Specifying a Consistent Joint Maximum-Likelihood (JMLE) Approach to Testing Bond Models

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Abstract: In this paper we extend the results derived in our earlier work to develop a methodology to employ the maximum-likelihood estimation technique for the pricing of interest rate instruments. In order to price bonds and their derivative assets, researchers must identify a preference parameter in addition to the dynamics for the interest rate process. There are two approaches to obtaining estimators for both preference and dynamics parameters: (1) a two-stage approach and (2) a single-stage joint maximum-likelihood (JMLE) approach. The first approach, while tractable, suffers from serious drawbacks, primarily those relating to the use of the estimates from the first stage in estimating parameters in the second stage. In this paper, we develop the theory necessary for joint maximum-likelihood (JMLE) over the set of bond prices and the interest rate. We operationalize the theory by assuming that the error processes for all coupon bonds are mutually independent and uniformly distributed with a mean of zero. This specification is at least partially justifiable by the observation that since market prices are quoted in 1/32 of a dollar, theoretical prices must always be rounded either up or down. JML estimators can be obtained from the joint log-likelihood function by the methods of sequential quadratic programming.

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I. Introduction

The study of interest rates and the valuation of bonds and related derivative instruments form a significant part of current investment research. Academic literature is replete with proposed bond-pricing models based on a single interest-rate factor evolving according to a simple stochastic differential equation (SDE). Although the proposed models all admit closed-form mathematical solutions, few meet the following simple consistency requirements for economic reasonableness:

a) the underlying interest rate (factor) should be observable (nominal) and hence nonnegative;

b) if the underlying rate reaches zero, it should bounce back to positive values (0 should be a reflective barrier);

c) the underlying interest rate process should possess a steady-state distribution with bounded moments (in other words, rates should not wander inexorably to either zero or infinity in the long run); and

d) the model should have general-equilibrium underpinnings.

Technical factors (a) - (c) can be insured by selecting an interest-rate process whose volatility increases with the level of the rate and whose drift exhibits mean-reversion.

Cox, Ingersoll, and Ross' (CIR) [5] one-factor, three-parameter model based on a mean-reverting, square-root (MRSR) interest-rate process is one of the few models to meet the foregoing consistency requirements. Unfortunately, practitioners have been unable to implement even the CIR model since empirical researchers have reported disappointing results attempting to estimate required pricing parameters.

Empirical investigators have spent over a decade trying to estimate parameters for the underlying interest-rate process and pricing preference parameter. In general, results have been miserable: standard errors of estimation have been statistically insignificant to even the wildest enthusiasts. For example, Chan, Karolyi, Longstaff, and Sanders [4] used the generalized method of moments (GMM) to estimate parameters for a nested set of SDEs describing the interest rate in an attempt to improve disappointing results reported by Marsh and Rosenfeld [14] using maximum-likelihood estimation (MLE). Their research reinforced the disappointments of earlier investigations by Brennan and Schwartz [2] and Dietrich-Campbell and Schwartz [7] who used least-squares approaches.

Estimation failures using nominal interest rates led some investigators to the radical assumption that the underlying interest-rate process is unobservable. For example, Brown and Dybvig [3], Gibbons and Ramaswamy [9], or Heston [11] all attempted to estimate parameters for a popular version of the CIR model using either GMM or nonlinear regression techniques. Once again, the efforts failed.

Partially in reaction to the failure of practitionersto accurately estimate drift-related parameters in the CIR general-equilibrium model, Ho and Lee (HL) [12] and Heath, Jarrow, Morton (HJM) [10] proposed an alternative theoretical method for contingent-claim pricing called AR (arbitrage-free rate movements).

Although inapplicable to pricing the underlying bonds (whose prices must be exogenously specified), AR models purportedly eliminate the drift from the pricing equations for contingent claims by using forward rates. Unfortunately, the drift-related parameters are merely transferred in a highly nonlinear fashion into the volatility function for the forward rate which is then intractable for estimation.

In a companion paper [16], we demonstrate that many prior estimation attempts for mean-reverting processes may have failed due to unearthed computational inefficiencies in canned computer packages. By paying meticulous attention to computational detail, we were able to produce algorithms which provided statistically significant parameter estimates for **daily** interest-rate series using a norm-preserving MRSR transition density and for FX rates using both norm-preserving MRS and MRL transition functions. These MML estimators are sufficient to make simple forecasts and to price FX options and futures.

However, in order to price bonds and their derivative assets, researchers must identify a preference parameter in addition to the dynamics for the interest-rate process. At first blush, there are two approaches to obtaining estimators for both preference and dynamics parameters: 1) a two-stage approach, and 2) a single-stage joint maximum-likelihood (JMLE) approach.

The simpler two-stage approach would proceed by first calculating marginal ML estimators for interest-rate dynamics and then finding a value of the preference parameter to minimize the error sum of squares on the set of Treasury Notes over the same time period conditional on the MML estimators. This approach may have a significant flaw. In Lemma 1 of this paper, we review in detail general asset pricing theory for MRSR dynamics. The theory implies that investors price bonds *as if* their observations of the interest rate are drawings from a *risk-adjusted* density rather than as drawings from the exogenous dynamics. In other words, the risk-adjusted dynamic parameters are in a sense conditional on the preference parameter and may differ from true exogenous parameters. Hence it may be more expedient to **jointly** estimate preferences and dynamics.

In this paper, we develop the theory necessary for joint maximum likelihood (JMLE) over the set of bond prices and the interest rate. We operationalize the theory by assuming that the error processes for all coupon bonds are mutually independent and uniformly distributed with a mean of zero. This specification is at least partially justifiable by the observation that since market prices are quoted in \$1/32 of a dollar, theoretical prices must always be rounded either up or down. Other choices (like a truncated normal distribution) might be equally tenable for the error distribution. JML estimators can be obtained from the joint loglikelihood function by the methods of sequential quadratic programming. Once again, great care must be taken to use analytical derivatives and properly handle the infinite series components of the problem. We plan to report empirical results in a future paper.

II. A Review of the MRSR Pricing Model for Coupon Bonds

A popular single-factor model to describe *nominal* bond prices evolves from standard no-arbitrage arguments in an economy characterized by the following four assumptions.

A1. The *nominal* instantaneous rate r(t) is a *mean-reverting, square root (MRSR)* diffusion process which follows the autonomous stochastic differential equation:

$$(1) \qquad \begin{aligned} dr(t) &= \kappa[\theta - r(t)]dt + \sigma\sqrt{r(t)}\; dZ(t) \\ &\equiv \mu(r,t)dt + \sigma(r,t)dZ(t) \end{aligned}$$

where Z is a standard Brownian motion and $\theta \geq 0$ represents a "long-run mean." The spot rate is elastically pulled toward (pushed away from) θ if the "speed of adjustment" κ is positive (negative). The process is autonomous because the drift and diffusion parameters have no functional dependence on time.

A second major assumption insures that the interest-rate dynamics preclude either negative values or absorption at the origin (so that if the interest rate ever reaches a level r(t) = 0, it will subsequently become positive again):

A2. The parameters of the transition density implied by (1) either satisfy the inequalities

(2a) $\kappa, \theta > 0$ and $2\kappa\theta \geq \sigma^2$

so that the origin is unattainable (zero is an entrance boundary) ; or they satisfy the inequalities

(2b) $\kappa > 0$ and $0 < 2\kappa\theta \leq \sigma^2$

so the origin (i.e. zero) is attainable but *arbitrarily defined as a reflecting barrier.* In either case, random shocks cannot dominate the process' mean-reverting characteristics.[1]

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1 Ramamurtie, Prezas and Ulman [15], show that the barrier could also be arbitrarily described as an absorbing boundary for parameter values consistent with (2b). However, reflecting and absorbing barrier assumptions result in two entirely different density functions: the reflecting-barrier solution yields a proper distribution function while the absorbing-barrier solution results in a defective distribution. If the interest rate were absorbed at the origin, it could not subsequently become positive; i.e., it would remain forever at zero.

The third and fourth assumptions specify the single-factor structure of nominal bond prices:

A3. The nominal instantaneous interest rate is the only state variable necessary to describe the uncertainty affecting riskless government bond prices.

A4. To obtain a particularly tractable and testable model for nominal bond prices, assume that the *factor premium* for nominal short interest rate risk (as described in Cox, Ingersoll, and Ross [5, 6]) has the simple functional form

(3) $\phi(r,\lambda,t) = \lambda r(t)$

or that the *market price of risk* for the instantaneous interest rate process evolves as

(4) $\beta(r,\lambda,t) = -\lambda r(t)^{1/2} / \sigma$.

Given assumptions A1-A4, the prices of nominal pure discount bonds must satisfy the partial differential equation[2]

(5) $$\tfrac{1}{2}\sigma^2 r P_{rr} + \kappa(\theta - r)P_r + P_t - \lambda r P_r - rP = 0$$

subject to the terminal condition $P(r,\lambda,T,T) = 1$.

Cox, Ingersoll, and Ross [5] solved (5) to obtain an expression for the theoretical price of a pure discount bond maturing at T:

(6a) $$P(r,\lambda,\kappa,\theta,\sigma,t,T) = A(\lambda,\kappa,\theta,\sigma,t,T)e^{-B(\lambda,\kappa,\sigma,t,T)r(t)}$$

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2 (5) follows from a well-known "no arbitrage" argument specifying the existence of a function $\beta(r,\lambda,t)$, independent of bond maturity, such that the excess expected return on every bond may be reduced to zero according to Girsanov's Theorem. Accordingly, the *market price of risk* represents a transformation on the *factor premium* such that $\beta(r,\lambda,t)\sigma(r,t) = -\phi(\lambda,r,t)$. Technically, the argument utilizes a transformation of measure using the function $\beta(r,\lambda,t)$ to identify the Radon-Nikodym derivative. For a statement of Girsanov's Theorem, see Friedman [8]. Cox, Ingersoll, and Ross [5] use a similar argument to prove Theorem 2 in their paper. The technique is sometimes known as a "transformation to martingale measure." Assumptions A1-A4 result in a partial differential equation which is identical to the general equilibrium results of Cox, Ingersoll, and Ross [6]. However, the CIR results are in real terms, whereas here the assumed prices and interest rate are in nominal terms. However, if the CIR production processes are restated with nominal output, there will exist a transformation on the state variable Y(t) such that the nominal interest rate follows (1).

where

$$(6b)\qquad A(\lambda,\kappa,\theta,\sigma,t,T)=\left[\frac{2\gamma e^{\frac{(\kappa+\lambda+\gamma)(T-t)}{2}}}{(\kappa+\lambda+\gamma)e^{\gamma(T-t)}+2\gamma}\right]^{\frac{2\kappa\theta}{\sigma^2}}$$

$$(6c)\qquad B(\lambda,\kappa,\sigma,t,T)=\left[\frac{2}{\kappa+\lambda+\gamma\coth(\gamma(T-t)/2)}\right]$$

$$(6d)\qquad \gamma=\left[(\kappa+\lambda)^2+2\sigma^2\right]^{1/2}.$$

The theoretical pure discount bond prices in (6) may be interpreted as *present value factors* for future \$1 riskless payouts at various horizon dates, T. Consequently, theoretical values for *noncallable* Treasury notes and bonds may be obtained by discounting the coupon and principal payments with the present value factors in (6) and subtracting the accrued interest.[3] For a bond with a \$100 face value, a semiannual coupon payment of $i/2$ dollars, n(t) coupons remaining to maturity at the current time t, and cash flow payment dates of T_i (measured as years from the current date), the theoretical coupon bond price may be represented as

$$(7)\qquad \begin{aligned} &b(r,\lambda,\kappa,\theta,\sigma,t,T,i,n)\equiv b(t)= \\ &\sum_{i=1}^{n(t)}(i/2)A(\lambda,\kappa,\theta,\sigma,t,T_i)e^{-r(t)B(\lambda,\kappa,\sigma,t,T_i)}+100A(\lambda,\kappa,\theta,\sigma,t,T_{n(t)})e^{-r(t)B(\lambda,\kappa,\sigma,t,T_{n(t)})}-accint(t,i_1).\end{aligned}$$

Let ($i_1/2$) be the first coupon; let DAYS(i_1 ,t) be the number of calendar days to the first coupon; and let HALF(i_1 ,t) the number of days in the half year ending on the first coupon date. Then accrued interest may be calculated as

3 Callable bonds will be worth less than the discounted sum in (7) by precisely the value of the call option to the government. The value of a callable bond may be numerically calculated as the solution to a finite-difference approximation of (5) subject to a terminal condition that the bond pay \$100 plus accrued interest at maturity and subject to a boundary condition that the government will call the bond at the first permissible instant that the bond's price exceeds the call price.

(8) $$accint(t,i_1) = \left[1 - \frac{DAYS(i_1,t)}{HALF(i_1,t)}\right](i_1/2).$$

The single-factor pricing theory culminating in (7) implies that all coupon bond prices are perfectly locally correlated. This means that knowledge of the parameter set $\{\kappa,\theta,\sigma,\lambda\}$ and either the level of the instantaneous interest rate r(t) or the theoretical value $b_j(t)$ for an arbitrary jth bond is sufficient to describe the set of theoretical prices for all traded coupon bonds *as long as price units are infinitely divisible.*[4]

Unfortunately, coupon bond prices are generally quoted in minimum price units of \$1/32nd (\$0.03125) so that reported market prices lack the infinite divisibility required by the theory.[5] In other words, even if market participants actually determined prices from the theory, reported bid and ask prices would deviate from theoretical values by an amount required to round up or down to the nearest \$1/32.

Consequently, empirical tests of the single-factor theory in (6) and (7) using observed market prices require a fifth assumption:

A5. An any time t, every observed coupon bond price differs from its theoretical price in (7) by a mean-zero error term which is independent of the theoretical price (and hence independent of the instantaneous interest rate). Thus, for an arbitrary jth bond defined by its contractual terms:

(9) $$\begin{aligned} b_j^{\circ}(t) &= b_j(t) + u_j(t) \\ E[u_j(t)] &= 0 \end{aligned}$$

where $b_j^{\circ}(t)$ represents the observed price, $b_j(t)$ represents the theoretical price calculated from (7), and $u_j(t)$ is the independent error process for bond j at time t.

The fact that (at least a substantial portion of) the *error* reflects *round-off* suggests that a *uniform distribution* may characterize the error process better than a normal distribution (since a round-off error of zero is no more likely than a round-off error of, say, one-half cent).

4 This statement is proved in Lemma 2 below.

5 Market insiders occasionally trade bonds in price units of \$1/64. However, most reported prices are listed in units of \$1/32.

III. A Joint Maximum-Likelihood (JMLE) Approach to the Single-Factor Bond Model

To simultaneously estimate the four parameters $\{\kappa, \theta, \sigma, \lambda\}$ of the single-factor model in (7) using maximum-likelihood method, we make the following additional assumption:

A6. Market participants observe the instantaneous interest rate without error. Furthermore, they use the overnight repurchase (RP or Repo) rate as the physical representation of the instantaneous rate r(t).[6]

Under assumptions A1-A6, the desired joint transition density describing the dynamics of observed bond prices and the interest rate can be written in terms of a transformation on the transition density of the instantaneous rate process (exogenously implied by (1)) and any assumed joint density of the error processes in (9). The resulting nonlinear joint likelihood function can then be exploited to obtain simultaneous estimators for the parameters $\{\kappa, \theta, \sigma, \lambda\}$.

To prove the main result and develop a readily testable likelihood function, a pair of technical lemmas will be useful.

Lemma 1.

The present value factors in (6) and coupon bond prices in (7) also occur in an economy where assumptions A4 and A1 are replaced by assuming that market participants act *as if*

(A4') they require *no factor premium* for interest-rate risk; and

(A1') interest-rate dynamics may be described by the following *preference-adjusted* transition density

$$\hat{p}\left(r_j \mid r_{j-1}; \kappa, \theta, \sigma, \lambda, \Delta t\right) \equiv \hat{f}\left(r_j \mid r_{j-1}; \kappa, \theta, \sigma, \lambda\right) \equiv d\hat{\wp}(t+\Delta t) =$$

(10)

$$c e^{-c\left[r_j + r_{j-1} e^{-(\kappa+\lambda)\Delta t}\right]} \left\{ \frac{r_j e^{(\kappa+\lambda)\Delta t}}{r_{j-1}} \right\}^{q/2} I_q\left[2c\left(r_j r_{j-1} e^{-(\kappa+\lambda)\Delta t}\right)^{1/2}\right]$$

where

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6 The overnight RP market represents the primary source of funding for security dealers. As explained in Prezas and Ulman [15], A6 should provide superior results relative to either weekly or monthly T-bill rates previously used as proxies for the instantaneous interest rate.

$$c(\kappa,\sigma,\lambda) \equiv \frac{2(\kappa+\lambda)}{\sigma^2\left[1-e^{-(\kappa+\lambda)\Delta t}\right]}$$

and

$$q \equiv \frac{2\kappa\theta}{\sigma^2} - 1;$$

$I_q(\bullet)$ is a modified Bessel function of the first kind of order q; and Δt is the time interval between successive observations of the interest rate.[7]

In other words, the present value factors in (7) may be represented as either

$$\text{(11a)} \quad P(r,\lambda,\kappa,\theta,\sigma,t,T) = \hat{E}_t\left\{\exp\left[-\int_t^T r(u)du\right]\right\}$$

or

$$\text{(11b)} \quad P(r,\lambda,\kappa,\theta,\sigma,t,T) = E_t\left\{\exp\left[-\int_t^T r(u)du\right]\left[\frac{d\hat{\wp}(t,T)}{d\wp(t,T)}\right]\right\}$$

where $\hat{E}_t(\bullet)$ indicates the expectation operator with respect to the preference-adjusted density $d\hat{\wp}(t,T)$ given in (10); $d\wp(t,T)$ is the density for the exogenously specified interest-rate density in (1) given by

$$\text{(12)} \quad \begin{aligned} &p_j(r_j|r_{j-1};\kappa,\theta,\sigma,\Delta t) \equiv f(r_j|r_{j-1};\kappa,\theta,\sigma) \equiv d\wp(t,t+\Delta t) = \\ &be^{-b\left(r_j+r_{j-1}e^{-\kappa\Delta t}\right)}\left(\frac{r_je^{\kappa\Delta t}}{r_{j-1}}\right)^{q/2} I_q\left[2b\left(r_jr_{j-1}e^{-\kappa\Delta t}\right)^{1/2}\right]\end{aligned}$$

where

$$b(\kappa,\sigma) \equiv \frac{2\kappa}{\sigma^2\left(1-e^{-\kappa\Delta t}\right)}$$

and other variables have the interpretation given in (10); $E_t(\bullet)$ is an expectation relative to the exogenous density (12); and $\frac{d\hat{\wp}(t,T)}{d\wp(t,T)}$ is the Radon-Nikodym derivative.[8]

[7]For a description of modified Bessel functions, see Abramowitz and Stegun [1].

Proof.

Assume that

(13) $$dr(t) = \left[\kappa\theta - (\kappa + \lambda r(t))\right] dt + \sigma r^{1/2}\, dW(t).$$

Define the operator L as in Friedman [8], p. 139; then (5) is a Cauchy problem $L\wp + \frac{\partial \wp}{\partial t} = 0$ with solutions given by (11a) according to Theorem 5.3, p. 148. Using the inverse Laplace transform given in Richard [17], the transition density corresponding to (13) is (10). Alternatively, use separation of variables and Laguerre polynomials in the spectral representation as in Karlin and Taylor [13] to derive (13). The equivalence of (11a) and (11b) follows from Cox-Ingersoll-Ross [6], Theorem 4.

QED

Lemma 1 requires some interpretation. It says that investors price bonds *as if* their observations of the interest rate are drawings from a density described by (10) rather than as drawings from the exogenous dynamics in (12). Since the support[9] of the *preference-free* process is identical to the support of the exogenous process, investors who agree that (7) describes (theoretical) bond prices effectively perform a *risk-adjustment consistent with merely changing the probabilities they ascribe to future interest rate outcomes in a fashion to transform bond prices to discounted martingales under the new probability system*. This means that a time series of interest rates (when viewed simultaneously with a time series of bond prices) may be interpreted as drawings from the transition density (10); i.e., investors may act as if the observed time series emanates from a preference-adjusted process with *speed of adjustment* $(\kappa + \lambda)$ and *long-run mean* $\frac{\kappa\theta}{(\kappa + \lambda)}$.

This interpretation does *not* imply that (10) describes the dynamics of the exogenous interest-rate process viewed in isolation. However, knowledge of the parameters $\{\kappa, \theta, \sigma, \lambda\}$ associated with the risk-adjusted process (10) is sufficient to make probabilistic statements about the exogenous interest-rate process since the parameters in conjunction with observations on the interest-rate level are the only inputs

8 For a discussion of the Radon-Nikodym derivative in this context, see Friedman [8], p. 166.

9 The support of the process may be viewed as the set of outcomes which have positive probability. Here, outcomes belong to the set $[0, \infty)$ for drawings from either (10) or (12).

needed to calculate the Radon-Nikodym derivative and translate statements from one probability space to another.[10]

Lemma 1 leads naturally to the following relationship between the dynamics of the set of theoretical bond prices and the *risk-adjusted* interest-rate process.

Lemma 2.

Assume an economy characterized by A1-A6 (or the equivalent assumptions of Lemma 1) and a set of m traded coupon bonds. Then the transition density of the set of theoretical coupon bond values conditional on the instantaneous interest rate interpreted as drawings from the *preference-adjusted* density (10) may be written as

$$f_{\{\Gamma_m(s)|r(s)\}|\{\Gamma_m(t)|r(t)\}}\left[\left(b_{1,s},\cdots,b_{m,s}|r(s)\right)\left(b_{1,t},\cdots,b_{m,t}|r(t)\right)\right]= \tag{14}$$

$$\prod_{j=1}^{m} I_{\Xi_j(s)|\Xi_j(t)}(b_{js}|b_{jt})$$

where $I_{\Xi_j}\left(b_{js}\right)$ is the indicator function for the set $\Xi_j(s)$ defined as $\Xi_j(s)=\left\{b_{js}:b_j^{-1}(b_{js})=r_s\right\}$ for any realization r_s of the interest rate and $\Gamma_m(s)=\{b_1(s),b_2(s);\cdot b_m(s)\}$, the set of all theoretical bond values at s.

Proof.

Under assumptions A1-A6 (or the alternative in Lemma 1), $r(t)\in\Re^+$. Consequently, at any time t, the theoretical coupon bond price for any bond j, defined by (7) lies in a compact set $B^j(t)$ and represents a 1:1 monotone decreasing transformation from the underlying interest rate process: $b_j:\Re^+\mapsto B^j$ for each t and fixed parameter and contract values due to the convexity of present value factors in (7). Therefore, the inverse transform exists and coincides with the observed instantaneous rate at equilibrium. Hence, given any observed interest rate consistent with (10), only one value is feasible for each bond at equilibrium so the conditional density in (14) must pertain. **QED**

Note that the result in Lemma 2 depends critically upon the specification of the factor premium specified in A4. In fact, (14) does not necessarily hold in terms of the *exogenous* marginal density (12) since the theoretical prices specified require the factor premium defined in A4. The exogenous density (12) is, of course, consistent with almost any specification for preference adjustment and there is no guarantee that the theoretical prices in (7) will result unless A4 holds.

[10] The Radon-Nikodym derivative here is merely the ratio of the preference-adjusted transition density to the exogenous transition density. If the pricing model (7) holds, both densities are known so the derivative may be easily calculated to translate one measure to another.

Given the technical results in Lemmas 1 and 2, we derive Theorem 1 (below), which specifies a general form for the density of bond prices observed jointly with the overnight interest rate in the single-factor economy described by A1-A6. Next, we derive Theorems 2 and 3, to develop the maximum-likelihood estimators and marginal transition density for the coupon bond described by (6) and (7) when *observation errors* have (assumed) independent uniform distributions (consistent with *round-off* inherent in quoting bond prices in units of \$1/32).

Theorem 1.

Assume that at time s there are m traded riskless coupon bonds traded in an economy characterized by assumptions A1-A6 (or the alternative assumptions of Lemma 1). Then, at equilibrium, the joint transition density function characterizing the dynamics of observable bond prices and the instantaneous interest rate may be specified solely in terms of the *preference-adjusted transition density given in (10), with any hypothesized joint density of the errors. In particular, if the errors are mutually independent, then*

$$(15)\qquad f_{b_1^\circ(s),\cdots b_m^\circ(s),r(s)\mid b_1^\circ(s),\cdots b_m^\circ(t),r(t)}(b_{1s}^\circ,\cdots b_{ms}^\circ, r(s)\mid b_{1t}^\circ,\cdots b_{mt}^\circ, r(t))=$$

$$\hat{f}_{r(s)\mid r(t)}(r_s\mid r_t)\; f_{u_1(s)\mid r(s)}(u_{1s}\mid r_s)\, f_{u_m(s)\mid r(s)}(u_{ms}\mid r_s)$$

where

$$\hat{f}_{r(s)\mid r(t)}(r_s\mid r_t)$$

is the *preference-adjusted* transition density (10) which investors associate with observations of the interest rate properly adjusted for risk and

$$f_{u_k(s)\mid r(s)}(u_{ks}\mid r_s)= f_{u_k(s)\mid r(s)}[b_{ks}^\circ - b_k(r_s)]$$

is any hypothesized transition density for the error associated with the kth bond (evaluated as the difference between the kth observed price and the kth theoretical value calculated at the observed interest rate using (7)).

Proof.

For notational convenience only, we assume m = 2. The general case follows easily by induction. With the instantaneous interest rate observable without error and its evolution consistent with the preference-adjusted density (10), theoretical values for all bonds are uniquely determined by the invertibility of the pricing equation (7) as proved in Lemma 2 where solutions lie in the sets $\Xi_{12}(s)$ for m = 2. Suppressing the conditioning of the joint transition function on values at previous time t for notational convenience only, we write

$$f_{b_1^\circ(s),b_2^\circ(s),r(s)}\left(b_{1s}^\circ, b_{2s}^\circ, r(s)\right)=$$

$$\frac{\partial^3}{\partial b_{1s}^\circ \partial b_{2s}^\circ \partial r_s} \int_{\{r\}} \int_{\{U_2\}} \int_{\{U_1\}} \int_{\{b_2\}} \int_{\{b_1\}} f_{b_1 b_2 u_1 u_2 r}(\bullet,\bullet,\bullet,\bullet,\bullet)\, db_1\, db_2\, du_1\, du_2\, dr =$$

(by using Leibnitz' rule)

$$\frac{\partial}{\partial r_s} \int_{\{r\}} \int_{\{b_2\}} \int_{\{b_1\}} f_{b_1 b_2 u_1 u_2 r}(b_{1s}, b_{2s}, b_{1s}^\circ - b_{1s}, b_{2s}^\circ - b_{2s}, r_s) db_1 db_2 dr =$$

(by using independence of errors and theoretical values)

$$\frac{\partial}{\partial r_s} \int_{\{r\}} \int_{\{b_1\}} \int_{\{b_2\}} f_{b_1 b_2 | r}(b_{1s}, b_{2s} | r_s) \hat{f}_r(r_s) f_{u_1 u_2}(b_{1s}^\circ - b_{1s}, b_{2s}^\circ - b_{2s}) db_1 db_2 dr$$

Substituting (14) from Lemma 2 for the conditional density, representing $b_{js}^*(r_s) \in \Xi_{j2}$ and evaluating the integrals over the indicator functions shows that

$$f_{b_1(s), b_2(s), r(s)}(b_{1s}^\circ, b_{2s}^\circ, r_s) =$$

$$\frac{\partial}{\partial r_s} \int_{\{r\}} \hat{f}_{r(s)}(r_s) f_{u_1 u_2}(b_{1s}^\circ - b_{1s}^*, b_{2s}^\circ - b_{2s}^*) dr$$

$$= \hat{f}_{r_s}(r_s) f_{u_1 u_2}(b_{1s}^\circ - b_{1s}^*, b_{2s}^\circ - b_{2s}^*).$$

When the error terms are mutually independent (as well as independent of the interest rate), the joint error density may be replaced by the product of the marginal densities as in (15). **QED**

Theorem 2. (Estimating Maximum-Likelihood Parameters)

In an economy characterized by A1-A6, assume that the error processes for all bond prices described by (7) and (9) are mutually independent and uniformly distributed with a mean of zero:

(16) $$f_{u_k(t)}(u_{kt}) = \frac{1}{2\upsilon_k} I_{[-\upsilon_k, \upsilon_k]}(u_{kt}) \text{ for all k, t}$$

where υ_k parameterizes the error distribution for bond k at all times t. Then the maximum-likelihood estimators of $\{\kappa, \theta, \sigma, \lambda, \upsilon_1, \cdots, \upsilon_k\}$ corresponding to the joint transition density (15) are the solutions to

$$\max_{\{\kappa,\theta,\sigma,\lambda\}} \log L_n^* = n\log c + \frac{q}{2}\left[\log\frac{r_n}{r_0} + n(\kappa+\lambda)\Delta t\right] - \sum_{j=1}^{n} c\left[r_j + r_{j-1}e^{-(\kappa+\lambda)\Delta t}\right]$$

(17)

$$+\sum_{j=1}^{n}\log\left\{I_q\left[2c\left(r_j r_{j-1}e^{-(\kappa+\lambda)\Delta t}\right)^{1/2}\right]\right\} - \sum_{k=1}^{m} n\log\left[2\hat{\upsilon}_k\left(y_{k1}, y_{kn} \mid r,\kappa,\theta,\sigma,\lambda\right)\right]$$

where

(18) $$\hat{\upsilon}_k\left(y_{k1}, y_{kn} \mid r,\kappa,\theta,\sigma,\lambda\right) = \left(\frac{2n+2}{2n+1}\right)\max\left[|y_{k1}|, |y_{kn}|\right]$$

with y_{k1} and y_{kn} the minimum and maximum order statistics corresponding to the observed errors of the kth bond.

Proof.

In an economy characterized by assumptions A1-A6, let $\left\{r_j\right\}_{j=0}^{n}$ represent a sample of size n+1 from the instantaneous interest rate process and let $\left\{b_{kj}\right\}_{j=1,n}^{k=1,m}$ be samples of size n for each of the m traded coupon bonds over the corresponding time intervals. The observed errors correspond to the set

$$\left\{u_{kj}\right\}_{j=1}^{n} = \left\{u_{kj} : u_{kj} = b_{kj}^{c} - b_{kj} \text{ and } b_{kj} \in \Xi_{km}(j)\right\}_{j=1}^{n}$$

where $\Xi_{km}(j)$ is defined as in Lemma 2. Let $\left\{y_{kj}\right\}_{j=1}^{n}$ represent the order statistics corresponding to the n observed errors of bond k. According to Theorem 1, the log likelihood function reflecting the joint density of observed bond prices with the instantaneous interest rate may be written as

$$\log L_n = n\log c + \frac{q}{2}\left[\log\frac{r_n}{r_0} + n(\kappa+\lambda)\Delta t\right] - \sum_{j=1}^{n} c\left[r_j + r_{j-1}e^{-(\kappa+\lambda)\Delta t}\right]$$

(19)

$$+\sum_{j=1}^{n}\log\left\{I_q\left[2c\left(r_j r_{j-1}e^{-(\kappa+\lambda)\Delta t}\right)^{1/2}\right]\right\} - \sum_{k=1}^{m} n\log(2\upsilon_k) + \sum_{k=1}^{m}\sum_{j=1}^{n}\log\left[I_{[-\upsilon_k,\upsilon_k]}\left(u_{kj}\right)\right]$$

where $I_{[-\upsilon_k,\upsilon_k]}(u_{kj})$ is the indicator function with value 1 if $u_{kj} \in [-\upsilon_k, \upsilon_k]$ and 0 otherwise; $I_q(\bullet)$ is a modified Bessel function of the first kind of order q (with q and c defined in (10)); there are m bonds; and υ_k parameterizes the error distribution for the kth bond as described in (16). The log likelihood function (19) is bounded away from $-\infty$ only if $\upsilon_k \geq \max[|y_{k1}|, |y_{kn}|]$. Furthermore, (19) will take on its maximum when υ_k equals the max since $-\log\upsilon_k$ is a monotone decreasing function for any given values of $r, \kappa, \theta, \sigma, \lambda$. Since the joint density of y_{k1} and y_{kn} (for any given sample values and selection of $\{\kappa, \theta, \sigma, \lambda\}$) is given by

$$(20) \qquad f_{y_{k1}, y_{kn}}(y_{k1}, y_{kn}) = \left(\frac{n(n+1)}{(2\upsilon_k)^n}\right)(y_{kn} - y_{k1})^{n-2} I_{[-\upsilon_k,\upsilon_k]}(y_{k1}) I_{[y_{k1},\upsilon_k]}(y_{kn})$$

(18) will be the *unbiased* ML estimator of υ_k for any observed errors consistent with the selected sample and parameters. Therefore, the values $\{\kappa, \theta, \sigma, \lambda\}$ which solve (17) for the observed sample subject (18) will also maximize (19). **QED**

Theorem 3. (Calculating the Marginal Transition Density for an Observed Bond Price)

In an economy characterized by A1-A6 so that observed coupon bond prices may be characterized by (7) and (9), if the error term is uniformly distributed as described in (16), then the transition density function for the observed bond price process may be written as

$$(21) \qquad f_{b^\circ(s)|b^\circ(t)}(b_s^\circ | b_t^\circ) = \frac{1}{2\upsilon}\left[\begin{array}{l} \chi^2\left\{2cr^*(b_s^\circ - \upsilon);\, 2q+2;\, 2c\left[r^*(b_t^\circ - \upsilon)e^{-(\kappa+\lambda)(s-t)}\right]\right\} \\ - \\ \chi^2\left\{2cr^*(b_s^\circ + \upsilon);\, 2q+2;\, 2c\left[r^*(b_t^\circ + \upsilon)e^{-(\kappa+\lambda)(s-t)}\right]\right\} \end{array}\right]$$

where b_s° is the observed bond price at time s, b_t° is the observed price at time t, c and q are as defined in (10), χ^2 is the non-central chi-squared distribution function, v is the parameter of the uniform error distribution, and $r^*(\bullet)$ is the inverse transform proved to exist in Lemma 2.

Proof.

Since, by assumption, the error term in (9) (and (16)) is independent of the theoretical price, the density of an observed price is merely the convolution of the density of the theoretical value with the density of the error. **QED**

IV. Conclusion

In this paper, we have developed the requisite theory to estimate both the preference and dynamics parameters needed to implement the CIR bond pricing model using joint maximum-likelihood methods (JMLE). We have made the theory operational by hypothesizing a joint uniform distribution for bond pricing errors. Other error specifications (e.g., truncated normal) could also be used. Additionally, the approach could be applied to other interest-rate dynamics (like the radial bessel process) which generate economically reasonable state-variable paths. In an ensuing paper, we plan to empirically test the distributional assumption (and CIR model) by using sequential quadratic programming methods and daily time series of Treasury Notes and the overnight RP (repo) rate to obtain the JMLE estimators from (17). Other permutations could also be tested by employing truncated normal errors. Our experience with marginal maximum likelihood (MMLE) has taught us that great care will be required in computational methods to successfully implement JMLE.

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