Asymmetric Information about Volatility and Option Markets

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Abstract: This paper develops a model of asymmetric information in which an investor has information regarding the future volatility of the price process of an asset but not the future asset price. It is shown that there exists an equilibrium in which the investor trades an option on the asset and expressions for the equilibrium option price and the dynamic trading strategy of the investor are derived endogenously. It is found that the expected volatility of the underlying asset increases in the net order flow in the option market. Also, the depth of the option market is smaller when there is more uncertainty about the variance of the underlying asset, which is conceptually consistent with empirical findings in the equity option market.

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1 Introduction

It is widely recognized that volatility (as represented by the variance or standard deviation) of the underlying asset is one of the most important determinants of option price and, yet, it is the one that is very unpredictable. Therefore it is likely that information asymmetry about the volatility of the underlying asset will have a significant impact on the option market, perhaps, in the same way that information asymmetry about the mean of the asset value impacts prices and liquidity in the market for the underlying asset, as in Kyle (85), Glosten and Milgrom (85) and others. Market makers in the Chicago Board Options Exchange (henceforth CBOE) are known to use different volatility packages for forecasting purposes and take positions in options on the basis of their volatility estimates [Chapter 6 - The Options Institute, (90)], indicating the existence of asymmetry of beliefs about volatility in the option market. Schwager (92) and Cookson (Risk - 93) also hint at the existence of information asymmetry about volatility in the equity options and currency options markets respectively.

While it is certainly true that information asymmetry about the level of the asset price or its mean may exist simultaneously with information asymmetry about volatility, it seems that the primary concern of options traders is the volatility of the underlying asset. In fact, according to Natenberg (90), "*In futures or cash trading, the primary determinant of whether one is successful is correctly picking the direction of the market. An options trader is not trying to pick the direction of the market; he is trying to pick an appropriate volatility.*" This seems to suggest that information asymmetry about the mean of the asset price, while it may exist may only be of secondary importance in the option market. On the other hand, information asymmetry about volatility can certainly have a significant impact on the option market

and private information about volatility may exist without any private information about the future price of the asset as discussed in Cox and Rubinstein (85, p. 55): "*Suppose you are confident that a paper company will soon unexpectedly change its plans to sell the mineral rights on part of its land for a fixed fee and will instead take a large participatory interest in their development. You have no ideas about the likely success of this venture, but you do know that as a result the stock will be much more volatile in the future than the market had anticipated. You know that as soon as this becomes known, options will rise in price, relative to the stock. But you do not know how the stock price will respond. You could not take advantage of your information in the stock market, but you could in the options market*".

The theoretical literature on asymmetric information in options includes John, Koticha and Subrahmanyam (91), Back (93), Biais and Hillion (94), Cherian and Jarrow (95) and Brennan and Cao (95). In these papers (with the exception of Cherian & Jarrow), the information asymmetry is about the mean of the underlying asset's value. Cherian and Jarrow consider the impact of asymmetric information about volatility of the asset price on the option market. They show that the Black-Scholes (73) pricing formula can arise in an equilibrium from self-fulfilling beliefs that it is the correct pricing formula and the information asymmetry can cause implied volatilities to be different from future volatilities. However, their model is static and the optimal demands of the informed trader (assumed to be exogenous) cannot be modeled. As a result the issues related to the dynamics of order flow and its impact on option price, liquidity and the resolution of information asymmetry (about volatility) through time cannot be addressed in that framework.

This paper considers a setting where one investor has private information about the future volatility of the price process (that evolves exogenously as a geometric Brownian motion) of an asset, but does not know the underlying asset price at any point in time. More specifically, the volatility of the underlying asset is going to jump at a point of time in the future and an informed investor has information about

the jump. However, the investor does not have any information about the level of the future asset price at any time. Thus, there is information asymmetry in the market about the volatility of the underlying asset, but not about the level of the asset price/mean.

In this type of setup a discrete time multi-period model is not analytically tractable in the Kyle (85) framework because to my knowledge, the probability distributions for volatility, a non-negative random variable (for example, lognormal or chi-square) do not fall within the elliptical class of distributions for which conditional expectation is linear in the conditioning variable. I utilize the approach to the continuous time Kyle model developed in Back (92) to address the issue of equilibrium in this model. However, the model and the associated information structure are very different from Back (92, 93) and other multi period models that have information asymmetry about mean. In this model, the informed investor knows the future volatility, but does not know the future asset price at any point in time and the option price depends on both the asset price and the volatility of the asset price. As a result, the informed investor, a priori, does not know the value of the option at the time the jump in volatility occurs. In equilibrium, the trading strategy of the informed investor has to be such that there is no jump in the option price immediately following the jump in volatility as otherwise it would imply that the option was mispriced at the end which is inconsistent with equilibrium. Therefore, in this model the informed investor has a moving target that he has to push the option price to and it is not a priori obvious whether an optimal trading strategy and therefore an equilibrium exists.

It is shown that there exists an equilibrium under these circumstances and the optimal trading strategy of the investor is derived explicitly as part of the equilibrium. I derive expressions for the equilibrium option price, the price pressure parameter in the option market and the expected average variance (until time to expiration of the option) of the underlying asset on the information set of the uninformed traders at a given point in time. The option price is a non-linear function of trading volume in the

option, variance of future variance and the variables that appear in the Black-Scholes formula. Interestingly enough, it is found that the price pressure in the option market (a measure that is inversely related to the depth of the option market) increases in the variance of future variance of the underlying asset. Therefore this model captures the important phenomenon that uncertainty about the volatility process lowers the depth of the option market and this result is consistent with the empirical findings of Jameson and Wilhelm (92) and Nandi (95) who find that there is a positive relationship between bid-ask spreads in the equity options market and a measure of uncertainty about the variance of the underlying asset. Also the model predicts that the expected average variance of the underlying asset (until time to expiration of the option) on the information set of the uninformed traders is an increasing function of the net order flow in the option market. This suggests that in the presence of information asymmetry about volatility, options trading volume may contain useful information about the evolution of the expected volatility process and there is certainly scope for empirical work along these lines that can yield meaningful insights on the interaction between the expected volatility process of the underlying asset and trading volume in the option market.

This paper is organized as follows. Section 2 sets up the formal model. Section 3 describes the equilibrium pricing rule and the optimal trading strategy as well as the Bellman equation (arising from the optimization problem of the informed investor) that yields the necessary and sufficient conditions for optimality of a trading strategy. Section 4 discusses the depth of the option market and the expected variance process of the underlying asset in our equilibrium. Section 5 concludes.

2 The Model

The price process of the underlying asset, S_t, evolves exogenously as a geometric Brownian motion represented by

$$\frac{dS_t}{S_t} = v\ dW_t \tag{1}$$

where W_t is a standard Wiener process and v is the instantaneous standard deviation of the asset returns. The absence of a drift term in (1) will be explained in the next paragraph. Also traded continuously is a risk-free asset on which the rate of return is r_f. Without loss of generality, r_f is normalized to zero and assume that the underlying asset pays no dividends.

At time 1, there will be a jump in the volatility of the price process such that the asset price process from time 1 onwards will be given by

$$\frac{dS_t}{S_t} = \tilde{\sigma}\ dW_t \tag{2}$$

A call option is traded simultaneously with the underlying asset and the option (with an exercise price of K) will expire at time 2. A risk-neutral investor has information about $\tilde{\sigma}$ at time 0 i.e., the investor knows that $\tilde{\sigma} = \sigma$.[1] The rest of the market treats $\tilde{\sigma}$ to be a random variable and knows that there is an investor who is informed about the future volatility. Since the risk-free rate has been normalized to zero and the informed investor is risk-neutral, it is required for absence of arbitrage that there be no drift term in (1), i.e., that the instantaneous expected return on the underlying asset be zero. Let $\tilde{\sigma}$ be lognormally distributed with distribution function G. Assuming that $\tilde{\sigma}$ is lognormal implies that there exists a normal random variable ξ such that $\tilde{\sigma} = \exp(\xi)$. Let $\bar{\xi}$ and σ_ξ denote the mean and standard deviation of ξ. Assume that $\tilde{\sigma}$ is independent of W (the Wiener process that drives the asset price) as otherwise, the investor's information about $\tilde{\sigma}$ will provide him with some information about the

level of the asset price, S_t, and the insider will start trading both the underlying asset and the option, a setup that is beyond the scope of this model. Although Back (93) solves for an equilibrium in which an informed investor trades in both the underlying asset and the option, the insider knows perfectly the liquidation value of the asset and there is no information asymmetry regarding the volatility of the underlying asset.

Since $\tilde{\sigma}$ and W are independent, the informed investor does not enjoy any informational advantage over the rest of the market about the level of the future asset price/mean. Therefore it is reasonable to assume that the informed investor, who is risk-neutral will not trade the underlying asset. If the investor was risk-averse averse, he might hedge the risk that comes from the unknown future value of the underlying asset by constructing a delta hedge portfolio (shorting $\frac{1}{delta}$ units of the call and buying 1 unit of the underlying asset) that is instantaneously risk-free and requires trading the underlying asset. However, risk-neutrality implies that the risk of the underlying asset does not matter to the investor. The equilibrium that I will construct is based on the assumption that the risk-neutral investor (who has information about volatility) will not trade the underlying asset and the asset price process exogenously evolves as in (1). It is possible that there are other equilibria in which the risk-neutral investor trades the underlying asset on the basis of his information about volatility. It is conjectured that such equilibria do not exist, but I cannot formally prove it within the scope of this paper.

Prices in the option market are set by risk-neutral competitive market makers (making zero expected profits). The model does not take into account the inventory risk faced by the market makers in the option market. However, in most major options exchanges, for example the CBOE, where there are multiple market makers, the inventory risk can be substantially reduced through pooling. In this model, the market maker (making zero expected profits) essentially provides liquidity service to the market and it is well known that market makers in exchanges such as the CBOE trade on their own account and employ a broad array of trading strategies to

hedge and speculate. However, it must be noted that according to the rules of an important options exchange like the CBOE, the primary task of the market maker is to provide liquidity to the market by making available bid-ask quotes that contribute towards maintaining an orderly market with no abrupt price discontinuities [Cox and Rubinstein (85) and George and Longstaff (93)] and thus our assumption about the market maker may not be unrealistic. In the rest of the paper, it will be assumed that there is a representative market maker who sets prices in the option market.

Also, there are noise/liquidity traders in the option market. The cumulative noise trades in the option market at time t are represented by Z_t, a Brownian motion process, with instantaneous variance σ_z^2. Z is independent of W and $\tilde{\sigma}$ and the reason for this assumption will be explained later in this section. The demands of these noise traders are exogenous to the model and completely inelastic with respect to the prices set in the option market. Also, all the orders (i.e., those of the investor and the noise traders) are market orders and prices are set by the market maker upon observing the order flows. Although there is only one option in this model, one could have multiple options with different exercise prices and times to maturity and the results will remain unchanged as long as the informed investor does not have any wealth constraints.

Let X_t be the cumulative trades of the informed trader in the option market, $Y_t(= X_t + Z_t)$ be the total cumulative trades in the option market, $Y_0 = 0$ and $X_0 = 0$. The information set of the investor, generated by $\tilde{\sigma}$, Z_t, and S_t will be denoted as $\mathcal{F}_t^I$ and the information set of the market maker, generated by Y_t and S_t will denoted as $\mathcal{F}_t^M$. Both filtrations/information sets are augmented in the usual sense. Assume that X is a continuous semimartingale [Karatzas and Shreve (88), Definition 3.3.1] and that the order strategy X_t of the investor in the option market is given by $dX_t = \alpha_t dt$ for some process α_t adapted to $\mathcal{F}_t^I$. Although it is assumed here that X_t is of bounded variation and continuous, it will be shown to be necessary in the equilibrium that will be constructed.

At time 1 the volatility jumps to σ and thereafter the value of σ is common knowledge. In other words, we assume that upon the realization of the event (that changes the volatility) at time 1, the new volatility becomes known to everyone and uncertainty about the volatility process is resolved fully. Thus, starting from time 1, when the information asymmetry about volatility ceases to exist, option prices will be set according to the Black-Scholes formula by the risk-neutral market maker. Prior to time 1, the future volatility is unknown and the order flows provide information to the market maker about σ, the future volatility. Therefore, during this time interval, the option price will be a function of the asset price and the order flow. Let $BS(S, \sigma)$ represent the Black-Scholes function for call options with the asset price equal to S, the exercise price equal to K, standard deviation of instantaneous asset returns equal to σ, time to maturity equal to 2 and the risk-free rate equal to zero.

Let the price of the option at time t be denoted by C_t and the pricing rule in the option market be given by $C_t = H(S_t, Y_t, t)$, where H is twice continuously differentiable in S and Y, continuous in t and strictly monotone in Y and S for each t. Since the pricing rule is monotone, the investor can infer the past liquidity trades and therefore as in Back (92), knows Z_t at the time he chooses α_t. If Z_t is correlated either with $\tilde{\sigma}$ (about which the investor has information) or with W, then the investor will know something about S (the level of the asset price) and may start trading the underlying asset, a scenario that has been mentioned before to be outside the scope of this model. Let $h(S_1, Y_1) \equiv H(S_1, Y_1, 1)$ and define $h(S_1, Y_1) = BS(S_1, \tilde{\sigma})$. Note that this is just a definition of the function $h()$ and does not imply that the values of $h(S_1, Y_1)$ and $BS(S_1, \tilde{\sigma})$ are equal for all possible realizations of S_1, Y_1 and $\tilde{\sigma}$.

The wealth dynamics of the investor are defined by

$$d\mathcal{W}_t = X_t dC_t \tag{3}$$

Without loss of generality, assume that $\mathcal{W}_0$ =0. Since the informational advantage of the investor lasts till time 1, it is assumed that the investor will trade until time

1.[2] The investor could trade after time 1 also (the option expires at time 2), but would be indifferent to trading after time 1 because the volatility of the asset price and hence the true option price is common knowledge. Therefore assume that the investor would not trade after time 1. Given this and the fact that the trading strategy of the investor is of bounded variation, it can be shown as in Back (92) that the wealth of the investor at time 1 is given by

$$\mathcal{W}_1 = \int_0^1 (BS(S_1, \tilde{\sigma}) - H(S_t, Y_t, t))dX_t \tag{4}$$

The above equation basically says that the wealth of the informed investor at time 1 is the market value of his option position $[BS(S_1, \tilde{\sigma})X_1]$ minus the cost incurred in acquiring it $[\int_0^1 H(S_t, Y_t, t)dX_t]$.

An equilibrium in the option market consists of a rational pricing rule of the market maker and an optimal trading strategy of the informed investor. These concepts are defined as follows:

Given a trading strategy X, a pricing rule is rational if it satisfies,

$$H(S_t, Y_t, t) = E\left[BS(S_1, \tilde{\sigma})|(S_u)_{u\leq t}, (Y_u)_{u\leq t}\right] \tag{5}$$

Given a pricing rule H, a trading strategy is optimal if it maximizes the expected end of period wealth $\mathcal{W}_1$ as given by

$$E\left[\int_0^1 \left(E(BS(S_1, \tilde{\sigma})|\mathcal{F}_t^I) - H(S_t, Y_t, t)\right) dX_t\right] \tag{6}$$

The above equation follows from (4) by using iterated expectations. Henceforth, an optimal trading strategy of the informed investor will refer to the trading strategy X that maximizes $\mathcal{W}_1$ given the the pricing rule H and the assumption that the investor does not trade the underlying asset.

3 Equilibrium

In this section, the main theorem that characterizes the equilibrium in this model is provided. Also I describe the Bellman equation associated with the informed trader's optimization problem. In this section and for the rest of the paper, if D denotes a stochastic process then the realization of D_t will be denoted by d_t.

Let $N()$ denote the normal $(0, \sigma_z^2)$ distribution function. Let $f_1(a_1, a_2, t)$ and $f_2(b_1, b_2, t)$ be two functions defined as follows:

$$
\begin{aligned}
f_1(a_1, a_2, t) &= \left(\frac{1}{a_1}\right) \frac{1}{\sqrt{2\pi v^2(1-t)}} \exp\left[-\frac{\left\{\log(a_1) - \log(a_2) - \frac{1}{2}v^2(1-t)\right\}^2}{2v^2(1-t)}\right] \\
f_2(b_1, b_2, t) &= \left(\frac{\sigma_z}{\sigma_\xi b_1}\right) \frac{1}{\sqrt{2\pi\sigma_z^2(1-t)}} \exp\left[-\frac{\left\{\left(\log(b_1) - \bar{\xi}\right)\frac{\sigma_z}{\sigma_\xi} - b_2\right\}^2}{2\sigma_z^2(1-t)}\right]
\end{aligned}
$$

Lemma 1 *Let $\phi = N^{-1} \circ G(\tilde{\sigma})$. Then ϕ is distributed normal $(0, \sigma_z^2)$ and*

$$\phi = (\log(\tilde{\sigma}) - \bar{\xi})\frac{\sigma_z}{\sigma_\xi} \tag{7}$$

Theorem 1 *Define for $0 \leq t \leq 1$*

$$H(s, y, t) = \int_0^\infty \int_0^\infty BS(s_1, \sigma) f_1(s_1, s, t) f_2(\sigma, y, t) ds_1 d\sigma \tag{8}$$

With ϕ as in Lemma 1, define

$$X_t = t\phi + (1-t)\int_0^t -\frac{Z_s}{(1-s)^2} ds \tag{9}$$

Then (H, X) is an equilibrium.

The proof of Lemma 1 is given later in this section and that of Theorem 1 is in the

appendix. It will be shown that in this equilibrium $Y_1 = \phi$ $a.s.$ Also $f_1()$ and $f_2()$ are probability density functions and I will explain later in this section what they exactly represent. Note that s_1 (that represents the realization of the asset price at time 1)and σ are integrated out of (8) making the right hand side a function of the asset price at time t, cumulative option order flow at t and t itself. One can numerically integrate (8) to arrive at an explicit option price given the asset price and the cumulative option order flow at time t as well as the other parameters like t, σ_z, σ_ξ, $\bar{\xi}$ and v. The formulas (8) and (9) depend on the assumption that $\tilde{\sigma}$ is lognormal. If $\tilde{\sigma}$ is any random variable with a distribution function L satisfying the following assumptions - (i) Support of L is an interval in $\Re^+$, (ii) L is continuous on this interval, and (iii) L does not have a mass point (i.e., not degenerate) then the equilibrium trading strategy (9) will be the same except $\phi = N^{-1} \circ L$ (see the appendix) and pricing rule will be the same as (8) except that $f_2(\sigma, y, t)$ will be replaced by a different density function.

As mentioned before, starting from time 1, the information asymmetry about $\tilde{\sigma}$ disappears and it is common knowledge that $\tilde{\sigma} = \sigma$. Thus from time 1, option prices as set by the risk-neutral market maker are given directly by the Black-Scholes function. Before time 1, the value of $\tilde{\sigma}$ is not known to the market maker. However, he can make inferences about $\tilde{\sigma}$ by observing the order flow Y_t that contains information about $\tilde{\sigma}$ because $Y_t = X_t + Z_t$, and X_t, the cumulative order flow of the informed trader at time t incorporates the information of the informed investor about $\tilde{\sigma}$. In other words, the market maker, at time t observes S_t and Y_t and on the basis of these observations, arrives at the best estimate of the option price by taking the conditional expectation of $BS(S_1, \tilde{\sigma})$ (recall that $BS(S_1, \tilde{\sigma})$ is the option price at time 1 given by the Black-Scholes function) with respect to the joint distribution of S_1 and $\tilde{\sigma}$ on his information set at time t. Since S and $\tilde{\sigma}$ are independent, the joint density is the product of the marginal densities of S_1 and $\tilde{\sigma}$ on $\mathcal{F}_t^M$ in this equilibrium and these marginal densities are represented by $f_1(s_1, s, t)$ and $f_2(\sigma, y, t)$ respectively. Following

are some remarks on this equilibrium:

Remark 1: The optimal trading strategy of the investor does not depend on the contemporaneous asset price or any function of it. This is intuitive because the investor does not have any information about the asset price.

Remark 2: The optimal trading strategy of the investor ensures that $H(S_1, Y_1, 1) = BS(S_1, \tilde{\sigma})$ $a.s$ $\forall$ s_1 and σ i.e., there is no jump in the option price immediately following the resolution of information asymmetry. A jump implies that the option was mispriced at the end of trading and the investor did not take advantage of the mispricing which is incompatible with equilibrium. Note that the jump does not occur despite the fact that the informed investor has no information about the future asset price.

Remark 3: The probability distribution of the future volatility i.e., $\tilde{\sigma}$ on the information set of the market maker is determined endogenously in this equilibrium and the information that determines the distribution is filtered through the order flow in the option market. This is different from stochastic volatility models (with no information asymmetry about the future volatility) of option pricing where the distribution of future volatility is exogenous. Thus unlike other stochastic volatility models, this model yields interactions between options trading activity and the expected volatility process of the underlying asset.

Remark 4: It is not claimed here that this particular equilibrium is unique. However, following Back (92) it can be shown that this equilibrium is unique among the class of equilibria in which the option price is a strictly monotone function of the underlying asset price and the total cumulative option order flow and the informed investor does not trade the underlying asset.

Proof of Lemma 1 : I will first prove the lemma for any $\tilde{\sigma}$ whose distribution function L satisfies the assumptions mentioned before. In other words it will be shown that if $\phi = N^{-1} \circ L(\tilde{\sigma})$, then ϕ is distributed normal $(0, \sigma_z^2)$. If $\tilde{\sigma}$ is lognormal, then (7) will follow automatically.

Let us denote $C()$ to be the cumulative distribution function of $\tilde{m} \equiv L(\tilde{\sigma})$. Now,

$$\begin{aligned} C(a) &= \text{Prob}(\tilde{m} \leq a) \\ &= \text{Prob}(L(\tilde{\sigma}) \leq a) \\ &= \text{Prob}(\tilde{\sigma} \leq L^{-1}(a)) \\ &= L(L^{-1}(a)) \\ &= a \end{aligned}$$

This implies that $\tilde{m}$ is distributed uniformly on (0,1). Now $\phi = N^{-1} \circ L(\tilde{\sigma})$. Therefore if R denotes the distribution function of ϕ, we have that

$$\begin{aligned} R(a) &= \text{Prob}(\phi \leq a) \\ &= \text{Prob}(N^{-1}(\tilde{m}) \leq a) \\ &= \text{Prob}(\tilde{m} \leq N(a)) \\ &= N(a) \end{aligned}$$

The last equality follows from the fact that $\tilde{m}$ is uniformly distributed on (0,1). This implies that ϕ i.e., $N^{-1} \circ L(\tilde{\sigma})$ is distributed normal $(0, \sigma_z^2)$ given that $\tilde{\sigma}$ is a random variable whose distribution function L satisfies the assumptions mentioned before. For the lognormal case, denoting the distribution of a standard normal random variable by N^*,

$$\begin{aligned} \phi &= N^{-1} \circ G(\sigma) \\ or,\ G(\tilde{\sigma}) &= N(\phi) \\ or,\ N^*\left(\frac{\log(\tilde{\sigma}) - \bar{\xi}}{\sigma_\xi}\right) &= N^*\left(\frac{\phi}{\sigma_z}\right) \end{aligned}$$

From the last of the above equations,

$$\phi = (\log(\tilde{\sigma}) - \bar{\xi})\frac{\sigma_z}{\sigma_\xi} \quad a.s. \tag{10}$$

Note that the relation between ϕ (that will be shown to be equal to Y_1 in our equilibrium) and $\tilde{\sigma}$ was derived independently of the level of the asset price S_t.

3.1 Bellman Equation

This section describes the Bellman equation associated with the optimization problem of the informed investor. The Bellman equation yields the necessary and sufficient conditions for a trading strategy of the informed investor to be optimal.

Let $P_t = \log(S_t)$. It follows from Ito's lemma that $dP_t = vdW_t - \frac{1}{2}v^2dt$. For the rest of this sub-section, I will use P_t (instead of S_t) as an argument in the option pricing rule, $H()$ and the Black-Scholes function, $BS()$. Set $Q(p,c) = h^{-1}(p,.)(c)$ i.e., $Q(p,c) = y$ iff $h(p,y) = c$.

As mentioned before, the trading strategy of the informed investor is given by $dX_t = \alpha_t dt$, for some α_t adapted to $\mathcal{F}_t^I$. Given an option pricing rule $H(P_t, Y_t, t)$, the informed investor solves

$$\max_{\alpha_t} E\left[\int_0^1 \left(E(BS(P_1,\tilde{\sigma})|\mathcal{F}_t^I) - H(P_t, Y_t, t)\right)\alpha_t dt\right] \tag{11}$$

Let $J(P_t, Y_t, t)$ denote the value function associated with the optimization problem of the informed investor as mentioned above (the value function at time t yields the expected profits to the investor if an optimal trading strategy is followed from time t until time 1). J is non-negative and satisfies the following Bellman equation and the associated boundary condition:

$$\max_{\alpha_t}\Big[J_t + \frac{1}{2}\sigma_z^2 J_{yy} + \frac{1}{2}v^2 J_{pp} + \alpha_t J_y - \frac{1}{2}v^2 J_p$$
$$+\Big(E(BS(P_1,\tilde{\sigma})|\mathcal{F}_t^I) - H(P_t,Y_t,t)\Big)\alpha_t\Big] = 0 \tag{12}$$

$$J(p,y,1) > J(p,Q(p,BS(p,\sigma)),1) = 0\ (\forall y \neq Q(p,BS(p,\sigma))) \tag{13}$$

The boundary condition (13) can be interpreted as saying that J being continuous at time 1, the remaining value $J(p,y,t)$ at times t near 1 is close to zero if and only if y is close to $Q(p,BS(p,\sigma))$. In other words, there is an optimal value of the order flow at time 1 that is necessary for the optimization problem of the informed investor.

Performing the maximization in (12) yields the following two equations

$$J_y = H(P_t,Y_t,t) - E(BS(P_1,\tilde{\sigma})|\mathcal{F}_t^I) \tag{14}$$

$$J_t + \frac{1}{2}\sigma_z^2 J_{yy} + \frac{1}{2}v^2 J_{pp} - \frac{1}{2}v^2 J_p = 0 \tag{15}$$

If J is non-negative and smooth and satisfies (15) then,

$$J(p,y,t) = E\left[J(p+P_u-P_t, y+Z_u-Z_t, u)\right]\ (\forall 0 < t < u \leq 1) \tag{16}$$

where the expectation is taken with respect to P and Z [Karatzas & Shreve (88), p. 366, Theorem 7.6].

The following lemma is analogous to Lemma 1 of Back (92). It calculates the value that can be attained by waiting until the "last instant"and trading "immediately" such that $h(P_1,Y_1) = BS(P_1,\tilde{\sigma})$ *a.s.* and also provides a solution to the Bellman equation.

Lemma 2 *Let h be a function that is strictly monotone in its arguments and $j(.,.) \equiv J(.,.,1)$. Let the option pricing rule be*

$$H(p,y,t) = E\left[h(p+P_1-P_t, y+Z_1-Z_t)\right] \tag{17}$$

Define

$$j(p,y) = \int_y^{Q(p,BS(p,\sigma))} (BS(p,\sigma) - h(b,x))dx \tag{18}$$

and

$$J(p,y,t) = E\left[j(p+P_1-P_t, y+Z_1-Z_t)\right] \tag{19}$$

where the expectation is being taken with respect to P and Z. Then J is a smooth solution of (12) *and* (13).

The Bellman equation and the associated value function are instrumental in proving the optimality of the trading strategy as proposed in Theorem 1. It can be shown as in Lemma 2 of Back (92) that any solution of the Bellman equation and the boundary condition is also the value function for the optimization problem of the informed investor. Also it can be shown following Lemma 2 of Back (92) that an optimal trading strategy of the informed investor implies that the trading strategy is continuous (i.e., no discrete orders), of bounded variation (i.e., no martingale component), and that there will not be any jump in the option price following the resolution of information asymmetry about volatility at time 1. Each of these conditions is individually necessary and they are jointly sufficient for a trading strategy to be optimal. The proofs of these are straightforward extensions of Lemma 2 of Back (92) and are therefore omitted. The following proof of Lemma 2 makes use of the fact that the distributions of Y_1 on $\mathcal{F}_t^M$ and Z_1 on $\mathcal{F}_t^I$ are equivalent (see the appendix).

Proof of Lemma 2: It is clearly the case that $J(.,.,1) = j(.,.)$ is continuous, ≥ 0 and satisfies the boundary condition (13). Also the function J satisfies (15) by [Karatzas & Shreve (88), p. 366)]. Thus it has to be made sure that (14) is satisfied for which one needs to differentiate under the expectation operator.[3] Differentiating,

$$\begin{aligned} J_y(p,y,t) &= E\left[j_y(p+P_1-P_t, y+Z_1-Z_t)\right] \\ &= E\left[h(p+P_1-P_t, y+Z_1-Z_t)\right] - E(BS(P_1,\tilde{\sigma})|\mathcal{F}_t^I) \\ &= H(p,y,t) - E(BS(P_1,\tilde{\sigma})|\mathcal{F}_t^I) \end{aligned}$$

4 Properties of the Equilibrium

4.1 Depth of the Option Market

The price pressure parameter (one of the primary determinants of market liquidity) in the option market is represented by the sensitivity of the option price to the order flow in the option market and is given by $\lambda_t \equiv H_y(S_t, Y_t, t)$. Depth of the market is given by $\frac{1}{\lambda}$. Given the pricing rule in (8), we differentiate under the integral sign to get[4]

$$\lambda_t = \int_0^\infty \int_0^\infty \frac{(\log(\sigma) - \bar{\xi})\frac{\sigma_z}{\sigma_\xi} - y_t}{\sigma_z^2(1-t)} BS(s_1,\sigma) f_1(s_1, s_t, t) f_2(\sigma, y_t, t) ds_1 d\sigma \tag{20}$$

Since (20) does not yield a closed-form solution for the price pressure parameter, I compute (by numerical integration) the price pressure parameter in the option market between time 0 and time 1 for various values of σ_ξ (note that the variance of $\tilde{\sigma}$ is directly proportional to σ_ξ). The results of this computation are displayed in Figure 1. In conformance with our intuition, it is seen that the price pressure parameter in the

option market is an increasing function of the variance of variance. In other words, an increase in the variance of variance, that in our model, measures the amount of private information about variance, lowers the liquidity of the option market. This is consistent with the empirical evidence of Jameson and Wilhelm (92) and Nandi (95) who find that bid–ask spreads in the equity options market is increasing in a measure of uncertainty about the variance of the underlying stock.

4.2 Expected Average Variance

Until time 1, the future volatility σ is unknown to the market maker and the order flows in the option market provide information about σ. Let ν_t denote the expected average variance of the underlying asset until time to expiration of the option on the information set (as of time t) of the market maker. It can be shown that

$$\begin{aligned} \nu_t &= \frac{1}{1-t}\int_t^1 v^2 du + E(\tilde{\sigma}^2 \mid \mathcal{F}_t^M) \\ &= v^2 + e^{2(\bar{\xi}+\frac{\sigma_\xi}{\sigma_z}y_t+\sigma_\xi^2(1-t))} \end{aligned} \tag{21}$$

The derivation of (21) is shown in the appendix.

Since y_t measures the net cumulative order flow in the option market at time t, (21) shows that in the presence of information asymmetry about volatility, *ceteris paribus*, the expected average variance of the underlying asset is increasing in the net order flow in the option market. The intuition behind this is that an increase in the value of σ causes the informed trader to buy more options and the market maker correctly anticipates it in equilibrium. It can be easily verified that ν_t is a martingale on the information set of the market maker ($\mathcal{F}_t^M$) i.e., $E(\nu_t|\mathcal{F}_s) = \nu_s \, \forall s \leq t$. Figure 2 shows how the ν_t process evolves for $t \in (0,1]$ for $\sigma_z = \sigma_\xi = 0.0005$, $v = 0.01$ and $\bar{\xi} = 0$.

Although this model is a simplification of the real world, it shows that in the presence of information asymmetry about volatility, options trading volume contains

useful information about the evolution of the expected volatility process of the underlying asset, an area that may be worth exploring empirically. It must be noted, however, that options trading volume in derivatives markets is also related to the amount of hedging demand (assumed exogenous in this model) that may depend on the magnitude of actual/implied volatility and volatility innovations.

5 Conclusion

I developed a model of asymmetric information in which a risk-neutral investor has private information about the future volatility of the asset price process, but does not know the future asset price at any point of time. As long as information asymmetry about the future volatility exists, the option price (set by a competitive risk neutral market maker) is a function of the options trading volume, variance of future variance and the variables that appear in the Black-Scholes formula. The model predicts that the option market becomes less liquid (in that the price pressure parameter increases) if the amount of private information about the future volatility increases, i.e., if the variance of variance increases. This is consistent with empirical results that the bid-ask spreads (a measure of liquidity) in the equity options market are positively related to a measure of the uncertainty about the variance. Thus private information about the volatility of the underlying asset lowers the depth/liquidity of the option market in the same way that private information about the mean of the asset value lowers the depth of the asset market in Kyle (85) and Glosten and Milgrom (85). Also, it is found that on the information set of the non-informed traders, the expected average variance (until time to expiration of the option) of the underlying asset is an increasing function of the net order flow in the option market, a result that has empirical implications in terms of the linkage between the expected volatility process of the underlying asset and trading volume in the option market.

Future research can be directed towards developing models in which the informed

investor is risk-averse and trades the underlying asset to hedge his position, a feature that will be very interesting to model and may yield valuable insights into the workings of the option market amidst information asymmetry about volatility. Also it will be desirable to endogenize liquidity trading and model hedging demands explicitly.

Appendix

This section provides a proof of Theorem 1 and derives the expression for the expectation of $\tilde{\sigma}$ on the information set of the market maker at time t i.e., (21). I will sometimes use $P_t(= \log(S_t))$ as an argument of the pricing rule of the market maker instead of S_t.

Proof of Theorem 1 : It was mentioned in section 3.1 that the optimal order strategy of the informed investor has to be of bounded variation (i.e., no martingale component), continuous and should not result in any jump in the option price at time 1.The trading strategy (9) in Theorem 1 is continuous and of bounded variation (of order dt). The remaining thing is to show that this trading strategy implies the last of the above condition, i.e., no jump.

If we add Z_t to (9) i.e., $Y_t = Z_t + X_t$, it follows that Y is a Brownian bridge (on the time interval [0,1], with instantaneous variance σ_z^2) on the information set of the informed investor ($\mathcal{F}_t^I$) that starts at 0 and ends at ϕ i.e., $N^{-1} \circ G(\tilde{\sigma})$ [Karatzas & Shreve (1988, p. 358)]. Therefore, given the trading strategy (9), $Y_1 = N^{-1} \circ G(\tilde{\sigma})$ $a.s.$ and equivalently $\tilde{\sigma} = G^{-1} \circ N(Y_1)$ $a.s.$ In order to check that the trading strategy (9) implies that there will not be any jump in the option price at time 1 given the pricing rule, we have to verify that $h(S_1, Y_1) = BS(S_1, \tilde{\sigma})$ $\forall$ s_1 and σ. Using the definition of function $h()$, (see section 2)

$$h(S_1, Y_1) = BS(S_1, G^{-1} \circ N(Y_1))$$

Therefore,

$$\begin{aligned} h(S_1, N^{-1} \circ G(\tilde{\sigma})) &= BS(S_1, G^{-1} \circ N \circ N^{-1} \circ G(\tilde{\sigma})) \\ or,\ h(S_1, N^{-1} \circ G(\tilde{\sigma})) &= BS(S_1, \tilde{\sigma}) \end{aligned}$$

It has already been shown that given the trading strategy (9) of the informed investor, $Y_1 = N^{-1} \circ G(\tilde{\sigma})$ $a.s.$, and this is true for any realization of S_1. Therefore it is always

true that given the trading strategy (9), $h(S_1, Y_1) = BS(S_1, \tilde{\sigma})$ $a.s.$ $\forall$ s_1, σ. This proves the optimality of (9) under the assumption that $\tilde{\sigma}$ is lognormal. For any other random variable whose distribution function L satisfies the assumptions mentioned in section 3, we can show following the steps above that $h(S_1, Y_1) = BS(S_1, \tilde{\sigma})$ $a.s.$ $\forall$ s_1, σ.

If a stochastic process is a Brownian bridge, then its finite dimensional distributions are identical to those of a Brownian motion, conditional on the ending point of the process being known [Karatzas and Shreve (1988, Problems 5.6.11 and 5.6.13)]. The ending point of the process is $N^{-1} \circ G(\tilde{\sigma})$, which is distributed normal $(0, \sigma_z^2)$. Therefore the unconditional (unconditional on S and $\tilde{\sigma}$) finite dimensional distributions of Y are those of a Brownian motion with instantaneous variance σ_z^2. This, in turn implies that on the information set of the market maker ($\mathcal{F}_t^M$), Y is a Brownian motion with instantaneous variance σ_z^2. Since Z is a Brownian motion on $\mathcal{F}_t^I$ with instantaneous variance σ_z^2, this implies the equivalence of the distribution of Z on $\mathcal{F}_t^I$ and the distribution of Y on $\mathcal{F}_t^M$.

Now we will prove the rationality of the pricing rule of the market maker given the trading strategy of the informed investor. Let $E^I[.]$ denote expectation with respect to the information set of the informed investor and $E^M[.]$ denote the expectation with respect to the information set of the market maker. Differentiating (16) with respect to Y and using (14) and (18) we get

$$H(P_t, Y_t, t) = E^I\left[h(p + P_1 - P_t, y + Z_1 - Z_t)\right] \tag{22}$$

This implies that

$$H(P_t, Y_t, t) = E^I\left[H(P_1, Z_1, 1) | P_t = p, Z_t = y\right] \tag{23}$$

It has been shown above that given the trading strategy of the informed investor, the distribution of Y on the information set of the market maker is the same as the

distribution of Z on the information set of the informed investor. Therefore,

$$H(P_t, Y_t, t) = E^M\left[H(P_1, Y_1, 1)|P_t = p, Y_t = y\right] \tag{24}$$

It has also been established that if the informed investor follows the trading strategy (9) then, $H(P_1, Y_1, 1) = BS(P_1, \tilde{\sigma})$ $a.s.$ Therefore,

$$H(P_t, Y_t, t) = E^M\left[BS(P_1, \tilde{\sigma})|P_t = p, Y_t = y\right] \tag{25}$$

or equivalently,

$$H(S_t, Y_t, t) = E^M\left[BS(S_1, \tilde{\sigma})|S_t = s, Y_t = y\right] \tag{26}$$

$$or,\ H(S_t, Y_t, t) = E^M\left[BS(S_1, \tilde{\sigma})|(S_u)_{u\leq t}, (Y_u)_{u\leq t}\right] \tag{27}$$

The last of the above equalities follows from the Markov properties of S and Y. The pricing rule in equation (8) is the conditional expectation as shown above that can be evaluated by integrating $BS(S_1, \tilde{\sigma})$ against the joint density of S_1 and $\tilde{\sigma}$ on $\mathcal{F}_t^M$, that by the independence of S_1 and $\tilde{\sigma}$ is the product of the marginal densities of S_1 and $\tilde{\sigma}$ on $\mathcal{F}_t^M$, represented by $f_1()$ and $f_2()$ respectively. Hence the rationality of the pricing rule (8) will be proved upon showing that $f_1()$ and $f_2()$ are the densities of S_1 and $\tilde{\sigma}$ respectively on the information set of the market maker (at time t) given the trading strategy of the informed investor.

Now, $dP_t = vdW_t - \frac{1}{2}v^2dt$. Therefore,

$$P_1 = P_t + v(W_1 - W_t) - \frac{1}{2}v^2(1-t) \tag{28}$$

This implies that the time t conditional distribution of P_1, given that $P_t = p$ is $N(p - \frac{1}{2}v^2(1-t), v^2(1-t))$. Since $S_1 = \exp(P_1)$, the expression for $f_1()$ follows.

It has been established before that an optimal trading strategy of the informed

investor implies that Y is a Brownian motion on the information set of the market maker. Therefore on the information set of the market maker at time t, given that $Y_t = y$, the distribution of Y_1 is $N(y, \sigma_z^2(1-t))$. Given the relation between Y_1 and $\tilde{\sigma}$ from Lemma 1 (recall that in equilibrium $\phi = Y_1$), the expression for $f_2()$ follows immediately from the distribution of Y_1 on $\mathcal{F}_t^M$. This completes the proof of Theorem 1.

The last part is to derive the expression for $E(\tilde{\sigma}^2|\mathcal{F}_t^M)$.

$$E(\tilde{\sigma}^2 \mid \mathcal{F}_t^M) = \int_0^\infty \sigma^2 f(\sigma|y,s)d\sigma \tag{29}$$

where $f(\sigma|y,s)$ is the conditional density of $\tilde{\sigma}$ at time t. As $\tilde{\sigma}$ and S are independent, the above integral is equivalent to

$$\int_0^\infty \sigma^2 f(\sigma|y)d\sigma \tag{30}$$

Since $f(\sigma|y)$ is the same as $f_2(\sigma, y, t)$, substituting Y_1 for $\tilde{\sigma}$ from (7) (recall that in our equilibrium $Y_1 = \phi$ *a.s.*), a straight calculation using the moment generating function of normal random variables shows that the result is $e^{2(\bar{\xi}+\frac{\sigma_\xi}{\sigma_z}y+\sigma_\xi^2(1-t))}$.

Footnotes

1. The informed investor could also know σ with some error. However, as long as the investor is risk neutral, the results remain essentially unchanged with no additional insights. The formulae become more cumbersome since one has to integrate out the error term.

2. The investor could trade after time 1 (the option expires at time 2), but would be indifferent to trading after time 1 because the volatility of the stock price and hence the true option price is common knowledge. Therefore it is assumed that the investor would not trade after time 1.

3. Back (92) shows that differentiation under the expectation operator is allowed in a similar case.

4. Differentiation under the integral sign is allowed in this case because the integral is a Riemann integral and the integrands as well as their derivatives are continuous [Trench (78), p. 576]

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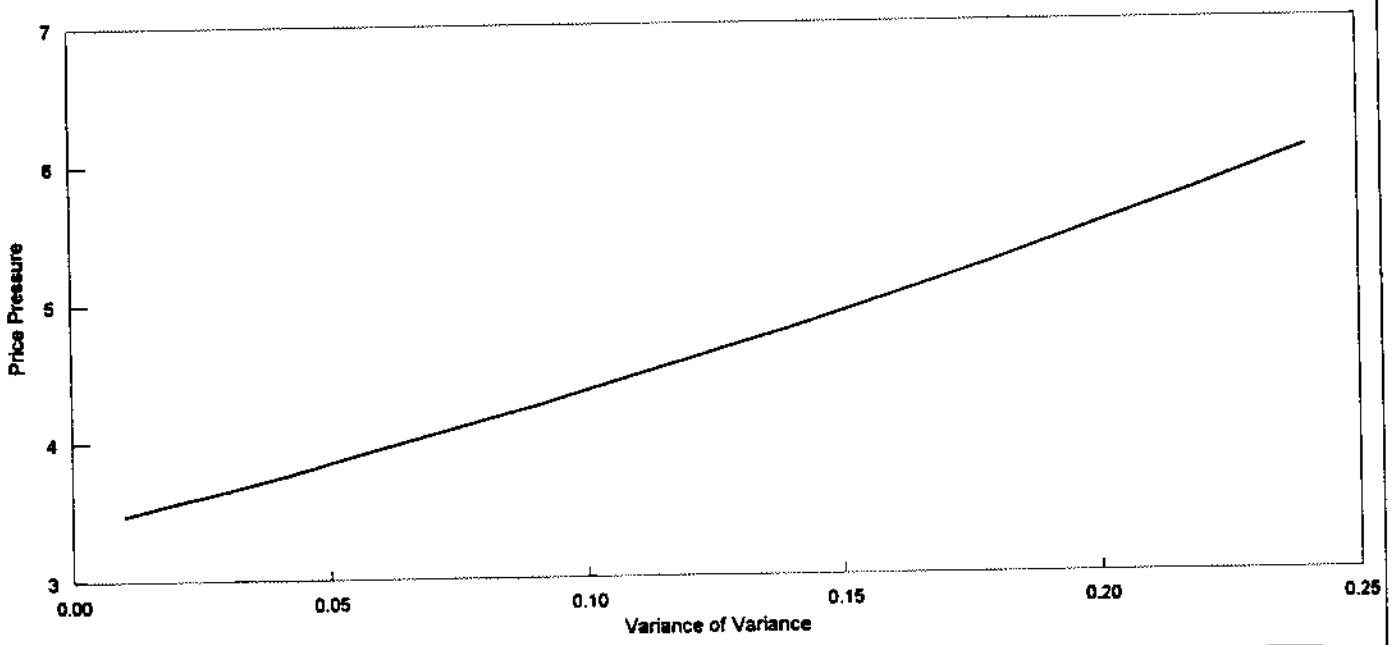
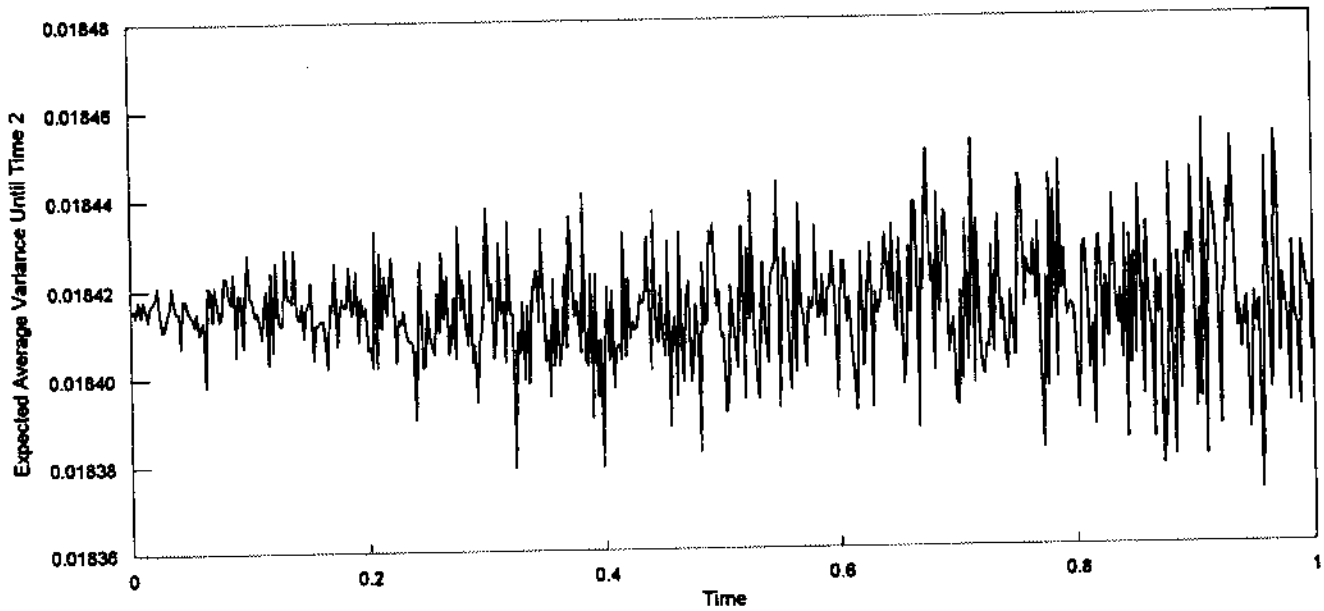
Figure 1



The above figure shows that price pressure in the option market is increasing in the variance of variance.

Figure 2



The above figure shows how the expected average variance process (until time to expiration of the option) evolves on the information set of the market maker for t in (0,1].