theta}_2\, a_1^2 - \bar{\theta}_1\, a_1\right]$.

Note: There are many feasible solutions to the above set-up, each corresponding to a *different* assumption re, a_0, and a_1, that can be arbitrarily specified.

For example, with $a_0 = 0$ and $a_1 = 1$, we get:

$$a_2 = \frac{1}{\bar{\theta}_3}\left[\left(\bar{\theta}_2 - \bar{\theta}_1\right) - R\right],$$

which together with a_1 then specifies a_3, and henceforth all the other a_n's.

Alternatively, with $a_0 = 0$ and $a_1 = 0$, we get $a_2 = \frac{-R}{\bar{\theta}_3}$, which gives

$$a_2 = \frac{1}{2\bar{\theta}_3}\left[-R + \frac{R\bar{\theta}_1}{\bar{\theta}_3}\right] = \frac{R\left(\bar{\theta}_1 - \bar{\theta}_3\right)}{2\bar{\theta}_3^2},$$ and so on.

Again, with $a_0 = 2$ and $a_1 = 0$, we get

$$a_2 = \frac{R}{\bar{\theta}_3}, \text{ and } a_3 = \frac{1}{2\bar{\theta}_3}\left[-R - \frac{R\bar{\theta}_1}{\bar{\theta}_3}\right] = \frac{-R\left(\bar{\theta}_1 + \bar{\theta}_3\right)}{2\bar{\theta}_3^2},$$

and so on.

The reason for setting $a_1 = 0$ is that it makes some of the steps easier. Nevertheless, once a_0 and a_1 are specified, then they automatically specify all the values of subsequent coefficients a_n for the function $h(x)$.

We could equally well set

$a_1 = \frac{\overline{\theta}_1}{\overline{\theta}_2}$ and $a_0 = 2$, which again gives $a_2 = \frac{R}{\overline{\theta}_3}$, or with $a_0 = 0, a_2 = \frac{-R}{\overline{\theta}_3}$.

We have therefore with an appropriate choice of a_0, a_1, and hence all the other a_n's, obtained the expression for $h(x)$ as:

$$h(x) = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{i} x^i . \tag{28}$$

Now we have:

$x \equiv Lnv$:

$$\Rightarrow g(v) = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{i} [Lnv]^i \tag{34}$$

$\Rightarrow$ The value function $f(v) = g(v) - v = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{i} [Lnv]^i - v.$

Once we have $f(v)$, we then have f_v, f_{vv} which enables us to find the expressions for the optimal assumption rate, $c^*(t)$, and the portfolio weight vector, $W\underline{w}^*$. We are also then in a position to undertake comparative statics of the problem.

II. Power Utility Function:

For the power utility case, we assume the following forms for the utility function and the derived utility function:

$U(c) \equiv \frac{[c-\hat{c}]^\gamma}{\gamma}$ $\quad \gamma < 1 \quad$ $\hat{c}$: Minimum consumption level

$U(c) \equiv \frac{-[\hat{c}-c]^\gamma}{\gamma}$ $\quad \gamma > 1 \quad$ $\hat{c}$: Maximum consumption level

$J[w,v] \equiv \Delta \frac{f^\gamma}{\gamma} \quad \gamma > 1; \gamma < 1$

$f \equiv f(w,v)$

Upon substitution, simplification, and rearrangement of terms, the HJBE per equation (6) is transformed to the following nonlinear PDE below:

$$
\begin{aligned}
& f_w\left[\Delta f_w\right]^{\frac{1}{\gamma-1}}\left[\tfrac{1}{\gamma}-1\right]f^2-\tfrac{\rho}{\gamma}f^2+RWf_wf-\tfrac{1}{2}\frac{f_w^2}{\left[f_{ww}f+(\gamma-1)f_w^2\right]}\hat{\theta}_0f^2-\hat{c}\Delta\, f_wf \\
& +v\left[\overline{Z}_v\left(f_w+f_v\right)-\hat{\theta}_1\frac{f_w\left[\left(f_{ww}+f_{wv}\right)f+(\gamma-1)f_w\left(f_w+f_v\right)\right]}{\left[f_{ww}f+(\gamma-1)f_w^2\right]}\right]f \\
& +\left(\tfrac{1}{2}\right)v^2\left[\begin{array}{l}\sigma_v^2\left[\left(f_{ww}+2f_{wv}+f_{vv}\right)f+(\gamma-1)\left(f_w+f_v\right)^2\right] \\ -\hat{\theta}_2\dfrac{\left[\left(f_{ww}+f_{wv}\right)f+(\gamma-1)f_w\left(f_w+f_v\right)\right]^2}{\left[f_{ww}f+(\gamma-1)f_w^2\right]}\end{array}\right]=0.
\end{aligned}
\tag{35}
$$

For the case when $\gamma < 1$, and for the case when $\gamma > 1$,

$$
\begin{aligned}
& f_w\left[\Delta f_w\right]^{\frac{1}{\gamma-1}}\left[1-\tfrac{1}{\gamma}\right]f^2-\tfrac{\rho}{\gamma}f^2+RWf_wf-\tfrac{1}{2}\frac{f_w^2}{\left[f_{ww}f+(\gamma-1)f_w^2\right]}\hat{\theta}_0f^2-\hat{c}\Delta\, f_wf \\
& +v\left[\overline{Z}_v\left(f_w+f_v\right)-\hat{\theta}_1\frac{f_w\left[\left(f_{ww}+f_{wv}\right)f+(\gamma-1)f_w\left(f_w+f_v\right)\right]}{\left[f_{ww}f+(\gamma-1)f_w^2\right]}\right]f \\
& +\left(\tfrac{1}{2}\right)v^2\left[\begin{array}{l}\sigma_v^2\left[\left(f_{ww}+2f_{wv}+f_{vv}\right)f+(\gamma-1)\left(f_w+f_v\right)^2\right] \\ -\hat{\theta}_2\dfrac{\left[\left(f_{ww}+f_{wv}\right)f+(\gamma-1)f_w\left(f_w+f_v\right)\right]^2}{\left[f_{ww}f+(\gamma-1)f_w^2\right]}\end{array}\right]=0.
\end{aligned}
\tag{36}
$$

And we also have:

$$
c^*=\left[\Delta f_W\right]^{\frac{1}{\gamma-1}}f+\hat{c}\ \text{for}\ \gamma<1,\ c^*=\hat{c}-\left[\Delta f_W\right]^{\frac{1}{\gamma-1}}f\ \ \text{for}\ \gamma>1. \tag{37}
$$

$$
W\underline{w}^*=-\underline{\underline{\Sigma}}^{-1}\left[\frac{f_wf\left(\underline{\overline{Z}}-R\underline{1}\right)}{\left[f_{ww}f+(\gamma-1)f_w^2\right]}+\frac{\left[\left(f_{ww}+f_{wv}\right)f+(\gamma-1)f_w\left(f_w+f_v\right)\right]}{\left[f_{ww}f+(\gamma-1)f_w^2\right]\Delta f^{(\gamma-2)}}\right]v\underline{\Sigma}_{Zv}, \tag{38a}
$$

or

$$
W\underline{w}^*=\frac{-\underline{\underline{\Sigma}}^{-1}}{\left[f_{ww}f+(\gamma-1)f_w^2\right]}\left[f_wf\left(\underline{\overline{Z}}-R\underline{1}\right)+\left[\begin{array}{l}\left(f_{ww}+f_{wv}\right)f \\ +(\gamma-1)f_w\left(f_w+f_v\right)\end{array}\right]v\underline{\Sigma}_{Zv}\right]. \tag{38b}
$$

We can see that resulting differential equation is nonlinear and quite hard to solve. We derive some solutions for the special cases below:

For the power utility function case, with $\gamma < 1$, we have the HJBE as,

$$(f_w)^{\gamma/\gamma-1}\left[\frac{1}{\gamma}-1\right]f^2 - \frac{\rho}{\gamma}f^2 - \tfrac{1}{2}\hat{\theta}_0\frac{f_W^2}{\left[f\cdot f_{WW}+(\gamma-1)f_W^2\right]}f^2 - \hat{c}f_w f$$

$$+RW\cdot f_W\cdot f + v\left[\overline{Z}_v(f_W+f_v)-\hat{\theta}_1\left[(f_w)\frac{\left[f(f_{WW}+f_{WV})+f_W(f_W+f_v)\right]}{\left[f\cdot f_{WW}+(\gamma-1)f_W^2\right]}\right]\right]f \qquad (35a)$$

$$+\tfrac{1}{2}v^2\left[\begin{array}{l}\sigma_v^2\left[f(f_{WW}+2f_{Wv}+f_{vv})+(\gamma-1)(f_W+f_v)^2\right]\\ -\hat{\theta}_2\dfrac{\left[f(f_{WW}+f_{Wv})+(\gamma-1)f_W(f_W+f_v)^2\right]}{\left[f\cdot f_{WW}+(\gamma-1)f_W^2\right]}\end{array}\right]=0.$$

with Δ set equal to 1.

Suppose that $\sigma_v^2 = \hat{\theta}_2$. Then we can set $f \equiv aW + bv + K$.

$\Rightarrow$ The HJBE becomes:

$$\left[(a)^{\gamma/\gamma-1}\left[\tfrac{1}{\gamma}-1\right]-\frac{\rho}{\gamma}-\tfrac{1}{2}\frac{\hat{\theta}_0}{(\gamma-1)}\right]f^2 - \hat{c}af$$

$$+RaWf + v\left[\overline{Z}_v(a+b)-\hat{\theta}_1(a+b)\right]f \qquad (39)$$

$$+\tfrac{1}{2}v^2\left[\sigma_v^2(\gamma-1)(a+b)^2-\hat{\theta}_2(\gamma-1)(a+b)^2\right]=0.$$

$$\Rightarrow\left[a^{(\gamma/\gamma-1)}\left(\tfrac{1}{\gamma}-1\right)-\frac{\rho}{\gamma}+R-\tfrac{1}{2}\frac{\hat{\theta}_0}{(\gamma-1)}\right]f^2 - \left[\hat{c}a+RK\right]f \qquad (40)$$

$$-Rbvf + v\left(\overline{Z}_v-\hat{\theta}_1\right)(a+b)f + \tfrac{1}{2}v^2\left[\sigma_v^2-\hat{\theta}_2\right](\gamma-1)(a+b)^2=0.$$

But with $\sigma_v^2 = \hat{\theta}_2$, we have the HJBE now as:

$$\left[a^{(\gamma/\gamma-1)}\left(\tfrac{1}{\gamma}-1\right)-\frac{\rho}{\gamma}+R-\tfrac{1}{2}\frac{\hat{\theta}_0}{(\gamma-1)}\right]f^2$$
$$-\left[Rb+\left(\overline{Z}_v-\hat{\theta}_1\right)(a+b)\right]vf=0. \tag{41}$$

We can now solve for *a*, *b*, and *K* as follows:

Define $\overline{Z}_v-\hat{\theta}_1\equiv\theta_1$.

Set $\left[a^{(\gamma/\gamma-1)}\left[\tfrac{1}{\gamma}-1\right]-\frac{\rho}{\gamma}+R-\tfrac{1}{2}\frac{\hat{\theta}_0}{(\gamma-1)}\right]=0,$

set $\left[-Rb+\theta_1(a+b)\right]=0,$ and

set $\left[\hat{c}a+RK\right]=0.$

From the first we obtain the value of *a* as:

$$a=\left[\left[R-\frac{\rho}{\gamma}-\tfrac{1}{2}\frac{\hat{\theta}_0}{(\gamma-1)}\right]\frac{\gamma}{(\gamma-1)}\right]^{\frac{\gamma-1}{\gamma}} \tag{42a}$$

and we assume that the "transversality" condition holds, and we obtain the value of *b*, as:

$$b=\left(\frac{\theta_1 a}{R-\theta_1}\right)\text{, and} \tag{42b}$$

$$K=-\frac{\hat{c}a}{R}. \tag{42c}$$

So that we have:

$$f(W,v)=aW+bv+K=a\left[W+\left(\frac{\theta_1}{R-\theta_1}\right)v-\frac{\hat{c}}{R}\right]$$

$$\text{and } I[W,v]=\frac{\left[f(W,v)\right]^\gamma}{\gamma}=a^\gamma\frac{\left[W+\left(\frac{\theta_1}{R-\theta_1}\right)-\frac{\hat{c}}{R}\right]^\gamma}{\gamma} \tag{42d}$$

$$\text{and } J[W,v,t]=e^{-\rho t}I[W,v]=e^{-\rho t}a^\gamma\frac{\left[W+\left(\frac{\theta_1}{R-\theta_1}\right)-\frac{\hat{c}}{R}\right]^\gamma}{\gamma}$$

When $\gamma > 1$, we use equation (36) and again setting $\Delta = 1$, and with the condition $\sigma_v^2 v^2 = \hat{\theta}_2$ holding, we obtain a functional form *fW f*(*W*,*v*) as below, with $\gamma > 1$:

$$f[W,v] = aW + bv + K,$$

where

$$a = \left[\left(\frac{\gamma}{\gamma-1}\right)\left[\frac{\rho}{\gamma} + \frac{1}{2}\frac{\hat{\theta}_0}{(\gamma-1)} - R\right]\right]^{\frac{\gamma}{\gamma-1}}, \tag{42e}$$

b and K the same as per equations (42b and 42c).

One can interpret the capitalized value of the wage income stream as $\left[\frac{\theta_1}{R-\theta_1}\right]v$.

An interesting aspect of this set up is that this value, $\left[\frac{\theta_1}{R-\theta_1}\right]v$, is independent of the investor characteristics, which is not too surprising given the assumption that $\sigma_v^2 = \hat{\theta}_2$, a spanning condition is satisfied. However, it <u>must</u> be noted here that there is no explicit assumption that the "market" asset returns span the "wage" stream. It is the "augmented" market <u>and</u> non-market asset set that is assumed to span the wage stream dynamics.

The coefficient a is a function of the investor characteristics and the capital asset parameters. It does not depend upon the wage dynamics. Again, a feature of the spanning condition imposed here.

Finally, note that the "augmented wealth," $\left[W + \left(\frac{\theta_1}{R-\theta_1}\right)v\right]$, is separable in actual wealth, W, and the wage stream, v. In general, it will not be the case that the "capitalized value" of the wage stream is independent of the investor's wealth and his risk characteristics. For the general case, we are forced to solve the highly non-linear, second order differential equation in $f(W, v)$.

The optimal consumption and portfolio allocations are therefore given as:

For optimal consumption, c^*, we have ?? equation (37), when $\gamma < 1$:

$$c^* = [a]^{\frac{1}{\gamma-1}}[aW + bv + K] + \hat{c},$$

and when $\gamma > 1$,

$$c^* = \hat{c} - [a]^{\frac{1}{\gamma-1}}[aW + bv + K] \text{ or}$$

$$c^* = [a]^{\frac{\gamma}{\gamma-1}}\left[W + \left(\frac{\theta_1}{R-\theta_1}\right)v - \frac{\hat{c}}{R}\right] + \hat{c} \quad \gamma < 1 \tag{43a}$$

$$c^* = \hat{c} - [a]^{\frac{\gamma}{\gamma-1}}\left[W + \left(\frac{\theta_1}{R-\theta_1}\right)v - \frac{\hat{c}}{R}\right] \quad \gamma > 1 \tag{43b}$$

with a as derived before for each case and the optimal portfolio allocation vector as:

$$W\underline{w}^* = -\underline{\underline{\Sigma}}^{-1}\begin{bmatrix} \left(\frac{1}{\gamma-1}\right)\left[W + \left(\frac{\theta_1}{R-\theta_1}\right)v - \frac{\hat{c}}{R}\right](\underline{\overline{Z}} - R\underline{1}) \\ -\left(\frac{R}{R-\theta_1}\right)v\underline{\Sigma}_{Zv} \end{bmatrix} \tag{44a}$$

for both $\gamma < 1$ and $\gamma > 1$.

Therefore, with the setup where $\sigma_v^2 = \hat{\theta}_2$, the problem is tractable and has an optimal solution.

For the general case, we revert back to our HJBE per equation (35) or (36).

General case result for the case when $\gamma = 2$:

When $\gamma = 2$, the derivations are as given below, equation (36) becomes:

$$\begin{aligned} &(I_W)^2\left(1 - \frac{1}{2}\right) - \frac{\rho I}{2} - \frac{1}{2}\hat{\theta}_0\frac{I_W^2}{I_{WW}} + RWI_W - \hat{c}I_W \\ &\theta_1 v I_W + \overline{Z}_v I_v - \hat{\theta}_1 v I_W \frac{I_{Wv}}{I_{WW}} + \frac{1}{2}\sigma_v^2 v^2(I_{WW} + 2I_{Wv}) \\ &+\frac{1}{2}\sigma_v^2 v^2 I_{vv} - \frac{1}{2}\sigma_v^2 v^2\frac{I_{Wv}^2}{I_{WW}} = 0. \end{aligned} \tag{44b}$$

If we assume

$I(W,v) = \frac{f^2}{2} + g(v)$; $f \equiv f(W,v)$, upon substituion we get,

$$(f_W)^2\left(1-\frac{1}{2}\right)f^2-\frac{\rho}{\gamma}f^2-\rho g(v)-\frac{1}{2}\hat{\theta}_0\frac{f_W^2}{\left[f\cdot f_{WW}+f_W^2\right]}f^2$$
$$+RWf_Wf-\hat{c}f_Wf+\theta_1 vf_Wf+\overline{Z}_v vf_vf+\overline{Z}_v vg_v$$
$$-\hat{\theta}_1 v\frac{f_W\left[f\cdot f_{Wv}+f_W\cdot f_v\right]}{\left[f\cdot f_{WW}+f_W^2\right]}f$$
$$+\frac{1}{2}\sigma_v^2v^2\left[f\left(f_{WW}+2f_{Wv}\right)+f_W\left(f_W+2f_v\right)\right] \tag{44c}$$
$$+\frac{1}{2}\sigma_v^2v^2\left[f\cdot f_{vv}+f_v^2\right]+\frac{1}{2}\sigma_v^2v^2g_{vv}$$
$$-\frac{1}{2}\sigma_v^2v^2\frac{\left[f\cdot f_{Wv}+f_W\cdot f_v\right]^2}{\left[f\cdot f_{WW}+f_W^2\right]}=0.$$

By setting $f(W,v)=aW+bv+K_1$; and $g(v)=cv^2$, we have
$f_W=a;\quad f_v=b;\quad g_v=2cv;\quad g_{vv}=2c;\quad f_{WW}=0;\quad f_{Wv}=0;\quad f_{vv}=0;$

so that we have

$$a^2\left(1-\tfrac{1}{2}\right)(aW+bv+K_1)^2-\frac{\rho}{2}(aW+bv+K_1)^2-\rho cv^2-\frac{1}{2}\hat{\theta}_0(aW+bv+K_1)^2$$
$$+R[aW+bv+K_1]^2-Rbv[aW+bv+K_1]-RK_1[aW+bv+K_1]-\hat{c}a[aW+bv+K_1]$$
$$+\theta_1av[aW+bv+K_1]+\overline{Z}_vbv[aW+bv+K_1]+2\overline{Z}_vcv^2-\hat{\theta}_1bv[aW+bv+K_1] \tag{44d}$$
$$+\frac{1}{2}\sigma_v^2v^2a(a+2b)+\frac{1}{2}\sigma_v^2v^2b^2+c\sigma_v^2v^2-\frac{1}{2}\sigma_v^2v^2b^2=0.$$

Note: $\theta_2=\sigma_v^2-\hat{\theta}_2$; $\theta_1=\overline{Z}_v-\hat{\theta}_1$.

Collecting like terms, we get:

$$\left[\frac{1}{2}a^2-\frac{\rho}{2}-\frac{1}{2}\hat{\theta}_0+R\right][aW+bv+K]^2$$
$$[Rbv-RK_1-\hat{c}a+\theta_1av+\theta_1bv](aW+bv+K_1) \tag{44e}$$
$$+\left[-\rho cv^2+2\overline{Z}_vcv^2+\frac{1}{2}\sigma_v^2v^2(a+b)^2+\sigma_v^2cv^2\right]=0.$$

Setting the terms in the square brackets to zero, we get, respectively:

$$\therefore a^2 = 2\left[\frac{\rho}{2} + \frac{1}{2}\hat{\theta}_0 - R\right] = \left[\rho + \hat{\theta}_0 - 2R\right] \text{ and} \tag{44f}$$

$$a = \left[\rho + \hat{\theta}_0 - 2R\right]^{1/2}.$$

Secondly:

$$-RK_1 - \hat{c}a = v$$

$$\Rightarrow K_1 = \frac{-\hat{c} \cdot a}{R} \tag{44g}$$

Thirdly:

$$\left[-(R-\theta_1)b + \theta_1 a\right]v = 0$$

$$\Rightarrow b = \frac{\theta_1 a}{(R-\theta_1)}. \tag{44h}$$

And finally:

$$\left[\left[-\rho + 2\overline{Z}_v + \sigma_v^2\right]c + \frac{1}{2}\theta_2(a+b)^2\right]v^2 = 0$$

$$\Rightarrow c = \frac{1}{2}\frac{\theta_2(a+b)^2}{\left[\rho - 2\overline{Z}_v - \sigma_v^2\right]}. \tag{44i}$$

So we have solved for everything, for once we obtain the specification for $I[W,v]$, then we can obtain the expressions for c^*, $W\underline{w}^*$, which in turn enable us to do comparative statics and measure welfare level(s). For $\gamma \neq 2$, we have to revertto equation (35) or (36) to solve for the general case. We have not yet been able to get the result for $\gamma \neq 2$.

III. Log Utility Function:

For the log utility function case, we assume the following forms of the utility function and the derived utility function:

$$U(c) = Ln(c - \hat{c}); \; I(W,v) = \Delta Ln(f(W,v)).$$

Upon substituting the assumed functional forms in equation (6), we get the HJBE as:

$$
\begin{aligned}
&Ln(f) - Ln(\Delta) - Ln(f_W) - \rho \cdot \Delta \cdot Ln(f) - 1 \\
&+RW \cdot f_W \cdot \frac{\Delta}{f} - \tfrac{1}{2}\hat{\theta}_0 \frac{\Delta f_W^2}{\left[f_{WW} f - f_W^2\right]} - \hat{c} f_W \cdot \frac{\Delta}{f} \\
&+v\left[(f_W + f_v)\overline{Z}_v - \hat{\theta}_1 \frac{f_W\left[(f_{WW} + f_{Wv})f - f_W(f_W + f_v)\right]}{\left[f_{WW} f - f_W^2\right]}\right]\frac{\Delta}{f} \\
&+\left(\tfrac{1}{2}\right)v^2 \begin{bmatrix} \left[(f_{WW} + 2f_{Wv} + f_{vv})f - (f_W + f_v)^2\right]\sigma_v^2 \\ -\left[\dfrac{\left[(f_{WW} + f_{Wv})f - f_W(f_W + f_v)\right]^2}{\left[f_{WW} f - f_W^2\right]}\right]\hat{\theta}_2 \end{bmatrix} \frac{\Delta}{f^2} = 0,
\end{aligned}
\tag{45}
$$

and the optimal choices are:

$$
c^* = \frac{f}{\Delta f_W} + \hat{c} \tag{46a}
$$

$$
W\underline{w}^* = -\underline{\underline{\Sigma}}^{-1}\left[\frac{f\, f_W\left(\underline{\overline{Z}} - R\underline{1}\right)}{\left[f_{WW} f - f_W^2\right]} + \frac{v\left[(f_{WW} + f_{Wv})f - f_W(f_W + f_v)\right]\underline{\Sigma}_{Zv}}{\left[f_{WW} f - f_W^2\right]}\right]. \tag{46b}
$$

Again, we derive the solution to the HJBE above for a special case.

Solution for the special case: $\sigma_v^2 = \hat{\theta}_2$.

We have $U(c) \equiv Ln[c - \hat{c}]$, and we have $I\,[W,v] \equiv \Delta\, Ln\,[f\,(W,v)]$. For the special case, we guess the solution for $f\,[W,v]$ as $f\,[W,v] = aW + bv + K$. This gives us the following partial derivatives:

$f_W = a$; $f_{WW} = 0$; $f_v = b$; $f_{vv} = 0$; and $f_{Wv} = 0$.

Substituting for these in the HJBE at the top of the page, we have:

$$Ln[aW+bv]-Ln[\Delta]-Ln[a]$$

$$-\rho\Delta Ln[aW+bv+K]-1+R\frac{aW\Delta}{(aW+bv+K)}$$

$$+\tfrac{1}{2}\frac{\hat{\theta}_0}{a^2}\Delta a^2-\frac{\hat{c}\Delta a}{[aW+bv+K]}+v\left[(a+b)\overline{Z}_v-\hat{\theta}_1(a+b)\right]\frac{\Delta}{(aW+bv+K)} \tag{45}$$

$$+\tfrac{1}{2}v^2\left[-(a+b)^2\sigma_v^2+(a+b)^2\hat{\theta}_2\right]\frac{\Delta}{(aW+bv+K)^2}=0.$$

Rearranging the terms, we get:

$$\left[Ln(\Delta)+Ln(a)+1-\tfrac{1}{2}\hat{\theta}_0\Delta-R\Delta\right]-\left[Ln(aW+bv+K)-\rho\Delta Ln(aW+bW+K)\right]$$

$$-[\hat{c}a+RK]\frac{\Delta}{(aW+bv+K)}+\left[bR-(\overline{Z}_v-\hat{\theta}_1)(a+b)\right]\frac{\Delta}{(aW+bv+K)} \tag{46}$$

$$+\tfrac{1}{2}v^2\left[\sigma_v^2-\hat{\theta}_2\right]\frac{\Delta}{(aW+bv+K)^2}=0.$$

From the way we have rearranged the terms above, we can solve separately and sequentially for Δ, a, and b, respectively, given that $\sigma_v^2=\hat{\theta}_2$.

Thus, we have $\Delta=\dfrac{1}{\rho}$ and, (47a)

$$Ln[a\Delta]=\Delta\left[R+\tfrac{1}{2}\hat{\theta}_0-\frac{1}{\Delta}\right]=\frac{1}{\rho}\left[R-\rho+\tfrac{1}{2}\hat{\theta}_0\right]$$

$$\Rightarrow a=\frac{1}{\Delta}exp\left[\frac{1}{\rho}\left[R-\rho+\tfrac{1}{2}\hat{\theta}_0\right]\right]=\rho\ exp\left[\frac{1}{\rho}\left[R-\rho+\tfrac{1}{2}\hat{\theta}_0\right]\right], \tag{47b}$$

$$K=\frac{\hat{c}}{R}a, \tag{47c}$$

and finally we have:

$$b=\frac{(\overline{Z}_v-\hat{\theta}_1)a}{R-(\overline{Z}_v-\hat{\theta}_1)}=\frac{\theta_1 a}{R-\theta_1},\ \text{where } \theta_1\equiv\overline{Z}_v-\hat{\theta}_1. \tag{47d}$$

Note: These results follow from the transformed HJBE above. Therefore, we have:

$$f[w,v] = aW + bv + K = a\left[W + \left(\frac{\theta_1}{R-\theta_1}\right)v - \frac{\hat{c}}{R}\right], \tag{48a}$$

where

$$a = \rho \cdot exp\left[\frac{1}{\rho}\left[R - f + \tfrac{1}{2}\hat{\theta}_0\right]\right].$$

$$\hat{\theta}_0 = (\underline{Z} - R\underline{1})^T \underline{\underline{\Sigma}}^{-1}(\underline{Z} - R\underline{1}).$$

$$\theta_1 = \overline{Z}_v - \hat{\theta}_1 = \overline{Z}_v - (\underline{Z} - R\underline{1})^T \underline{\underline{\Sigma}}^{-1}\underline{\Sigma}_{Zv}.$$

So, we have:

$$I[W,v] = \Delta Ln[f(w,v)] = \frac{1}{\rho}Ln\left[a\left[W + \frac{\theta_1}{R-\theta_1}v - \frac{\hat{c}}{R}\right]\right] \tag{48b}$$

with a as defined above. This, in turn, enables us to obtain the expressions for the optimal consumption rate c^* and portfolio choice vector $W\underline{w}^*$. We have, from the expressions on page 20, upon substitution:

$$c^* = \frac{f}{\Delta f W} + \hat{c} = \rho\left[W + \left(\frac{\theta_1}{R-\theta_1}\right)v - \frac{\hat{c}}{R}\right] + \hat{c} \tag{49}$$

$$W\underline{w}^* = \underline{\underline{\Sigma}}^{-1}\left[\left[W + \left(\frac{\theta_1}{R-\theta_1}\right)v\right](\underline{Z} - R\underline{1}) - \left(\frac{R}{R-\theta_1}\right)v\underline{\Sigma}_{Zv}\right]. \tag{50}$$

Finally,

$$J[W,v,t] = e^{-\rho t}I[W,v] = e^{-\rho t}\frac{1}{\rho}\left[Ln[aW + bv + K]\right] \tag{51}$$

with a and b as derived above.

When $\sigma_v^2 \neq \hat{\theta}_2$, we revert back to the general expressions for the HJBE, c^* and $W\underline{w}^*$ per equations (45a), (46b), and (46c). We will again have to first solve for $f(W,v)$ before we can characterize the optimal c^* and the $W\underline{w}^*$ vector.

We stop our exercise at this stage and leave the study of comparative statics for another paper. The latter task is rendered quite transparent, especially in the case of the negative exponential class of utility functions, where the solution has been obtained for the next general setup. In the case of the power utility function, we have a solution only for the special case.

The negative exponential case is easier to handle, as the capitalized value of the wage stream does not depend upon the wealth endowment, W.

SECTION IV: CONCLUSION

Once we have fully characterized the optimal consumption choice, $c^*(t)$, and the optimal portfolio vector, $W\underline{w}^*$, the next step would be to examine their functional dependence upon the various parameters (or variables) of the system. We defer this comparative-statics exercise to a later paper and restrict ourselves here to the observation that investor behavior in the light of "private" assets and/or wage income stream, whether "spanned" or "non-spanned," will be radically different from a case where he has only the set of "market" assets to choose from.

This is not a radical or surprising observation and has, in fact, been known for quite some time; but to our knowledge, this is the first successful attempt to combine both private assets and wage stream in one model and derive the optimality relationships, where the wage stream follows a more economically appealing geometric Brownian motion instead of an arithmetic Brownian motion. We are also in a position to solve explicitly, for the first time, the problem for the negative exponential class of utility functions.

References

Bodie, Z., R.C. Merton, and W.F. Samuelson, (1992). "Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model," *Journal of Economic Dynamics and Control* 16, 427-449.

Merton, R., (1971). "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory* 3, 373-413.

Svensson, L.E.O., (1988). "Portfolio Choice and Asset Pricing with Nontraded Assets," NBER Working Paper No. 2774.

Svensson, L.E.O., and I. Werner, (1993). "Nontraded Assets in Incomplete Markets: Pricing and Portfolio Choice," *European Economic Review* 37, 1149-1168.