Credible Monetary Policy with Long-Lived Agents: Recursive Approaches

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Abstract: This paper develops recursive methods that completely characterize *all* the time-consistent equilibria of a class of models with long-lived agents. This class is large enough to encompass many problems of interest, such as capital-labor taxation and optimal monetary policy. The recursive methods obtained are intuitive and yield useful algorithms to compute the set of all time-consistent equilibria.

These results are obtained by exploiting two key ideas derived from dynamic programming. The first—developed by Abreu, Pearce, and Stachetti in the context of repeated games and by Spear and Srivastava and Green in the context of dynamic principal agent problems—is that incentive constraints in infinite horizon models can be handled recursively by adding as a state variable the continuation value of the equilibrium. The second insight, due to Kydland and Prescott, is that the set of competitive equilibria of infinite horizon economies can often, in turn, be characterized recursively.

I illustrate my methods by discussing optimal and credible monetary policy in a version of Calvo's (1978) model of time inconsistency. The set of time-consistent outcomes can be completely characterized as the largest fixed point of either of two well-defined operators, one motivated by Abreu, Pearce, and Stachetti (1990) and the other by Cronshaw and Luenberger (1994). In addition, recursive application of either of these two operators provides an algorithm that is shown to always converge to the set of time-consistent outcomes. Finally, the recursive method developed here yields valuable information about the nature of the time-inconsistency problem.

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1. Introduction

Studies of macroeconomic policy in models of long lived agents are of utmost importance for both theoretical and practical reasons. Key examples are the taxation of capital and labor in an infinite horizon growth model [1] and the optimal conduct of monetary policy in a Sidrauski or cash-in-advance framework.[2] These models are considerably complex, partly because they typically involve solving for infinite horizon competitive equilibria for each of a (sometimes large) set of government policies. Progress has been achieved, by and large, by assuming that the government can commit at the beginning of time to a policy specifying its actions for all current and future dates and states of nature. With this assumption, impressive advances have been made recently, in particular, in characterizing optimal macroeconomic policy.[3]

However, the significance of the results thus obtained is unclear if governments cannot commit to date-state contingent policies. If instead governments are assumed to choose policies sequentially, optimal policies under commitment may be time inconsistent, as first pointed out by Kydland and Prescott (1978) and Calvo (1978). As a consequence, it would seem urgent to check whether policies derived under the commitment assumption are time consistent and, more generally, to characterize the set of time consistent outcomes. But this goal has proven to be very elusive in models with long lived agents, presumably due to the difficulty of the issues involved.

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[1] The extensive literature examining this problem includes, in particular, Chamley (1986), Judd (1985), Lucas and Stokey (1983), and Chari, Christiano and Kehoe (1994).

[2] See in particular Calvo (1978), Woodford (1990), Chari, Christiano and Kehoe (1995) and Ireland (1995).

[3] In particular, see Chari, Christiano, and Kehoe (1994).

This paper develops a way to deal with all these issues that can be applied to a wide class of models with long lived agents, including the capital-labor taxation and the optimal money supply problems mentioned at the beginning. Its key achievement is to completely characterize the set of all time consistent outcomes in a recursive fashion. To be more precise, I show that the set of all time consistent outcomes is the fixed point of two related but different operators, one inspired by the work of Abreu, Pearce, and Stachetti (1990) and the other by the work of Cronshaw and Luenberger (1994). The approach in the paper is in the spirit of dynamic programming (Bellman 1957) and yields valuable insights about the time consistency problem; it is discovered, for example, that all time consistent outcomes have a Markovian structure. In addition, the approach yields algorithms that always converge to the set of time consistent outcomes. Hence, the recursive methods developed in this paper amount to an essentially complete solution of the time consistency problem in models of long lived agents.

In order to understand the intuition for the recursive methods of this paper, it is natural to ask first why it is that characterizing time consistent outcomes is so difficult when there are long lived agents. The short answer is that there are "too many infinities" to take care of. Somewhat more precisely, a time consistent solution[4] must include a description of government behavior and market behavior such that the continuation of such behavior after any history is a competitive equilibrium and is optimal for the government. Hence, given any history, checking for time consistency involves solving for a nontrivial infinite horizon competitive equilibrium problem;

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[4]As described below, the concept of time consistency employed in this paper is the appropriate generalization of the "sustainable plans" concept developed by Chari and Kehoe (1990) and Stokey (1991).

moreover, this has to be done for every one of an infinite number of histories.

The approach in this paper exploits two key ideas that help reducing the problem of "too many infinities" to more manageable dimensions. The first is that the need to check for time consistency after each of an infinity of histories can be managed more effectively by introducing as a (fictional) state variable the continuation value of the equilibrium. This insight was first developed for repeated games by Abreu, Pearce and Stachetti (1990) and for dynamic principal agent problems by Spear and Srivastava (1987), Green (1987), and Thomas and Worrall (1990), to obtain recursive characterizations of the solutions of their respective models.

The second key idea underlying my approach is that, in checking that the continuation of a candidate for a time consistent solution is consistent with an infinite horizon competitive equilibrium, one can exploit the fact that the set of competitive equilibria can itself be expressed recursively. The crucial observation is that, for a wide class of models, competitive equilibria can be expressed as the solution of a sequence of Euler-type equations. Although there are an infinity of such equations, each one connects only a small number of periods (say, today and tomorrow); a plausible guess, then, is that infinite horizon competitive equilibria can be characterized very simply by introducing an adequate state variable. This variable turns out to be the "right hand side" of the Euler equation, a conjecture suggested first by Kydland and Prescott (1980) in the context of capital-labor taxation with commitment.[5]

I present my results in the context of the monetary model proposed in

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[5]The Kydland-Prescott approach has been recently been extended by Marcet and Marimon (1995).

Calvo's (1978) seminal study in time consistency. This model is very simple; in particular, there is no physical state variable. However, solving for competitive equilibria is a nontrivial infinite horizon problem, and hence Calvo's model presents the crucial difficulties associated with characterizing time consistency in models with long lived agents. As a consequence, the intuition, the power, and the possible limitations of the methods proposed below are well illustrated in Calvo's setup. The price may be that Calvo's model is not very "realistic"; however, it should become clear that my methods can be readily adapted to more complicated and "realistic" models, which may include physical capital, uncertainty, and so on, as long as their competitive equilibria can be expressed as a system of (possibly stochastic) Euler equations.

This paper is, or course, related to a very large literature. In particular, the concept of time consistency employed in this paper is the appropriate generalization of the "sustainable plans" concept proposed by Chari and Kehoe (1990) and Stokey (1991). Both the Chari-Kehoe and Stokey papers adapt results from Abreu (1988) to characterize the set of all the sustainable plans of their models. Abreu's method involves finding the worst continuation time consistent outcome, which may be very difficult in many models of interest. In contrast, the recursive methods developed in this paper do not require finding the worst continuation; in fact, they yield the worst and the best continuations as part of the solution.[6]

A very similar recursive approach to time consistency in models with long lived agents has been independently developed by Phelan and Stachetti (1996)

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[6] Pearce and Stachetti (1995) have recently applied the methods of Abreu, Pearce, and Stachetti (1990) to a time consistency problem. Since Pearce and Stachetti assume that there is no borrowing or lending of any kind, their problem is essentially a repeated game between the government and the public.

in the context of a capital-labor taxation problem. As in my analysis, they combine the work of Abreu, Pearce and Stachetti (1990) and Kydland and Prescott (1980) to arrive to a recursive characterization of the set of sustainable plans. Their papers and mine are obviously complementary. Two differences deserve special mention. Their setup includes a physical state variable and may therefore by more realistic than my monetary setup. On the other hand, my application of the methods of Cronshaw and Luenberger (1994) is new.

Two recent noteworthy attempts at characterizing time consistent monetary policy in infinite horizon models are Obstfeld (1991) and Ireland (1995). Obstfeld (1991) studies a dynamic seigniorage problem and, in characterizing time consistent outcomes, focuses on the Markov perfect equilibria of the model, taking as state variables the previous real quantity of money and the inherited government debt. Hence his approach only provides a partial characterization of the set of time consistent outcomes. My analysis emphasizes that, in a similar problem, a small enlargement of the set of state variables yields the set of all time consistent outcomes, and that all such outcomes are Markovian.

Ireland (1995) analyzes a cash-in-advance model and is able to characterize all of its time consistent equilibria. This is achieved by showing that the the worst allowable hyperinflation is a time consistent outcome, which is then used to support all other time consistent outcomes adapting the arguments of Abreu (1988). As previously emphasized, Ireland's paper is insightful but his solution method depends on very special features of his environment, in particular that the worst possible hyperinflation is a dominant strategy for the government. Hence his arguments are not generally useful in finding the worst time consistent outcome in other models. In contrast, the recursive approach pursued below always works for a wide class of models and makes it unnecessary to look for the worst time consistent outcome.

The paper proceeds as follows. Section 2 sets up the economic environment under study. Section 3 discusses competitive equilibria; it is emphasized, in particular, that the set of competitive equilibria is recursive in a precise sense. Section 4 examines optimal government policy under commitment. Following arguments of Kydland and Prescott (1980), I show that the Ramsey problem can be written as a dynamic programming problem, by the introduction of a fictional state variable. In addition, the set of fictional states is shown to be the largest fixed point of a particular operator and can be computed recursively. Section 5 discusses the solution concept, sustainable plans, that is used later to characterize time consistency. Section 6 shows the existence of some sustainable plans and provides a partial characterization. Sections 7 and 8 contain the paper's main results. Section 7 studies an operator inspired by Abreu, Pearce, and Stachetti (1990), whose largest fixed point yields the set of all sustainable outcomes. It is also shown there that the repeated application of that operator yields a sequence of sets that converges to the sustainable set. Section 8 studies a second operator, motivated by the work of Cronshaw and Luenberger (1994), whose largest fixed point also yields the set of sustainable outcomes, and whose repeated application also converges to that set. To demonstrate the computational feasibility of the theory, Section 9 computes and discusses the solutions for a parametric version of the model. Section 10 concludes. Some proofs are delayed to an Appendix.

2. The Model

For concreteness, I will analyze a discrete time version of a model first proposed by Calvo (1978). Before proceeding, it must be emphasized again that Calvo's model is not chosen because of its realism: that model is clearly too simple to be "realistic". However, for my purposes its simplicity is a virtue: while Calvo's model is fairly manageable, it is a truly infinite horizon dynamic model. I believe that the intuition and the power of the recursive methods proposed in this paper are best illustrated in this setup. It should become clear that my analysis will carry over to many other, more "realistic" models.

Time is discrete and indexed by t = 0,1,2,...In each period there is only one consumption good and currency is the only asset. The economy is populated by a large number of identical households and a government. The representative household lives forever and has preferences over consumption and real money holdings given by:

$$\sum_{t=0}^{\infty} \beta^t \, [\, u(c_t) + v(m_t) \,] \tag{2.1}$$

where c_t denotes consumption in period t, $m_t \equiv q_t M_t$ is the real value of money holdings, M_t denotes currency holdings at the end of period t, and q_t the price of currency in terms of the consumption good (the inverse of the price level). The functions u and v satisfy:

[A1] $u : \mathbb{R}_+ \to \mathbb{R}$ is C^2, strictly concave, and strictly increasing.

[A2] $v: \mathbb{R}_+ \to \mathbb{R}$ is C^2, and strictly concave.

[A3] $\lim_{c \to 0} u'(c) = \lim_{m \to 0} v'(m) = \infty$

[A4] There is a finite $m = m^f > 0$ such that $v'(m^f) = 0$

restriction is that money creation must be bounded above my some (arbitrarily large) number $\bar{\mu}$. Although this assumption can probably be defended on the basis of realism, it may be interesting to see what happens if it is dropped. This is left for future research.

The assumptions [A1]-[A3] are fairly standard.[7] [A4] defines m^f as the satiation level of money. It will become clear that these four assumptions can be generalized substantially, as long as the model has a recursive structure and some boundedness conditions hold.

The household will maximize (2.1) subject to c_t, $M_t \geq 0$ and

$$q_t M_t \leq y_t - x_t - c_t + q_t M_{t-1}, \tag{2.2}$$

for all $t \geq 0$. The household takes the sequences $\{q_t\}$, $\{x_t\}$, $\{y_t\}$, and its initial currency holdings M_{-1} as given. In (2.2), y_t denotes period t endowment of the consumption good and x_t is a lump sum tax (or transfer, if negative).

The government chooses how much money to create or to withdraw from circulation and must satisfy:

$$q_t (M_t - M_{t-1}) = - x_t \tag{2.3}$$

According to (2.3), money newly printed in period t is used to give a transfer to or levy a tax on each household. For technical reasons, I will impose some bounds on the admissible rates of money creation or destruction:

[A5] For some $\underline{\mu}$, $\bar{\mu}$ such that $0 < \underline{\mu} < \beta < 1 < \bar{\mu}$, $M_t/M_{t-1} \in [\underline{\mu}, \bar{\mu}]$.

It is probably uncontroversial to impose that the supply of money be positive; imposing that M_t/M_{t-1} be not less than $\underline{\mu} > 0$, where $\underline{\mu}$ is arbitrarily small, is only a mild strengthening of that requirement. A stronger

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[7]Note that [A2] implies that v(0) is finite, which in turn implies that $mv'(m) \to 0$ as $m \to 0$; see Obstfeld and Rogoff (1983).

letters denote sequences, and a subscript (resp. superscript) denotes the first (resp. last) date of the sequence. Thus $x_t^s = (x_t, x_{t+1}, \ldots, x_s)$. If the subscript is omitted, the first date is understood to be t = 0, while an omitted superscript implies that the last date is s = ∞. Thus $x^t = (x_0, \ldots, x_t)$, $x_t = (x_t, x_{t+1}, \ldots)$ and $x = (x_0, x_1, \ldots)$.

3. Competitive Equilibria

In this section competitive equilibrium is defined in the usual way. In order to prepare the ground for our main discussion, some facts about equilibria are collected. In particular, this section makes precise the idea that competitive equilibria have a recursive structure.

A <u>policy</u> is a sequence of money growth rates, $g = (g_0, g_1, \ldots)$, and a sequence of tax rates, $x = (x_0, x_1, \ldots)$, such that $g_t \in [\underline{\mu}, \bar{\mu}]$ and $x_t \in \mathbb{R}$, all $t \geq 0$. An <u>allocation</u> is a set of nonnegative sequences of consumptions, c, real money demands, m, endowments, y, and inverse price levels, q. Given M_{-1}, a policy (g,x) and an allocation (c, m, y, q) form a <u>competitive equilibrium</u> if:

(i) Markets clear in every period $t \geq 0$: $m_t = q_t M_t$ and $y_t = f(x_t)$.

(ii) The government budget constraint (2.3) is satisfied and $M_t = g_t M_{t-1}$.

(iii) The pair (c,M) solves the consumer's problem, given the sequence of prices q, endowments y, and taxes x.

A slight investment in notation will simplify our analysis somewhat. First, define $h_t = 1/g_t = M_{t-1}/M_t$; in equilibrium, $h_t \in [1/\bar{\mu}, 1/\underline{\mu}] \equiv [\underline{\pi}, \bar{\pi}] \equiv \Pi$. The government budget constraint (2.3) can now be written as:

$$-x_t = q_t M_t \ (1-h_t) = m_t \ (1-h_t) \tag{3.1}$$

Since, in any competitive equilibrium, $m_t \in [0,m^f]$ and $h_t \in \Pi$, x_t must belong to the interval $[(1-\bar{\pi})m^f, (1-\underline{\pi})m^f] \equiv X$. Equation (3.1) emphasizes that h_t, the inverse of the money growth rate, can be thought of as the (gross) rate of the inflation tax.

I shall assume that output is strictly positive for any policy:

[A8] f is strictly positive on X.

This assumption is mainly for simplicity. Without it, we would have to worry about what happens if taxes are so high that there is no production.

Under the stated assumptions, one can prove that:

<u>Proposition 1.</u> A competitive equilibrium is completely characterized by a a sequence $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ such that, for all t, $m_t \in [0,m^f]$, $h_t \in \Pi$, $x_t \in X$, and:

$$-x_t = m_t \ (1-h_t) \tag{3.2}$$

$$m_t \ \{u'[f(x_t)] - v'(m_t)\} = \beta \ u'[f(x_{t+1})] \ (m_{t+1} + x_{t+1}) \tag{3.3}$$

<u>Proof:</u> See Appendix.

Let $E = [0,m^f] \times X \times \Pi$, and $E^\infty = E \times E \times E \times \ldots$ Proposition 1 says that a sequence $(\mathbf{m},\mathbf{x},\mathbf{h})$ is consistent with a competitive equilibrium if it belongs to E^∞ and if it satisfies the government budget constraint (3.2) and the household's Euler condition (3.3) in all periods. Hence the set of competitive equilibria can be described as the solution of an infinite sequence of equations, each of which connects at most two periods. This observation is crucial for understanding our approach later.

Note that E^{∞} is compact when endowed with the product topology. Given Proposition 1, an element of E^{∞} such that (3.2)-(3.3) is satisfied will be called a competitive equilibrium sequence. The set of all such sequences will be denoted by $CE = \{(\mathbf{m}, \mathbf{x}, \mathbf{h}) \in E \mid (3.2) \text{ and } (3.3) \text{ are satisfied}\}$.

The following facts are now are easy to prove:

<u>Corollary 1:</u> CE is not empty.

In particular, there is a competitive equilibrium with constant money growth.

<u>Corollary 2:</u> CE is compact.

<u>Proof:</u> See Appendix.

<u>Corollary 3:</u> The continuation of a competitive equilibrium is a competitive equilibrium. In other words, if $(\mathbf{m},\mathbf{x},\mathbf{h}) \in CE$, then $(\mathbf{m}_t,\mathbf{x}_t,\mathbf{h}_t) \in CE$ for all t.

The proof of Corollary 3 follows immediately from Proposition 1. In spite of the simplicity of the proof, Corollary 3 is a crucial aspect of the model: it makes precise a sense in which the set of competitive equilibria has a recursive structure. This property will be heavily exploited later.

4. A Recursive Treatment of The Ramsey Problem

From now on we shall assume that the government's objective is to maximize the welfare of its representative citizen. The government's menu of choices to achieve its objective depends, however, on the "commitment technology" available to it. A natural starting point is to suppose that the government can fix the entire path of money growth rates once and for all at the beginning of time. This case, which will be referred to as the commitment case, is the subject of this section.

The government's problem under commitment is to choose $\mathbf{g} = (g_0, g_1, \ldots)$ such that $\mathbf{g}$ is consistent with the existence of a competitive equilibrium and such that there is no other $\mathbf{g}'$, also consistent with competitive equilibrium, that results in consumer's welfare. This problem can be restated more precisely, given the results of the previous section, as that of choosing ($\mathbf{m}$, $\mathbf{x}$, $\mathbf{h}$) in CE to maximize (2.1), with $c_t = f(x_t)$. Following previous authors,[8] this problem will be called the Ramsey problem.

Since (2.1) is continuous on E^{∞}, and CE is compact, we know that the Ramsey problem has a solution. Also, given Proposition 1, we know that the solution must solve:

$$\text{Max (2.1) subject to (3.2)-(3.3) and } c_t = f(x_t)$$

where the maximization is over sequences in E^{∞}.

The Ramsey problem, as stated above, can be solved (at least in principle) with a variety of methods. Since our objective is ultimately to look at recursive methods, next we describe a procedure that solves the Ramsey problem in a recursive way. My procedure is a variant of that originally proposed by Kydland and Prescott (1980).

The key to the procedure is to use a recursive description of competitive equilibria. A competitive equilibrium can be seen as the collection of a

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[8] For example, Chari and Kehoe (1990).

policy and an allocation today, together with a "promise" of policies and allocations in the future that satisfies some conditions. By Proposition 1, it follows that the essential feature of the "promise" made in period t is given by the scalar $u'[f(x_{t+1})](m_{t+1} + x_{t+1}) \equiv \theta_{t+1}$ in the Euler equation. Roughly speaking, θ_{t+1} can be seen as the period (t+1) marginal utility of money "promised" by the equilibrium in period t.

Hence we need to study a set Ω defined by:

$$\Omega = \{ \theta \in \mathbb{R}: \theta = u'[f(x_0)](m_0 + x_0) \text{ for some } (\mathbf{m}, \mathbf{x}, \mathbf{h}) \in CE \} \tag{4.1}$$

Ω is the set of initial marginal utility of money "promises" consistent with competitive equilibria.

Proposition 2. Ω is a nonempty and compact subset of $\mathbb{R}_+$.

Proof: Since CE is not empty, Ω is not empty. In any competitive equilibrium, $(m_t + x_t) = h_t m_t \in [0, \bar{\pi} m^f]$. Since $u'[f(x_t)]$ is a positive, continuous function on X, its range is a bounded subset of $\mathbb{R}_+$. Hence Ω is included in some compact interval $[0, \bar{\theta}]$, for some $\bar{\theta}$.

To see that Ω is compact, it is enough to show that Ω is closed. Let $\{\theta^n\}$ be a sequence in Ω converging to $\theta \in [0, \bar{\theta}]$. By definition, there is a sequence $(\mathbf{m}^n, \mathbf{x}^n, \mathbf{h}^n)$ in CE such that $\theta^n = u'[f(x_0^n)](m_0^n + x_0^n)$ for each n. Since CE is compact, $(\mathbf{m}^n, \mathbf{x}^n, \mathbf{h}^n)$ can be assumed without loss of generality to converge to some $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ in CE. Finally, continuity of u' and f implies that $\theta = u'[f(x_0)](m_0 + x_0)$. Hence Ω is closed and compact.∎

Now we can follow Kydland and Prescott (1980) and formulate the Ramsey problem in two stages. Suppose that the government was constrained not only be the requirement of equilibria, but also by an initial "promise" $\theta \in \Omega$. Then its problem would be:

$$w^*(\theta) = \text{Max} \sum_{t=0}^{\infty} \beta^t \, [u(f(x_t)) + v(m_t)] \qquad \text{s.t. } (m, x, h) \in \Gamma(\theta) \qquad (4.2)$$

where $\Gamma(\theta) = \{ (m, x, h) \in CE \mid \theta = u'[f(x_0)](m_0 + x_0) \}$.

Given any initial "promise" θ in Ω, $\Gamma(\theta)$ is a nonempty, compact subset of CE. Since the objective in (4.2) is clearly continuous, the function w^* is well defined on Ω. If we knew $w^*(.)$, the value of the Ramsey problem would simply be given by the max of $w^*(\theta)$ on Ω.

The usefulness of recasting the Ramsey problem in this way is that now we obtain a "dynamic programming" formulation:

Proposition 3: $w^*(\theta)$ satisfies the functional equation:

$$w(\theta) = \text{Max } u[f(x)] + v(m) + \beta \, w(\theta') \qquad (4.3)$$

s.t.

$$(m,x,h,\theta') \in E \times \Omega$$

$$\theta = u'(f(x))(m+x) \qquad (4.4)$$

$$-x = m(1-h) \qquad (4.5)$$

$$\text{and } m \, \{ u'[f(x)] - v'(m) \} = \beta \, \theta' \qquad (4.6)$$

Conversely, if a bounded function $w:\Omega \to \mathbb{R}$ satisfies the above functional equation, then $w = w^*$.

The proof is closely related to the proof of Bellman's optimality principle and given in the Appendix.

Proposition 3 provides a way in which the Ramsey problem can be solved --- provided that we can compute the set Ω. Now, I shall argue that Ω can be computed taking advantage of the fact that Ω must be the fixed point of a particular operator. This approach was also originally suggested by Kydland and Prescott (1980), and is related to that of Abreu, Pearce, and Stacchetti (1990) or Spear and Srivastava (1987).

Let W be a nonempty and bounded subset of $\mathbb{R}_+$. Define a new set $\mathbb{B}(W)$ as follows:

$$\mathbb{B}(W) = \{\theta \in \mathbb{R}: \text{there is } (m,x,h,\theta') \in E \times W \text{ such that (4.4)-(4.6) hold}\}$$

Then one can show that:

<u>Proposition</u> <u>4:</u>(i) $W \subseteq \mathbb{B}(W)$ implies that $\mathbb{B}(W) \subseteq \Omega$. (ii) $\Omega = \mathbb{B}(\Omega)$.

<u>Proof:</u> Left to the reader.

In other words, Ω is the largest fixed point of the operator $\mathbb{B}$. Following Abreu, Pearce, and Stachetti (1990), we will refer to property (i) in Proposition 4 as <u>self generation</u>, and to property (ii) as <u>factorization.</u>

The advantage of this formulation is that it delivers a way to compute Ω as follows. Let $W_0 = [0,\bar{\theta}]$, where $\bar{\theta}$ is as in Proposition 2. For $n = 1,2,\ldots$, define $W_n = \mathbb{B}(W_{n-1})$. Now, the definition of $\mathbb{B}$ clearly implies that $\mathbb{B}$ is monotone in the sense that $W \subseteq W'$ implies $\mathbb{B}(W) \subseteq \mathbb{B}(W')$. This implies that the sequence $\{W_n\}_{n=0}^{\infty}$ is decreasing. Also, one can show that $\mathbb{B}$ preserves compactness. Hence each W_n is a compact set. Define $W_\infty = \bigcap_{n=0}^{\infty} W_n$, i.e. W_∞ is obtained in the limit by repeated application of the operator $\mathbb{B}$, starting with $[0,\bar{\theta}]$. Now it turns out that:

<u>Proposition</u> <u>5:</u> $\Omega = W_\infty$.

<u>Proof:</u> Obviously it suffices to show that $W_\infty \subseteq \Omega$. We shall prove this by showing that $W_\infty \subseteq \mathbb{B}(W_\infty)$; the desired result will then follow by self

generation.

Suppose that $\theta \in W_\infty$. By definition of the sequence W_n, it follows that $\theta \in W_n = \mathbb{B}(W_{n-1})$, all n = 1,2,... By definition of $\mathbb{B}$, there is for each n = 0,1,2... a vector $(m^n, x^n, h^n, \theta'^n)$ in $E \times W_n$ that satisfies (4.4)-(4.6). The sequence $(m^n, x^n, h^n, \theta'^n)$, when seen as a sequence in $E \times [0,\bar{\theta}]$, can be assumed without loss of generality to converge to some $(m,x,h,\theta') \in E \times [0,\bar{\theta}]$. By the continuity of u', v', and f, (m, x, h, θ') satisfies (4.4)-(4.6). Finally, θ' can be shown in fact to belong to W_∞ by the following argument: fix any n = 0,1,...Then, given any $k > n$, $\theta'^k \in W_k \subseteq W_n$. Hence θ', which is the limit of the sequence $\{\theta'^k\}$, must also belong to W_n. Since this is true for any n, $\theta' \in W_\infty$.

The preceding argument shows that $W_\infty \subseteq \mathbb{B}(W_\infty)$. By self generation, $W_\infty \subseteq \Omega$ and the proof is complete. ∎

Together, Propositions 3 and 5 provide a procedure that one can use to solve for $w^*(\theta)$, and hence to compute the Ramsey outcome. First, one can compute Ω by iterating on $\mathbb{B}$, as described before. Once Ω is known, the functional equation in Proposition 4 can be solved by standard methods to obtain w^*. In this sense, the Ramsey problem can be solved recursively.

Before ending this section, it is worth emphasizing that our results imply that the Ramsey problem has a Markovian structure. Along an optimal path, the "state" can be defined to be θ_t; the optimal "action" (m_t, x_t, h_t) and next period state θ_{t+1} are time invariant functions of θ_t. The introduction of the state variable θ_t takes care of the requirement that a Ramsey plan be consistent with a perfect foresight competitive equilibrium.[9]

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[9]On the other hand, θ_t is a "fictional" state variable in the sense that the

5. Sustainable Plans: Definition

Now we will assume that the government does not have the ability to commit to an infinite sequence of money growth rates. Instead, we shall assume that the government sets period t's money growth at the beginning of the period. In this section I will define an appropriate equilibrium concept for this situation. The concept is a direct extension of that developed by Chari and Kehoe (1990) and Stokey (1991) for related environments, and will be called "sustainable plans" (SPs).

A history in period t, denoted by $\mathbf{h}^t = (h_0, h_1, \ldots, h_t)$ describes the actual sequence of money growth rates in every period up to t. Recalling that h_t is restricted by assumption to belong to a compact interval Π, a strategy for the government is a sequence $\{\sigma_t\}_{t=0}^{\infty}$ such that $\sigma_0 \in \Pi$ and $\sigma_t : \Pi^{t-1} \to \Pi$.

In order to have a well defined game we will impose an additional restriction on the strategy space available to the government. This is, roughly speaking, because some strategies may imply that, after some history, the continuation of the strategy be inconsistent with the existence of a competitive equilibrium. This will be ruled out as follows: let $CE_\pi =$ $\{\ \mathbf{h} \in \Pi^\infty$: there is some $(\mathbf{m},\mathbf{x})$ such that $(\mathbf{m},\mathbf{x},\mathbf{h}) \in CE\ \}$. CE_π is the set of infinite horizon sequences of money growths that are consistent with competitive equilibria; it is clearly nonempty and compact. A strategy σ will be called admissible if after any history $\mathbf{h}^{t-1}$ the continuation history $\mathbf{h}_t$, defined by the continuation of σ in the natural way, belongs to CE_π. In what follows we will restrict the government to choose an admissible policy. Intuitively, this says that, after any history, the government has to

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Ramsey problem allows θ_0 to be picked freely.

"announce" a policy for the infinite future that is consistent with the existence of a competitive equilibrium.

Note in particular that, after any history h^{t-1}, the above considerations restrict the government's choice in period t to the set $CE_{\pi}^{0} = \{ h \in \Pi$: there is $\mathbf{h} \in CE_{\pi}$ with $h = h_0 \}$.[10]

Now we are ready to describe market behavior. An <u>allocation rule</u> is a sequence of functions $\alpha = (\alpha_t)_{t=0}^{\infty}$ such that, for each t, $\alpha_t : \Pi^t \to [0,m^f] \times X$. Here, $\alpha_t(h^t) = (m_t(h^t), x_t(h^t))$ denotes the real value of money and taxes in period t, after history h^t has been observed.

Given an admissible government strategy σ, an allocation rule α will be called <u>competitive</u> if given any history h^{t-1} and $h_t \in CE_{\pi}^{0}$, the continuations of σ and α after (h^{t-1}, h_t) induce, in the obvious way, a competitive equilibrium sequence.[11]

Finally, a government strategy σ and an allocation rule α constitute a <u>sustainable plan</u> if (i) σ is admissible; (ii) α is competitive given σ; (iii) After any history h^{t-1}, the continuation of σ is optimal for the government, that is, the sequence $\mathbf{h}_t$ induced by σ after h^{t-1} maximizes (2.1) over CE_{π}, given α.

The definition of a sustainable plan has some nice properties. One of them is that the continuation of a sustainable plan is itself a sustainable plan. This in turn will enable us to apply recursive methods in Section 7.

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[10] Note that $CE_{\pi}^{0} = \{ h \in \Pi$: there is $(m, \theta') \in [0,m^f] \times \Omega$ such that $m[u'(f(h-1)m) - v'(m)] = \beta\theta' \}$

[11] That is, given h^{t-1} and $\mathbf{h}_t \in CE$, define $\mathbf{h}_{t+1}$ recursively by $h_{t+k} = \sigma_{t+k}(h^{t-1}, h_t, \mathbf{h}_{t+1}^{t+k-1})$, $k = 1,2,\ldots$ Then define $(m_{t+k}, x_{t+k}) = \alpha_{t+k}(h^{t-1}, h_t, \mathbf{h}_{t+1}^{t+k})$, $k = 0, 1, 2,\ldots$ For α to be competitive given σ, the sequence $(\mathbf{m}_t, \mathbf{x}_t, \mathbf{h}_t)$ must be in CE.

Proposition 6: Given any history $\mathbf{h}^{t-1}$, the continuation of a sustainable plan is itself a sustainable plan.

Proof: Left to the reader (just a matter of accounting).

To conclude this section, note that any sustainable plan induces, in the natural way, a competitive equilibrium sequence $(\mathbf{m}, \mathbf{x}, \mathbf{h})$. In view of this fact, a competitive equilibrium sequence will be called a sustainable outcome if it is induced by some sustainable plan.

6. Existence and Partial Characterization of Sustainable Plans

The natural place to start discussing sustainable plans is to examine their existence. This section shows that the set of sustainable plans is not empty and provides a partial characterization. I start by showing that there is a sustainable plan whose outcome is a constant supply of money. Then I show that there are sustainable plans that imply that money has no value. One can then find many other sustainable plans by using either of these SPs as a threat. Although informative, this approach will leave us short of characterizing the whole set of SPs. This will motivate the alternative, recursive methods pursued later.

To prove that there exists a SP with constant money, let $\hat{m}$ be the real quantity of money associated with zero money growth and no taxes; that is, $\hat{m}$ is the only solution to

$$u'[f(0)]\ (1-\beta) = v'(\hat{m}) \tag{6.1}$$

Then I claim ınat the following is a sustainable plan:

$\sigma_t(h^{t-1}) = 1$

$m_t(h^t) = z(h_t)$, where $z(h_t)$ is the (only) value of $z \in [0,m^f]$ that solves:[12]

$$z \{ u'[f(z(h_t-1))] - v'(z) \} = \beta u'[f(0)] \hat{m} \tag{6.2}$$

Finally, set $x_t(h^t) = (h_t - 1)m_t(h^t)$.

Before proceeding, note the intuition behind the candidate sustainable plan. The government's strategy σ prescribes keeping the money supply constant after any history. The allocation rule implicitly defined by (6.2) states that, after any history h^t, the private sector believes that the money supply will be constant from period (t+1) on. In other words, any deviation from the constant money supply rule is taken to be temporary. Hence, the price level in period t adjusts to ensure that the supply of money in period t is willingly held.

Proposition 7(i): The strategy σ and the allocation rule $\alpha_t(h^t) = (m(h^t), x(h^t))$ described above are a SP.

Proof: See Appendix.

Hence there is at least one SP in which money is valued. On the other hand, there are a lot of SPs that imply that money has no value. Formally:

Proposition 7(ii): Let σ be any admissible strategy and α be an allocation rule that prescribes that money has no value (that is, $\alpha_t(h^t)$ =

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[12] To see that (6.2) has a unique solution for each h_t in Π, let $T(z;h_t)$ be the LHS of (6.2). For given $h_t \in \Pi$, $T(.;h_t)$ is continuous and strictly increasing in $[0,m^f]$, with $T(0) = 0$ and $T(m^f) = m^f u'[f(m^f(h_t-1))] > m^f u'[f(0)] > \beta u'[f(0)]\hat{m}$. The result follows.

(0,0), all t, $\mathbf{h}^t$). Then (α,σ) is a SP.

<u>Proof:</u> Given any t and $\mathbf{h}^t$, the continuation of α after $\mathbf{h}^t$ is clearly a competitive equilibrium independently of σ. Conversely, the continuation of σ is trivially optimal for the government. ∎

Let $\hat{w} = u[f(0)]/(1-\beta)$ denote the value of the SP of Proposition 7(ii). Adapting results from Friedman (1971) one can now show that there are "a lot" of (monetary and nonmonetary) SPs:

<u>Proposition</u> <u>8:</u> Let $(\mathbf{m},\mathbf{x},\mathbf{h})$ be a competitive equilibrium sequence (an element of CE). Then $(\mathbf{m},\mathbf{x},\mathbf{h})$ is a sustainable outcome if , given any t,
$\sum_{s=0}^{\infty} \beta^s[u(f(x_{t+s})) + v(m_{t+s})] \geq \hat{w}$.

The proof adapts the trigger arguments of Friedman (1971) and given in the Appendix.

Proposition 8 is useful to characterize some sustainable outcomes. For example, sometimes we are interested in <u>stationary</u> outcomes, that is, outcomes $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ such that (m_t, x_t, h_t) are constant. From Proposition 8 we can deduce that:

<u>Corollary</u> <u>4:</u> (i) Let h and satisfy $u'[f((h-1)m](1-\beta h) = v'(m)$, and suppose that $u[f((h-1)m)] + v(m) \geq u[f(0)]$. Then there is a SP in which the real quantity of money is m and the rate of money growth is 1/h. (ii) An optimal <u>stationary</u> outcome is sustainable. (iii) A sufficient condition for the satiation level of money to be part of a stationary outcome is $u[f(m^f(1/\beta - 1))] + v(m^f) \geq u[f(0)]$.

On the other hand, it is important to note that Proposition 8 implies that there are "a lot" of sustainable outcomes that are not stationary. In particular, one cannot use Calvo's (1978) argument to show that a Ramsey policy is not a sustainable outcome.

Although Proposition 8 shows that there are many SPs, it does not go far enough. It only provides a sufficient condition for a path to be sustainable, but a given path may be sustainable under weaker conditions.

Can we strengthen Proposition 8? A possible route would be to try to calculate a "worst" continuation outcome. Then, an adaptation of the arguments in Abreu (1988) would presumably allow us to characterize every sustainable path, as in Chari and Kehoe (1990) and Stokey (1991). However, at this point it is not obvious how to characterize such a "worst" continuation. In fact, we cannot even be sure that such a worst continuation exists.

It turns out, however, that a change of perspective towards recursive methods is much more informative about the last two questions. This is pursued in the next section.

7. Sustainable Outcomes: A Recursive Approach

This section and the next discuss the key results of this paper. The idea of this section is to adapt the tools of Abreu, Pearce and Stachetti (1990) in order to characterize the set of sustainable outcomes in a recursive manner. As in other dynamic incentive problems, one of the key insights is that incentive constraints can be handled by introducing as a state variable the continuation value of the equilibrium. In our context this is not enough, though, because one has to ensure, after any history, that the continuation of a SP is consistent with a competitive equilibrium for the infinite future. But we have seen that, in the Ramsey problem, this constraint can be handled by introducing the promised marginal utility of money as a state variable. Hence one would guess that a recursive approach to the set of sustainable plans should include at least two state variables, one for the continuation values and another for the promised marginal utility of money. In addition, one would guess that it is possible to characterize the state space in a recursive fashion. We shall see that these guesses are in fact correct.

Let $\Theta = \{(\mathbf{m}, \mathbf{x}, \mathbf{h}) \in CE \mid$ there is a SP whose outcome is $(\mathbf{m}, \mathbf{x}, \mathbf{h})\}$ be the set of all sustainable outcomes. Then define S as:

$$S = \{(w,\theta) \mid \text{there is a sustainable outcome } (\mathbf{m}, \mathbf{x}, \mathbf{h}) \in \Theta \text{ with value } w,$$
$$\text{and such that } u'[f(x_0)](m_0+x_0) = \theta\}$$

By [A1]-[A2] and Proposition 1, the value of any competitive equilibrium must belong to some compact interval, say $W = [\underline{w}, \bar{w}]$. Hence S is a subset of the compact set $W \times \Omega$.

The set S is the set of all pairs of continuation values and promised marginal utilities of money that may emerge in the first period of a SP. Our main objective is to characterize S in a recursive fashion. To this end, the following remarks may be useful.

Any sustainable outcome implies an initial value w and an initial "promise" θ; (w,θ) must belong to S. Now think of what a SP must describe in the first period: it must describe an initial, "recommended" action, say $\hat{h}$ and, for each possible deviation h that the government may consider (i.e. for each h in CE^0_π), the SP must specify the real quantity of money m(h) and seigniorage x(h). Moreover, the SP will specify a continuation value, w'(h), and a continuation "promise" θ'(h), which must themselves belong to S.

These considerations motivate the following approach. Let Z denote any nonempty subset of $W \times \Omega$. Define a new set $\mathbb{B}(Z) \subseteq W \times \Omega$ by:

$\mathbb{D}(Z) = \{ (w,\theta) \mid$ there is $\hat{h} \in CE^0_\pi$ and, for each $h \in CE^0_\pi$, a four-tuple $(m(h),\ x(h),\ w'(h),\ \theta'(h))$ in $[0,m^f] \times X \times Z$ such that:

$$w = u[f(x(\hat{h}))] + v[(m(\hat{h})] + \beta\ w'(\hat{h}) \tag{7.1}$$

$$\theta = u'[f(x(\hat{h}))]\ [m(\hat{h}) + x(\hat{h})] \tag{7.2}$$

and for all $h \in CE^0_\pi$:

$$w \geq u[f(x(h))] + v[m(h)] + \beta\ w'(h) \tag{7.3}$$

$$x(h) = m(h)(h-1), \text{ and} \tag{7.4}$$

$$m(h)\ [\ u'(f(h)) - v'(h)\ ] = \beta\theta'(h)\ \} \tag{7.5}$$

The constraints (7.1)-(7.2) are usually called "regeneration constraints", while (7.3) is an "incentive" constraint. (7.4) and (7.5) are relatively novel, and are necessary to ensure that the continuation of a sustainable plan after any deviation is consistent with a competitive equilibrium.

As in Abreu, Pearce, and Stachetti (1991), the operator $\mathbb{D}$ has the following properties, which imply that S is the largest fixed point of $\mathbb{D}$:

<u>Proposition</u> <u>9:</u> (i) Self Generation: If $Z \subseteq \mathbb{D}(Z)$, then $\mathbb{D}(Z) \subseteq S$; (ii) Factorization: $S = \mathbb{D}(S)$.

<u>Proof:</u> (i) Suppose $Z \subseteq \mathbb{D}(Z)$ and let (w,θ) be in $\mathbb{D}(Z)$. Set $(w_0, \theta_0) = (w,\theta)$, and construct a SP (α,σ) recursively as follows. For any h^{t-1}, suppose that we can define $(w_t(h^{t-1}),\ \theta_t(h^{t-1}))$ in Z. Since $Z \subseteq D(Z)$, there is there is $\hat{h}_t \in CE^0_\pi$ and, for each $h_t \in CE^0_\pi$, a four-tuple $(m(h_t),\ x(h_t),\ w'(h_t),\ \theta'(h_t))$ in $[0,m^f] \times X \times Z$ such that (7.1)-(7.5) are satisfied. Define, then, $\sigma_t(h^{t-1}) = \hat{h}_t$, and $\alpha_t(h^t) = (m(h_t),\ x(h_t))$ if $h_t \in CE^0_\pi$, $= (0,0)$ if not.

Finally, define $(w_{t+1}(h^t),\ \theta_{t+1}(h^t)) = (w'(h_t),\ \theta'(h_t))$ if $h_t \in CE^0_\pi$, $= (w,\theta)$ if not. By definition, then, $(w_{t+1}(h^t),\ \theta_{t+1}(h^t)) \in Z$ for all h^t.

Checking that $(\alpha,\ \sigma)$ is a SP is straightforward with value w and initial promise θ is straightforward and left to the reader.

(ii) By self generation, it is enough to show that $S \subseteq \mathbb{D}(S)$. This is easy from the definitions and also left to the reader. ∎

It should be noted that the proof of part (i) of the Proposition shows how one can construct a sustainable plan given any element of S. Hence finding S would imply that we know all the relevant information about the set of SPs. In particular, the construction reveals that any sustainable outcome has essentially a Markovian structure.

Next we can try to study properties of S by looking at properties of $\mathbb{D}$. It is easy to show that $\mathbb{D}$ is nicely behaved, in the sense that it has a monotonicity property and it preserves compactness:

<u>Proposition</u> <u>10:</u> (i) Monotonicity: $Z \subseteq Z'$ implies $\mathbb{D}(Z) \subseteq \mathbb{D}(Z')$; (ii) If Z is compact, $\mathbb{D}(Z)$ is compact.

<u>Proof:</u> (i) is obvious from the definition of $\mathbb{D}$. To prove (ii), it is sufficient to show that if Z is compact, $\mathbb{D}(Z)$ is closed.

Let $(w^{(n)},\ \theta^{(n)})$ be a sequence in $\mathbb{D}(Z)$ converging to $(w,\theta) \in \mathcal{W} \times \Omega$. By definition, for each n there is a recommended action $\hat{h}^{(n)}$ in CE^0_π and, for each h in CE^0_π, a 4-tuple $(m(h)^{(n)},\ x(h)^{(n)},\ w'(h)^{(n)}, \theta'(h)^{(n)})$ in $[0,m^f] \times X \times Z$ that satisfies (7.1)-(7.5).

Since CE^0_π is compact, there is no loss of generality in assuming that the sequence $\hat{h}^{(n)}$ converges to some $\hat{h}$ in CE^0_π. Likewise, for each h in CE^0_π, $(m(h)^{(n)},\ x(h)^{(n)},\ w'(h)^{(n)}, \theta'(h)^{(n)})$ is a sequence in the compact set $[0,m^f]$ $\times X \times Z$ and can be assumed to converge to an element $(m(h),\ x(h),\ w'(h),$

$\theta'(h))$ of $[0,m^f] \times X \times Z$.

The continuity of u, f, v, u', and v' ensure that the recommended action $\hat{h}$ and the function $(m(h), x(h), w'(h), \theta'(h))$ thus defined can be easily seen to satisfy (7.1)-(7.5). Hence (w,θ) belongs to $\mathbb{D}(Z)$. Since $\mathbb{D}(Z)$ contains all its limit points, it is closed, and hence compact. ∎

In particular, the above two properties now imply that S is compact, thus answering one of the questions raised in the last section:

<u>Proposition</u> <u>11:</u> S is compact.

<u>Proof:</u> As discussed above, S is bounded. Hence it is enough to show that S is closed. Let cl(Z) denote the closure of a set Z. Factorization and monotonicity imply $S = \mathbb{D}(S) \subseteq \mathbb{D}(cl(S))$. Since S is bounded, cl(S) is compact; hence $\mathbb{D}(cl(S))$ is compact. It follows that $cl(\mathbb{D}(clS)) = \mathbb{D}(cl(S))$. But then $cl(S) = cl(\mathbb{D}(S)) \subseteq cl(\mathbb{D}(cl(S))) = \mathbb{D}(cl(S))$. Self Generation implies now that $cl(S) \subseteq S$, that is, S is closed.∎

<u>Corollary</u>: There is a best and a worst SP.

Finally, we can modify the proof of Proposition 5 to obtain an algorithm that computes S. Recall that S must be included in $\mathcal{W} \times \Omega$. If we define $S_0 = \mathcal{W} \times \Omega$, the monotonicity of $\mathbb{D}$ implies $S = \mathbb{D}(S) \subseteq \mathbb{D}(S_0) \equiv S_1$. Defining now, for each $n \geq 1$, $S_n = \mathbb{D}(S_{n-1})$, we obtain a decreasing sequence of sets $S_0 \supseteq S_1 \supseteq S_2 \supseteq \ldots$ Moreover, each S_n contains S and is compact (because S_0 is compact and $\mathbb{D}$ preserves compactness). One can then conjecture that the sequence $\{S_n\}$ must converge to S, in the sense that $S_\infty \equiv \bigcap_{n=0}^{\infty} S_n = S$. The following Proposition confirms the validity of such conjecture:

Proposition 12: Let $S_0 = W \times \Omega$ and $S_n = \mathbb{D}(S_{n-1})$, $n = 1,2,\ldots$ Then $S_\infty \equiv \bigcap_{n=0}^{\infty} S_n = S$.

Proof: The proof is essentially the same as that of Proposition 5 and left to the reader. ∎

The preceding results amount to a complete characterization of the set of sustainable plans. We have learned that the set of sustainable plans is compact. This means, in particular, that a best and a worst sustainable plan exist. Proposition 12 provides a way to compute the set S, and the proof of Proposition 9 provides a way to compute a sustainable plan corresponding to any (w,θ) in S.

The remaining issues are "only" computational. In particular, the difficulty of computing any sustainable plan arises solely from the difficulty of computing the mapping $\mathbb{D}$. For the problem at hand, computing $\mathbb{D}(Z)$ given Z seems fairly complicated in particular by the presence of the constraints (7.3)-(7.5). These constraints can be simplified somewhat; this is the subject of the next section.

8. An Alternative Recursive Method

In this section we study a second operator whose largest fixed point is the set S and whose repeated application also yields a decreasing sequence of sets that converges to S. The analysis is related to that developed by Cronshaw and Luenberger (1994) for repeated games.

The intuition for a simpler approach is that the government, when considering whether or not to obey an equilibrium "recommendation", need not consider the consequences of all alternative deviations, but only the payoff associated with the "best" deviation. On the other hand, in order to provide incentives for following equilibrium "recommendations", one can restrict attention to sustainable plans that prescribe the harshest available punishment in response to a government deviation. In this section I show how these considerations motivate an alternative operator whose largest fixed point is the set S which characterizes sustainable plans.

To start, let h be any element of CE^0_π; the reader can interpret h as a "deviation". Let Z be a compact set such that $S \subseteq Z \subseteq W \times \Omega$. Now define:

$$P(h;Z) = \text{Min } u[f(x)] + v(m) + \beta w' \text{ subject to} \tag{8.1}$$

$$-x = m(1-h) \tag{8.2}$$

$$m\,[u'[f(x)] - v'(m)] = \beta\theta' \tag{8.3}$$

$$(m,x,w',\theta') \in [0,m^f] \times X \times Z \tag{8.4}$$

If Z were equal to S, P(h;Z) would be the worst possible SP continuation after a deviation h in CE^0_π. This notion is extended to allow for punishments that can be supported by pairs of future (w,θ) in sets Z possibly larger than S.

In the above definition, the condition that Z be a subset of S is required mainly to ensure that the set defined by (8.2)-(8.4) be nonempty.

Now, let:

$$BR(Z) = \text{Max } P(h;Z) \text{ s.t. } h \in CE^0_\pi. \tag{8.5}$$

If Z were equal to S, BR(Z) would be the government's "best deviation".

Finally, define:

$E(Z) = \{(w,\theta) \in W \times \Omega \mid$ there is $(m,x,h,w',\theta') \in E \times Z$ s.t. (8.2)-(8.3) hold and:

$$w = u[f(x)] + v(m) + \beta w' \tag{8.6}$$

$$\theta = u'[f(x)](m+x) \tag{8.7}$$

$$\text{and } w \geq BR(Z) \;\} \tag{8.8}$$

The intuition behind the operator $\mathbb{E}$ should be clear from the observation that, if Z were equal to S, $\mathbb{E}(Z)$ would include all pairs (w,θ) that could be "enforced" by a threat of reverting to the least favorable continuation for the government. Hence the following properties of $\mathbb{E}$ should be intuitive:

Proposition 13: Let Z be a compact set such that $S \subseteq Z \subseteq W \times \Omega$. Then: (i) Self Generation: $Z \subseteq \mathbb{E}(Z)$ implies $Z = S$; (ii) Factorization: $S = \mathbb{E}(S)$.

Proof: In Appendix.

Hence the operator $\mathbb{E}$ has the key properties of self generation and factorization. As in previous sections, we can derive a lot of mileage from showing that $\mathbb{E}$ has other nice properties:

Proposition 14: (i) $\mathbb{E}$ is monotone in the sense that $S \subseteq Z_1 \subseteq Z_2$ implies $S \subseteq \mathbb{E}(Z_1) \subseteq \mathbb{E}(Z_2)$. (hi) If Z is compact and $S \subseteq Z$, then $\mathbb{E}(Z)$ is compact.

Proof: (i) Let $S \subseteq Z_1 \subseteq Z_2$ be given and suppose that (w,θ) is in $\mathbb{E}(Z_1)$. To show that (w,θ) is in $\mathbb{E}(Z_2)$ it is sufficient to show that $BR(Z_1) \geq BR(Z_2)$. This follows from the definition of BR. Hence $\mathbb{E}(Z_1) \subseteq \mathbb{E}(Z_2)$. Now, $S \subseteq \mathbb{E}(Z_1)$ follows by applying the preceding result to $S \subseteq S \subseteq Z_1$ and noting that $S = \mathbb{E}(S)$. (ii) The proof is easy and left to the reader. ∎

Finally, we can use Propositions 13-14 to obtain an algorithm to compute S as follows. We know that $S \subseteq W \times \Omega$. Set $Z_0 = W \times \Omega$ and, for all $n = 1,2,\ldots$, $Z_n = \mathbb{E}(Z_{n-1})$. By the preceding results, the sequence $\{Z_n\}$ is a decreasing sequence of compact sets which include S. Hence a plausible conjecture is that $S = Z_\infty \equiv \bigcap_{n=0}^{\infty} Z_n$. This is in fact true, as shown by:

Proposition 15: $S = Z_\infty$.

Proof: See Appendix.

In summary, the approach in this section also provides a useful characterization of the set of sustainable outcomes and yields a successful algorithm for computing it. The operator $\mathbf{E}$ seems somewhat simpler to implement than the operator $\mathbf{D}$ of the previous section, which may be useful in specific implementations.

9. Computational Issues

In order to examine computational issues related to the theory just advanced, this section presents and analyzes a parametric example. My objective will be to show that implementing the theory is feasible and to illustrate some of the difficulties involved. I have not attempted here to develop efficient and accurate computational algorithms for the theory; I believe that task to be a nontrivial endeavor and better left for future research.

I will focus on the question of computing the set S of SP (w,θ) pairs given functional forms and parameter values for the model of the previous sections. Given my objectives, I chose a parameterization in order to illustrate the implementation of the theory; hence my choices are not intended to be necessarily realistic. Functional forms and parameter choices are summarized in Table 1 below.

Table 1: Assumptions for Computed Example

$$u(c) = 10000 \log c$$

$$f(x) = 64 - (0.2\ x)^2$$

$$v(m) = 40m - m^2/2$$

$$\beta = 0.9$$

$$\Pi = [\underline{\pi}, \bar{\pi}] = [0.25, 1.75]$$

This parameterization of the model ensures that assumptions [A1]-[A7] are met. As Table 1 shows, I assumed u to be logarithmic in consumption, and f and v to be quadratic.[13] Maximal feasible output was set at 64 (= f(0)), and the satiation level of money was set at 40. The assumption on Π, the range of permitted values of $h_t = M_{t-1}/M_t$, implies that the nominal quantity of money can at most quadruple between periods, and that it can shrink by about 43 percent. The assumption that β is equal to 0.9 was made mostly for simplicity.

With these values, the set of possible values of seigniorage revenue is given by $X = [-30,30]$; then [A8] is satisfied. Now one can calculate ranges of values of w and θ that are consistent with this parameterization. The representative agent's utility, w, must be bounded above by $\bar{w} = \{u[f(0)] + v(m^f)\}/(1-\beta)$, which is the discounted value of extracting no seigniorage while enjoying the satiation level of real balances. A lower bound for w is in turn given by $\underline{w} = [u[f(\bar{x})] + v(0)]/(1-\beta)$, the discounted value of living with maximal seigniorage and worthless money. Hence any equilibrium value of w will belong to $W = [\underline{w}, \bar{w}]$. For our example, $\underline{w} = 144716$ and $\bar{w} = 188618$. Note, for

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[13] The description of v is almost but not exactly accurate. As stated by Table 1, v'(0) is finite, contradicting [A3]. To remedy this, v(m) was assumed to be a square root function for m very small; the parameters of this function were adjusted so as to satisfy [A2].

future reference, that the value of the nonmonetary equilibrium is 180618, the value of the constant money supply equilibrium is 187397, and the value of the competitive equilibrium associated with the Friedman rule (i.e. deflation at the rate of time preference) is 188078. Both the constant money supply rule and the Friedman rule come very close to achieving the maximum feasible utility level $\bar{w}$.

As for θ, recall that $\theta = u'[f(x)](m+x) = u'[f(x)]hm$. Since u', h and m are nonnegative, θ is bounded below by zero. An upper bound for θ is given by $\bar{\theta} = u'[f(\bar{x})]\bar{\pi}m^f$; for our example, $\bar{\theta} = 25000$.

The next task is to compute Ω, the set of θ's consistent with a competitive equilibrium. To do this, we can implement Proposition 5, taking $W_0 = [0,\bar{\theta}]$ and applying $\mathbb{B}$ repeatedly to obtain a decreasing sequence of sets $W_{n+1} = \mathbb{B}(W_n)$ which converges to Ω. This procedure presents two main difficulties. The first is that, although W_0 is a "nice" compact interval, the sets W_n need not as be well behaved. In particular, those sets may not be convex, which greatly complicates their representation in a computer. To deal with this, I approximated the interval $W_0 = [0,\bar{w}]$ by 101 equally spaced points. Any subset of W_0 can be then represented, in the obvious way, by a 101-tuple of ones and zeros, with ones denoting inclusion in the set.

The second difficulty is related to the definition of $\mathbb{B}$. Suppose that an approximation to W_n , call it $\hat{W}_n$, is given (by a vector of zeros and ones). The computation of $\hat{W}_{n+1} = \mathbb{B}(\hat{W}_n)$ amounts to checking, given any θ in $\hat{W}_n$, whether there is (m,x,h) in E and θ' in $\hat{W}_n$ that solve (4.4)-(4.6). Given the nonlinearity of (4.4)-(4.6), this is a nontrivial task, and I proceeded as follows. I eliminated x by inserting (4.5) in (4.4) and (4.6). Then, the ranges of values of m and h, given by $[0,m^f]$ and Π in the model, were discretized: the interval $[0,m^f]$ was approximated by 121 equally spaced points, and Π by 51 equally spaced points. In other words, the set $[0,m^f]\times\Pi$

was represented by a 121x51 matrix which I will refer to as Egrid for the discussion.

Now, given any θ in $\hat{W}_n$, a finite search suffices to check whether there is (m,h) in Egrid and θ' in $\hat{W}_n$ that solve (4.4)-(4.6). There is one more detail to deal with. Because of the discretization procedure, it is possible that no (m,h,θ') in Egrid x $\hat{W}_n$ solve (4.4)-(4.6) exactly even if there is (m,h,θ') in E x W_n that solve (4.4)-(4.6). To correct for this, I allowed an element θ of $\hat{W}_n$ to be an element of $\hat{W}_{n+1}$ if there was (m,h,θ') in Egrid x $\hat{W}_n$ such that (4.4)-(4.6) are satisfied approximately. For the example, the maximum margin of (combined) error was set at one tenth of the size of the intervals of the grid for θ.

Summarizing, given an approximation $\hat{W}_n$ (a vector of zeros and ones) one can compute $\hat{W}_{n+1}$ by checking, for each nonzero element θ of $\hat{W}_n$, whether there is an (m,h,θ') in Egrid x $\hat{W}_n$ that approximately solve (4.4)-(4.6). If there is such (m,h,θ'), θ is kept at one, and set to zero otherwise. One can iterate on this procedure until convergence to obtain an approximation to Ω; because of the discretization, convergence is guaranteed in a finite number of iterations.

To perform the computations, I wrote a GAUSS program and ran it in my personal computer (a Pentium 200).[14] These are relatively modest resources, in spite of which the computation of Ω was relatively quick, taking no more than 10 minutes. However, the amount of computation and the time required to compute Ω seem to increase quite fast as one increases the number of points used to approximate the different sets involved.

Figure 1 displays the computed value of Ω. As discussed above, the interval $[0, \bar{\theta}]$ is approximated by 101 points, which in the figure are

[14] The GAUSS programs for the calculations of this section are available on request.

measured in the horizontal direction; the first entry corresponds to $\theta = 0$ and the 101-th entry to $\theta = 25000 = \bar{\theta}$. The solution for Ω is given by a vector of zeros and ones, with one denoting inclusion. The figure shows that Ω is much smaller than $[0,\bar{\theta}]$. It also suggests that Ω is not an interval, although further investigation on this issue seems to be warranted. There are some theoretical reasons to suspect that Ω is not an interval in this class of models;[15] on the other hand, the emergence of seemingly isolated elements of Ω may be due to my approximation method.

With the estimate of Ω in hand (call it $\hat{\Omega}$), I proceeded to compute an approximation to S, the set of all (w,θ)'s consistent with a sustainable plan. To this end one can exploit Proposition 15, taking $Z_0 = \mathbf{W} \times \Omega$ and applying $\mathbf{E}$ recursively to obtain a sequence $Z_{n+1} = \mathbf{E}(Z_n)$ which converges to S. The computation of $\mathbf{E}$ presents essentially the same difficulties as the computation of $\mathbf{B}$, except that the amount of computation required is now much more demanding.

To start the computations one needs a finite approximation to $Z_0 = \mathbf{W} \times \Omega$. Ω was naturally approximated by $\hat{\Omega}$, while $\mathbf{W} = [\underline{w}, \bar{w}]$ was approximated by a grid of 51 equally spaced points, hereon called $\mathbf{W}$-grid. Hence the subsets of Z_0 that emerge in the iterative procedure can be represented by matrices of ones and zeros, with ones denoting inclusion. It will be seen that, for some questions, one would like to study a finer approximation of $\mathbf{W}$. This is where my computational constraints became importantly binding: even with a grid this coarse, the computation of S took about 24 hours, and I found that the amount of computing time grew very quickly with the fineness of $\mathbf{W}$-grid.

The other details related to the computation of S are very similar to

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[15] For instance, in the model of Obstfeld and Rogoff (1983), which is closely related to Calvo, the variable corresponding to θ assumes a discrete number of values.

those associated with computing Ω and need not be repeated. Figure 2 displays the resulting approximation to S. Note that only the "relevant" part of the computation is depicted: only the entries corresponding to the last ten elements of $\mathcal{W}$-grid are displayed and measured along the horizontal side of the box and, likewise, only the entries associated with the first seventy elements of $\hat{\Omega}$ are displayed and measured along the vertical side. The algorithm sets entries belonging to the approximation to S equal to one; they are the white squares in the figure. The remaining, omitted entries were found to be zeros in our calculations and hence need not be displayed. It has to be noted, though, that the algorithm starts with Z_0 and shrinks it to approximately one tenth its original size.

Figure 2 shows that the SP values of w are approximated by the last ten elements of $\mathcal{W}$. This range is very close to the interval [180618, 188618] given by the feasible values of w that represent at least as much utility as the nonmonetary equilibrium. Hence my computation suggests that the nonmonetary equilibrium is the worst sustainable outcome for our example. Perhaps one can arrive at the same conclusion by analytical means; it is nonetheless remarkable that the result emerges directly from the computation.

On the other hand, whether the value of the Ramsey plan is the value of a sustainable plan remains an open question. This is because the grid for $\mathcal{W}$ is too coarse to make a distinction between the value of the Ramsey plan and the value of the Friedman rule, which is very close to $\bar{w}$ and is known to be sustainable. Examining finer approximations seems to be required.

Finally, Figure 2 suggests that S is not convex. It also suggests that S may not be connected, although it is unclear whether this result is due to my approximation procedure. Again, a finer approximation would help.

Computing sustainable paths for money growth, real balances, output, etc. is now straightforward. I omit the details for brevity and because in this example the result seems to be, simply, that any competitive equilibrium path whose continuation value is at all times no less than the value of the nonmonetary equilibrium is sustainable.

Although the results for the example just discussed are quite simple, our discussion (I believe) has been fruitful. We know now that computing an approximation to the whole set of sustainable plans is possible. In this example, it has turned out that the nonmonetary equilibrium seems to be the worst sustainable outcome; that result may not hold for other parameterizations, though, while our procedures are still applicable and deliver the whole sustainable set. One can perhaps criticize the quality of my approximations, but improving them is just a matter of letting the computer run for a longer time or using a supercomputer. In any case, we have learned that computational constraints may be binding and, therefore, that developing alternative procedures is an important topic for future research. Progress has been made recently by Conklin and Judd (1993) and Cronshaw (1995),[16] but much more remains to be done.

10. Final Remarks

This paper has provided recursive methods that yield a complete characterization of the set of sustainable outcomes in an infinite horizon monetary economy. The recursive characterization yields valuable insights about the set of sustainable outcomes, and suggest algorithms for computing it.

It should be clear that the methods of this paper are applicable to a

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[16] Both Conklin and Judd (1993) and Cronshaw (1995) focus on the problem of computing the fixed point of the Abreu, Pearce, and Stachetti (1990) operator. They both assume that the fixed point is a convex set, which simplifies the computer representation and implies a much smaller computational burden.

wide variety of models. The essential requisite seems to be that the set of competitive equilibria should have a recursive structure. This will typically be the case if competitive equilibria can be described by the solution of (possibly stochastic) difference equations. Hence, it is clear that the recursive approach will be applicable to models that have physical state variables, uncertainty, etc. Indeed Phelan and Stachetti (1996) have arrived to a similar conclusion in the context of capital taxation.

The results of this paper reduce the time consistency problem to a question of computing two operators between sets. I implemented a brute force way to deal with this computation that clearly leaves room for improvement. It should be noted that this computing problem is still not very well understood, as witnessed by Conklin and Judd's and Cronshaw's recent attempts to create algorithms for applying the methods of Abreu, Pearce, and Stachetti (1990). Hence the development of simple and efficient computational procedures should be a priority of future research.

More fundamentally, one may ask whether the derivation of time consistent equilibria is of any value after all. I obviously believe it is, but it may be informative to assume that government behavior, in particular, is motivated by objectives other than maximizing public welfare. A rapidly expanding literature on political economy may yield insights on this issue. It has to be noted, though, that the recursive approach of this paper may be adaptable to models with very different assumptions about the government's objectives.

Finally, the approach in this paper may be adapted to some problems in which information may be imperfect, because of imperfect monitoring for example. However, it is unclear whether other asymmetric information time consistency problems can be handled with the same approach. These questions are also interesting for future research.

Appendix

Proof of Proposition 1: Given a competitive equilibrium, m_t must be nonnegative. Also, if $m_t > m^f$ for any t, the consumer could reduce his money holdings at t and increase his consumption at t, contradicting optimality. Hence $m_t \in [0, m^f]$.

That $h_t \in H$ follows from [A5]. It has already been shown that $x_t \in X$.

(3.2) follows from the government budget constraint. Finally, straightforward analysis implies that the Euler equation for the consumer is:

$$m_t \, \{u'[f(x_t)] - v'(m_t)\} = \beta \, u'[f(x_{t+1})] \, h_{t+1} m_{t+1} \tag{3.4}$$

Using (3.1), $h_{t+1}m_{t+1} = m_{t+1} + x_{t+1}$. Inserting this in (3.4) gives (3.3).

Conversely, suppose (**m**, **x**, **h**) satisfy (3.2), (3.3), and $(m_t, x_t, h_t) \in E$, all t. Define $M_t = M_{t-1}/h_t$, $q_t = m_t/M_t$, $c_t = f(x_t)$. Then it is easy to check that the policy (**g**,**x**) and the allocation (**c**,**m**,**y**,**q**) are a competitive equilibrium. To see this, note that (3.2) and (3.3) ensure that, respectively, the government budget constraint and the representative agent's Euler conditions are satisfied. It is then sufficient to prove that the transversality condition for the representative agent holds, that is, that $\beta^t u'[f(x_t)]m_t h_t \to 0$ as $t \to \infty$. Now, since E is compact, the continuity of u' and f ensures that u'[f(x)]mh must belong to a compact interval for any (x,m,h) in E. Hence $u'[f(x_t)]m_t h_t$ is a uniformly bounded sequence, and $\beta^t u'[f(x_t)]m_t h_t$ must indeed converge to zero.∎

Proof of Corollary 2: CE is a nonempty subset of the compact set E^∞. Let ($\mathbf{m}^n$, $\mathbf{x}^n$, $\mathbf{h}^n$) be a sequence in CE converging to some sequence (**m**, **x**, **h**). Since E is compact, it is closed, and hence $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ belongs to E^{∞}. By [A1]-[A2] and the fact that (3.2) and (3.3) are satisfied for each n we can conclude that $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ must satisfy (3.2)-(3.3) as well. Hence $(\mathbf{m},\mathbf{x},\mathbf{h})$ belongs to CE. This implies that CE is a closed subset of the compact set E^{∞}, hence it is compact.█

<u>Proof</u> <u>of</u> <u>Proposition</u> <u>3:</u> For any bounded function $w: \Omega \to \mathbb{R}$, let Tw be the sup of $u[f(x)] + v(m) + \beta\, w(\theta')$ over all $(m,x,h,\theta') \in E \times \Omega$ that satisfy (4.4)-(4.6). The first claim of the Proposition is that $w^* = Tw^*$, and that the "sup" is in fact achieved. To prove this, fix $\theta = \theta_0 \in \Omega$, and let $(\mathbf{m},\mathbf{x},\mathbf{h})$ attain the max in (4.2). Define $\theta_1 = u[f(x_1)](m_1+x_1)$. Then (m_0,x_0,h_0,θ_1) satisfies (4.4)-(4.6). Hence $Tw^*(\theta_0) \geq u[f(x_0)] + v(m_0) + \beta w^*(\theta_1) \geq w^*(\theta_0)$. Suppose the inequality is strict. Then there is $(m_0', x_0', h_0', \theta_1')$ in $E \times \Omega$ that satisfies (4.4)-(4.6) and such that $u[f(x_0')] + v(m_0') + \beta w^*(\theta_1') > w^*(\theta_0)$. But then there must be a plan $(\mathbf{m}', \mathbf{x}', \mathbf{h}') \in \Gamma(\theta_0)$ whose value is more than $w^*(\theta_0)$, which is a contradiction. Hence $w^* = Tw^*$, and the "sup" is achieved by $(m_0, x_0, h_0, \theta_1)$.

To prove the second claim, let w be bounded and satisfy (4.3). Given any $\theta = \theta_0$ in Ω, define a sequence $(\mathbf{m},\mathbf{x},\mathbf{h})$ recursively as follows: if θ_t is given in Ω, choose $(m_t,x_t,h_t,\theta_{t+1})$ in $E \times \Omega$ that satisfies (4.4)-(4.6); such choice is possible by (4.3). Clearly $(\mathbf{m},\mathbf{x},\mathbf{h}) \in \Gamma(\theta_0)$ and has value $w(\theta_0)$ by the boundedness of w. Hence $w^*(\theta_0) \geq w(\theta_0)$. The proof that $w(\theta_0) \geq w^*(\theta_0)$ is easy and left to the reader.█

<u>Proof</u> <u>of</u> <u>Proposition</u> <u>7:</u> σ is clearly admissible. To check that α is competitive given σ, note that the continuation of σ after any h^t implies that the money supply will be constant from period t on.

To check that σ is optimal given α, I shall show that the government cannot gain from any one shot deviation from the constant money supply rule, given any $\mathbf{h}^{t-1}$. Then the Principle of Optimality applies and implies that no (finite or infinite) deviation from the constant money supply rule can be profitable.

Note that the allocation rule and the government's strategy imply that, after any one shot deviation, the continuation of (α,σ) no matter the value of the initial deviation. Hence a one shot deviation is profitable if and only if there is h_t such that $u[f((h_t-1)m_t] + v(m_t) > u[f(0)] + v(\hat{m})$, i.e. if it improves current utility, with $m_t = z(h_t)$. Now, the definition of $z(h_t)$ implies that $z(h_t)$ is maximized at $h_t = 1$. But then the above inequality cannot hold for any h_t.∎

<u>Proof of Proposition 8:</u> (Sketch) Let $(\mathbf{m},\mathbf{x},\mathbf{h})$ satisfy the conditions of the Proposition. Construct a "trigger" argument as follows. In period t, if the government has not deviated from $\mathbf{h}^t$, continue with the path $(\mathbf{m}_t,\mathbf{x}_t,\mathbf{h}_{t+1})$. If the government has deviated from $\mathbf{h}^t$, follow the strategy and allocation rule of Proposition 1. It is clear that, if the government has not deviated from $\mathbf{h}$ up to period t, it has no incentive to deviate in period t, for it can at most obtain $\hat{w}$ which, by assumption, is less than the value of continuing with the proposed path.∎

<u>Proof of Proposition 13:</u> (i) Suppose $Z \subseteq \mathbb{E}(Z)$ and let $(w,\theta) \in Z$. We shall construct a SP that "delivers" (w,θ) as follows. Set $w_0 = w$, $\theta_0 = \theta$. Consider any period $t \geq 1$, and an arbitrary history h^{t-1}. To use an inductive step, assume that $(w_t(\mathbf{h}^{t-1}),\theta_t(\mathbf{h}^{t-1})) \in Z$. By hypothesis, $(w_t(\mathbf{h}^{t-1}),\theta_t(\mathbf{h}^{t-1})) \in \mathbb{E}(Z)$; hence there is $(\dot{m},\dot{x},\dot{h},\dot{w}',\theta')$ in E x Z s.t.(8.2)-(8.3) and (8.6)-(8.8) are satisfied. Set $\sigma_t(h^{t-1}) = \dot{h}$, and define $(m_t(\mathbf{h}^t)$, $x_t(\mathbf{h}^t)$, $w_{t+1}(\mathbf{h}^t)$,

$\theta_{t+1}(h^t))$ to be equal to $(\dot{m},\dot{x},\dot{w}', \theta')$ if $h_t = \dot{h}$, and equal to any solution to the problem (8.1)-(8.4) if $h_t \neq \dot{h}$ but $h_t \in CE^0_\pi$. If h_t is not in CE^0_π, set $(m_t(h^t), x_t(h^t)) = (0,0)$ and $(w_{t+1}(h^t), \theta_{t+1}(h^t)) = (w,\theta)$.

By induction, the strategy $\sigma_t(h^{t-1})$, and the allocation rule $\alpha_t(h^t) = (m_t(h^t),x_t(h^t))$ are well defined for all h^t. It can be easily checked that (α,σ) are a SP, and that it "delivers" (w,θ).

(ii) It suffices to show that $S \subseteq \mathbb{E}(S)$. This is easy and left to the reader. ∎

<u>Proof of Proposition 14:</u> (i) Let $S \subseteq Z_1 \subseteq Z_2$ be given and suppose that (w,θ) is in $\mathbb{E}(Z_1)$. To show that (w,θ) is in $\mathbb{E}(Z_2)$ it is sufficient to show that $BR(Z_1) \geq BR(Z_2)$. This follows from the definition of BR. Hence $\mathbb{E}(Z_1) \subseteq \mathbb{E}(Z_2)$. Now, $S \subseteq \mathbb{E}(Z_1)$ follows by applying the preceding result to $S \subseteq S \subseteq Z_1$ and noting that $S = \mathbb{E}(S)$. (ii) The proof is easy and left to the reader. ∎

<u>Proof of Proposition 15:</u> By Self Generation it suffices to show that $Z_\infty \subseteq \mathbb{E}(Z_\infty)$. Let (w,θ) be in Z_∞. By definition, (w,θ) belongs to $\mathbb{E}(Z_n)$ all n. Hence, for each n, there is $(m^n, x^n, h^n, w'^n, \theta'^n)$ in $E \times Z_n$ that satisfy (8.2)-(8.3), (8.6)-(8.7), and $w \geq BR(Z_n)$. Without lost of generality, assume that $(m^n, x^n, h^n, w'^n, \theta'^n)$ converges to some (m, x, h, w', θ') in $E \times \mathbb{W} \times \Omega$. Clearly (m, x, h, w', θ') satisfies (8.2)-(8.3) and (8.6)-(8.7). Moreover, it is easily shown that $(w', \theta') \in Z_\infty$.[11]

It remains to show that $w \geq BR(Z_\infty)$. Fix an arbitrary n. For $k \geq n$, $w^k \geq BR(Z_k) \geq BR(Z_n)$. Hence $w \geq BR(Z_n)$ for all n. Suppose that $w < BR(Z_\infty)$. Then there is an $h \in CE^0_\pi$ such that $BR(Z_\infty) = P(h;Z_\infty) > w$. But this implies a

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[11]Fix any n. For all $k \geq n$, $(w'^k,\theta'^k) \in Z_k \subseteq Z_n$. Hence $(w', \theta') \in Z_n$, all n.

contradiction, because $P(h;Z_n)$ must converge to $P(h;Z_\infty)$ for all $h \in CE^0_\pi$, as shown next.

Suppose that $P(h;Z_n)$ does not converge to $P(h;Z_\infty)$ for some $h \in CE^0_\pi$. Then there is an $\varepsilon > 0$ such that, given any N, there is $n \geq N$ such that $P(h;Z_\infty) - P(h;Z_n) > \varepsilon$. This means that for any $k = 1,2,\ldots$ there is $n(k)$ and a subsequence $(m, x, w', \theta')^{(n(k))}$ in $[0,m^f] \times X \times Z_{n(k)}$ such that $n(k) \geq n(k-1)$, (8.2)-(8.4) are satisfied, and $P(h;Z_{n(k)}) = u[f(x^{n(k)})] + v(m^{n(k)}) + \beta\, w'^{,n(k)}$ The subsequence $(m, x, w', \theta')^{(n(k))}$ can be assumed without loss of generality to converge to some (m, x, w', θ') in $[0,m^f] \times X \times Z_\infty$. This (m, x, w', θ') satisfies (8.2)-(8.4) and is such that $P(h;Z_\infty) \geq u[f(x)] + v(m) + \beta\, w' + \varepsilon$. But this contradicts the optimality of $P(h;Z_\infty)$. The proof is complete. ∎

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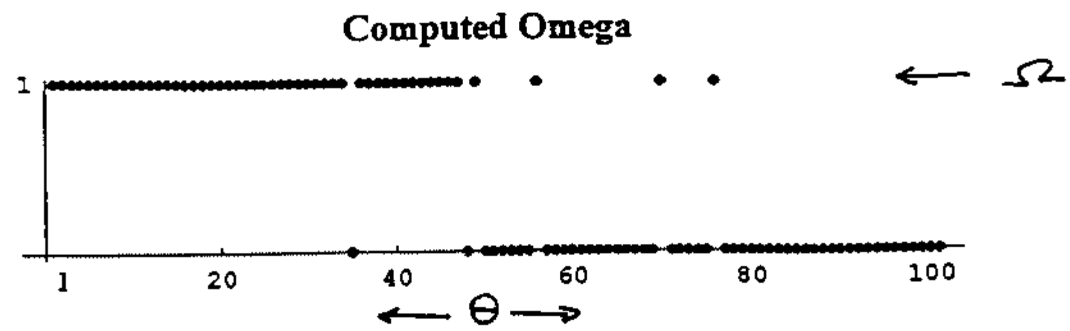

FIGURE 1

FIGURE 2

Computed S