Callable U.S. Treasury Bonds: Optimal Calls, Anomalies, and Implied Volatilities

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Abstract: Previous studies on interest rate derivatives have been limited by the relatively short history of most traded derivative securities. The prices for callable U.S. Treasury securities, available for the period 1926–95, provide the sole source of evidence concerning the implied volatility of interest rates over this extended period. Using the prices of callable, as well as non-callable, Treasury instruments, this paper estimates implied interest rate volatilities for the past seventy years. Our technique for estimating implied volatilities enables us to address two important issues concerning callable bonds: the apparent presence of negative embedded option values and the optimal policy for calling these, and similarly structured, deferred-exercise embedded option bonds.

In examining the issue of negative option value callable bonds, our technique enables us to extend significantly both the sample period and sample breadth beyond those covered by other investigators of this phenomenon and to resolve the inconsistencies in their results. We show that the frequency of mispriced bonds is time-varying and that there also exist irrationally underpriced bonds. Critically, both anomalies are shown to be related to volatility-insensitive, away-from-the-money bonds.

In contrast to the naive call decision rules suggested by previous authors, we develop the option-theoretic optimal call policy for deferred-exercised "Bermuda"-style options for which prior notification of intent to call is required. We do this by introducing the concept of "threshold volatility" to measure the point at which the time value of the embedded call option has been eroded to zero. By using this concept, we address the valuation of such bonds and document the frequent optimality of the Treasury's past call decisions for U.S. government obligations.

JEL classification: G13, H63

Key words: Optimal call decision, implied interest rate volatility, callable U.S. Treasury bond mispricing

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1 Introduction

Interest-rate contingent claims have grown substantially in variety, open interest, and trading volume in recent years. Concomitant with these developments have come a number of interest-rate valuation models. The equilibrium-based models of Cox, Ingersoll, and Ross (1985), and Vasicek (1977) have been adapted by Black, Derman, and Toy (1990), Black and Karasinski (1991), and Hull and White (1990) to fit arbitrary term structures of interest rates and volatilities. Arbitrage-based approaches, taking the term structure as a starting point, were developed by Ho and Lee (1986) and Heath, Jarrow, and Morton (HJM) (1990, 1992). Chan, Karolyi, Longstaff, and Sanders (1992) provide an empirical comparison of several models of deterministic volatility. These models have been extensively tested using exchange-traded and over-the-counter interest-rate derivatives which has necessarily restricted the analysis to recent years when these instruments were traded [e.g., Amin and Morton (1994) and Flesaker (1992)]. One interest-rate dependent contingent claim, however, has a long history that has not heretofore been analyzed: callable U.S. Treasury bonds.

We implement a variant of the Black and Karasinski (1991) interest-rate model to value callable bonds conditional on an assumed volatility for the short rate process. By finding the volatility that fits the observed bond price, the implied volatility is computed. Taking appropriately weighted cross-sectional averages of implied volatilities, we may analyze the history of interest-rate volatility over the 1926–1995 period, a far more extensive period than is possible with other derivative instruments. Using cross-sectionally derived estimates of interest-rate volatility, we document the changes in volatility through time.

The use of callable U.S. Treasury securities requires that we confront the issue of possible irrational pricing of these securities. A number of authors, beginning with Longstaff (1992), have found negative option values implicit in the market prices of callable U.S. Treasuries to varying degrees. Negative option value bonds do not have implied volatilities. This raises the twin questions of whether callable bonds are rationally priced and whether the cross-sectional averages of their implied volatilities are meaningful. The meaningfulness of implied volatility calculations also hinges, in part, on whether or not the U. S. Treasury is rational in exercising the embedded call option. Bonds with negative option values, and those with implausibly high implied volatilities, are shown to contain little reliable volatility information as measured by the sensitivity of bond value to changes in volatility. The negative option values noted by Longstaff and others exist predominantly when the option is deep in-the-money and may be attributed to pricing errors well within the normal range found in non-callable Treasury securities. *Inter alia* we extend both the sample period and sample breadth beyond that permitted by the techniques employed by other investigators of this phenomenon and resolve the inconsistencies in their results.

Lastly, we derive the optimal call policy for callable U.S. Treasury securities and then examine whether the Treasury's call decisions in the past have been consistent with such an optimal rule. Longstaff's (1992) analysis of callable Treasuries, based on Vu (1986), overlooks certain critical features of Treasury call provisions as well as option theory, a problem shared by other authors. These, and other, authors base their call/no-call decision on the market price of the callable bond. We argue that the market price is subject to distortions and that the value, based on an interest-rate model incorporating both interest rate movements and volatility, is needed to make an informed decision. Further, several authors, including Longstaff (1992), have ignored the 120-day call notification period, which the Treasury must provide to holders of callable bonds.[1]

Our optimal call rule is based on the principle of minimizing the expected value of the liability on the call date; either the principal to be repaid if the bond is called, or the expected value of the callable bond if it is not. To do this we introduce the "threshold volatility," or volatility at which the issuer should be indifferent between calling and not. By comparing this value to prevailing levels of implied volatility at the time, derived from our analysis of

[1]The predecessor paper, Vu (1986), also ignores the less-onerous requirement for corporate bonds, which typically have a prior notification period of between 30 and 90 days.

cross-sectional implied volatilities of all callable bonds, we are able to determine the optimal decision.

Analysis of past Treasury call decisions shows that most past calls decisions have been optimal. Those that have not may sometimes be explained using the naive call decision rules suggested by Vu, Longstaff and others. In a few cases, the call decisions were clearly irrational under any rule. Our valuation model, which embeds the optimal call decision rule, demonstrates that callable bonds can rationally trade at substantial premiums to par as observed, but not explained, in Longstaff (1992).

The remainder of the paper is organized as follows. Section 2 discusses the empirical techniques used for modeling interest rate movements, computing fitted values of callable bonds, and estimating implied volatilities. The time-series properties of implied volatility estimates are discussed and analyzed. Section 3 addresses in detail the issue of negative option values. We show that the effect is present throughout our sample period but that the frequency of mispriced bonds is time-varying. This, together with an analysis of how violations are counted, is shown to resolve the sometimes conflicting results of previous, more limited studies. Section 4 develops the optimal call policy for callable bonds and examines the history of U.S. Treasury call decisions in the light of this and the naive rules. Section 5 concludes.

2 Calculation of Implied Volatility

2.1 Stochastic Models of Interest Rate Movements

As reviewed above, the finance literature has provided an abundance of stochastic interest-rate models. We propose to use the following risk-neutral stochastic model for spot interest rate movements:[2]

$$dr = \mu_t \, r \, dt + \sigma \, r \, dz, \tag{1}$$

[2] Any reference hereafter to expectations $E(\cdot)$ or variance $\mathrm{Var}(\cdot)$ constitutes a reference to the variable's risk-neutral distribution.

where

$dr \equiv$ the change in the short-term rate of interest r;

$\mu_t \equiv$ the expected change in dr/r; the μ_t's for $t = 1, \ldots, T$ are chosen to precisely match the observable term structure of interest rates;

$\sigma \equiv$ a measure of the standard deviation; thus, $\mathrm{Var}(dr) = \sigma^2 r^2 \, dt$;

$dz \equiv$ is the stochastic Brownian process, with $E(dz) = 0$ and $\mathrm{Var}(dz) = dt$.

In implementing the above model, we precisely match the term structure of interest rates.[3] As is well-known, models such as equation (1) are widely used both in academia and industry. These models are not, however, without their academic critics. Specifically, in discrete time, the use of the vector $\boldsymbol{\mu}_0 \equiv [\mu_1, \mu_2, \ldots, \mu_T]'$ (designed to match the time-0 term structure exactly) typically gives rise to intertemporal inconsistency. Intertemporal consistency would require that the vector of the time drift parameters at time 1, $\boldsymbol{\mu}_1$, equal that at time 0 less μ_1, i.e., $\boldsymbol{\mu}_1 \equiv [\mu_2, \mu_3, \ldots, \mu_T]'$. This property is typically not satisfied with arbitrary term structures for times 0 and 1. Nevertheless, we adopt the usage of equation (1) due to its important ability to exactly match the observable term structure of interest rates. In so doing, we make maximal use of the information available in the term structure of interest rates and are able to match the prices of underlying non-callable assets.[4]

2.2 Data

The data for this study are taken from the CRSP monthly Government Bonds tapes. In order to perform our tests, we require an estimate of the term structure of interest rates, which estimate we obtain from the prices of non-callable Treasury securities. Bills of less than one month to maturity and notes and bonds whose maturity was less than the longest maturity bill in the same month were excluded from the term structure estimation subsample. These

[3] This means that under the risk-neutral expectation generated by eq. (1), for given $\sigma, \boldsymbol{\mu}_0$ is chosen so that

$$E\left(\exp\left\{-\int_0^t r_s \, ds\right\}\right) = \mathrm{PV}_t, \qquad \forall t = 1, \ldots, T$$

where PV_t is the price of a zero-coupon bond of maturity t.

[4] No single term structure is able to match exactly the prices of all non-callable Treasury securities, as shown in Bliss (1996).

bills, notes and bonds were used to create term structures using two different methods, as outlined below. In addition, a sample of coupon STRIPS prices was obtained from Merrill Lynch to create a third estimated term structure.[5]

A subsample of callable bonds was also extracted from the CRSP files. To avoid using term structure estimates which extrapolated beyond the available non-callable instruments used to create the term structures, we excluded those callable bonds whose maturities are not spanned by their non-callable counterparts. To avoid having additional embedded options beyond the call option being analyzed, we excluded callable bonds which were also flower bonds.[6] Figure 1 presents the availability and maturities of callable U.S. Treasury bonds before and after applying these filters for the period 1926–1995. In total there were 16,521 observations on 88 issues during the sample period. The filters reduced the usable sample to 8880 observations on 74 separate issues. This represents, by far, the largest sample and longest sample period of callable bonds studied to date.

The usable issues break into three periods. Prior to 1960, there was a large sample of callable bonds available for analysis; however, this is a period of little variation in interest rates. Between 1960 and 1980, there are many callable bonds, but few non-callable bonds. Thus, in this period we cannot construct a non-callable term structure. In addition, most of the callable bonds issued between 1945 and 1970 were flower bonds. Fortunately, few callable bonds were within their call period during this time, so our analysis of the Treasury's call history is not impacted materially. The post-1980 period saw the re-issuance of long maturity non-callable bonds and so the previously existing callable bonds could again be analyzed. Issuance of flower bonds ceased in 1965.

[5]STRIPS stands for "Separate Trading of Registered Interest and Principal Securities." Several studies have shown that stripped principal payments contain liquidity effects. Stripped coupons are free of these distortions and provide a more complete term structure.

[6]Flower bonds have an embedded put option which permits a person's estate to redeem the bonds at face value in payment of estate taxes if the decedent held the bonds at time of death, or six months before, depending on the issue. Flower bonds tend to trade at below-market yields-to-maturity and are presumably held by persons expecting to exercise the put option in the near future.

2.3 Numerical Implementation Techniques

To ensure that our analysis is robust to measurement errors in the estimated term structure, we estimate the term structure of interest rates using three alternative procedures:

1. Yields on stripped Treasury coupons. C-STRIPS data were available beginning in February 1985.

2. Implementation of the Fama-Bliss (1987) method of extracting the pure discount rate curve from a sample of non-callable bonds.[7] The Fama-Bliss term structures were computed over the entire post-1925 sample period.

3. Implementation of the Nelson-Siegel-Bliss method of extracting the pure discount rate curve from a sample of non-callable bonds. See Bliss (1996) for details. Sufficient density of non-callable bonds to estimated Nelson-Siegel-Bliss term structures was available beginning in 1970.

See Appendix A for a description of the latter two estimation methods.

This produces three alternate estimates of the current risk-free present value function, PV_t for $0 \leq t \leq T$; the value of T will be dictated by the maturity date of the longest-maturity non-callable bond in each monthly sample. Given the term structure, Jamshidian (1991) provides the forward-induction procedures used for estimating the vector of drifts $\boldsymbol{\mu}_0$ in an efficient manner in the context of a binomial tree for the short-term rate of interest.

The binomial tree is built through appropriate calculation of the centrality parameters of the tree for all future periods. Thus, for a period of length Δ, calculating the centrality parameter for the second time interval requires solving the nonlinear equation for u_Δ as given by

$$\mathrm{PV}_{2\Delta} = \mathrm{PV}_\Delta \, E\left(\exp\left\{-u_{2\Delta} e^{\sigma \tilde{z}}\right\}\right),$$

where the discretized Brownian motion $\tilde{z}$ has $E\left(\tilde{z}\right) = 0$ and $\mathrm{Var}(\tilde{z}) = \Delta$.

[7] The CRSP Fama-Bliss files extend only to a maturity of five years. The same underlying procedure was utilized to extend our calculations to the maturity date of the longest non-callable bond.

A recursive procedure is then used to construct the tree for successive time intervals. Appendix B provides a numerical example of the implementation of this procedure for the lognormal interest-rate distribution in eq. (1). In the implementation of the interest-rate tree, we utilize a six-month time step for Δ. However, we need to account for the time interval from the quote date to the first coupon date, which will generally be different from six months. Appendix B also demonstrates the numerical procedure implemented to account for the distinct length of the first time interval from subsequent ones.

Once we have completed the calculation of $u_{i\Delta}$ for all periods i, for a given σ we can calculate the value of the callable bond through backward induction. The implied volatility is chosen to equate the model's bond value with the market price.

2.4 Implied Volatility Results

To check the robustness of the estimated implied volatilities to the underlying term structure used to build the binomial tree, the summary statistics for the several estimates and their correlations are presented in Table 1. This table covers only the latter period when all three term structures were available. (During this period there were insufficient short-maturity callable bonds to compute meaningful summary statistics.) As discussed below, it was not always possible to estimate a usable implied volatility. Table 1 excludes those cases where the option value was negative or the implied volatility implausibly high.

There is substantial agreement across term structure estimation methods in the resulting implied volatilities for intermediate maturities. At longer maturities, the implied volatility levels differ somewhat across term structures but are still highly correlated. The lognormal interest-rate process assumed in the interest-rate model and the binomial-tree implementation is more sensitive to changes in interest rates at longer horizons.

Finally, the term structure-independent result that longer-maturity bonds have higher implied volatilities suggests the existence of a term structure of volatilities. This could not be captured by the methodology employed in this study.[8]

The vega, or partial derivative of callable bond price with respect to a change in implied volatility, provides a measure of the information content in the implied volatility. A small measurement error in a bond with a low vega will result in a large change in the measured implied volatility. For this reason, we discarded issues with vegas of less the 0.02 and then weighted the remaining implied volatilies by the associated vegas when averaging them.

Multiple observations of rationally priced callable bond prices with vegas exceeding .02 are available only for the period 1926–1946 and 1987–1995. In the period 1947–1986, such usable data occurred only infrequently.

Figure 2a presents usable callable bonds' vega-weighted implied volatilities for the period 1926–1955 using the Fama-Bliss method for the estimation of the term structures of interest rates. A cyclical pattern of variation in the implied volatilities is evident through 1949, along with occasional months where the weighted implied volatility appears to jump temporarily. Between 1926 and 1933, implied volatilities lie in a range between 5% and 30%. From 1933 through 1948, implied volatilities were considerably higher, in the range from 20% to 70%. Between 1949 and 1956 the consistent pattern breaks down and the implied volatilies show little consistency other than lying below 50%.

Figure 2b presents the vega-weighted implied volatilities for the period 1978–1995 for each of the three estimated term structures. Between 1978 and 1988 there is little consistency in implied volatilities, either across term structure estimation methods in a given month, or across months for a given term structure estimation method. Fortunately, in this period no bonds were currently callable. Post-1987 the pattern becomes more consistent with implied

[8] Interestingly, it also suggests an upward-sloping term structure of volatility which is contrary to received wisdom and the finding of other studies. Two explanations may obtain. First, the non-mean reverting interest-rate process may lead to numerical problems at long horizons. Second, longer maturity bonds differ systematically from intermediate maturity (and hence older) issues in terms of coupon rate and hence moneyness; so the two samples of implied volatilities may not be strictly comparable if a "smile effect" exists.

volatilities varying in the range of 6% to 18% and showing common time-series variation across estimation methods.

3 The Callable Treasury Paradox

The literature on apparent mispricing of callable U.S. Treasury bonds, the so-called "Callable Treasury Paradox," is extensive; at least six articles have been written on the subject with varying conclusions. In this section we analyze the issue for two reasons. First, our data set and method of determining which bonds are mispriced allows a much more extensive analysis, in time and bonds covered, than has been possible in the past. Second, our subsequent analysis of implied volatilities necessarily ignores mispriced bonds. We therefore wish to establish that the information in these excluded bonds regarding general levels of implied volatilities is minimal.

We begin by showing that the sometimes conflicting results in existing papers on the subject may be resolved by attention to differences in sample periods, bonds included, and methods of measuring the frequency of violations. We then demonstrate that the Treasury Paradox has substantial time variation and that there is a systematic pattern in the bonds found to be mispriced.

3.1 Arbitrage Bounds on the Pricing of Callable Bonds

There are two well-known arbitrage bounds for callable bonds that should be satisfied by any no-arbitrage valuation model. These pertain to the relationship between the market price of the callable (at par) bond B, on the one hand, and the values of two hypothetical non-callable bonds (with identical coupons) maturing at the first call date and the maturity date. We denote the values of these bonds as S and L, respectively. It can be shown that, in the absence of arbitrage,

$$B \leq \min\{S,\, L\}\,. \tag{2}$$

Intuitively, inequality (2) is motivated by the fact that the option not to precommit on the call/no-call decision is valuable.[9] Any violation of relation (2) gives rise to an arbitrage opportunity and hence cannot be rationally explained by an arbitrage-free valuation model.

The callable bond can be thought of as a long bond, L, with a call option, C, whose value is non-negative. Since the option belongs to the bond issuer who is short the underlying long bond, the callable bond's value must be $L - C \leq L$, since $C \geq 0$. If $B > L$, a long position in the long bond coupled with a short position in the callable bond will earn arbitrage profits equal at least to $B - L$. If interest rates increase, so that both bonds trade at a discount, do nothing. Since their coupon and principal amounts exactly offset each other, net future cash flows are zero. However, if at any call date interest rates have declined sufficiently so that the long bond is selling at a premium, then selling the long bond and calling the shorted bond will earn an additional profit equal to the premium.

The callable bond can also be thought of as a short bond, S, maturing on the next call date, together with a put option P allowing the issuer to put the bond back to the bond holder at its face value on the call date, effectively extending the maturity. The value of the embedded put option is non-negative, so the callable bond must also be valued at $S - P \geq S$. To profit from a negative implied put option value, if $B > S$, sell short the callable bond and buy the short bond, pocketing the difference. On the first call date, S will pay \$100, which may then used to call the shorted bond.

3.2 Previous Studies

Longstaff (1992) first examined the question of apparent negative option values in U.S. Treasury bonds using a subset of five callable U.S. Treasury bonds over the period June 1989 through September 1990. Longstaff's method was to construct a synthetic matched-maturity,

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[9] Jordan, Jordan, and Jorgensen (1995) point out that this relation must hold for all call dates, not merely the first one. In practice, this is unimportant since the prices of similar-coupon rate non-callable bonds is almost always monotonic in maturity (increasing or decreasing depending on the shape of the term structure). It is highly unlikely that a callable bond will satisfy $B \leq \min\{S, L\}$ but violate the arbitrage condition with respect to an intermediate call date. Nonetheless, our methodology for detecting mispricing implicitly accounts for all the intermediate call dates, not just the one at the extremes.

matched-coupon non-callable bond, L in our notation, using an existing matched-maturity non-callable bond together with a single coupon STRIPS maturing on the same date as the callable and non-callable bonds.[10] By combining the non-callable coupon bond with the STRIPS coupon in an appropriate portfolio, the coupon rate, as well as the maturity, of the callable bond can be matched. The call option value is then just the difference in the prices of the callable bond and the synthetic bond. Longstaff uses bid-asked midpoint prices in his calculations. He finds that, on average, in 61.5% of the observations the implied option value is negative; i.e., $B > L$. Percentages across individual bonds varied from 14.7% to 95.6%. Longstaff considers a number of possible explanations for this surprising phenomenon, but finds none satisfactory.

Edelson, Fehr, and Mason (1993) examine a subset of 14 callable bonds (including the five in Longstaff's sample) in the period March 1985 through September 1991. Edelson et al. build synthetic bonds in the same manner as Longstaff, but use bid prices for the callable bond and asked prices for the bonds in the synthetic portfolio. In addition to looking at the call prices implicit in the maturity-matched synthetic bond, Edelson et al. examine the put options implicit in prices of synthetic bonds maturing on the first call date. This permits examining options embedded in an additional 12 bonds.

Edelson et al. find negative call option values in 56.4% of their observations, with bond-specific frequencies varying from 9.9% to 87.8%. Taking account of the use of spreads, which will tend to price the synthetic higher and callable lower than using mid-prices, these results are comparable to Longstaff's. Put option violations, $B > S$, occurred in 20.2% of cases and varied from 0.02% to 40.9% across bonds. Edelson et al. do not report, and it is impossible to infer, violations of $B \leq \min\{S, L\}$ in the three cases where it was possible to create synthetic S and L for the same callable bond.

[10]Sufficient non-callable bonds are purchased to match the coupon payment amounts of the callable bond, and then the STRIPS are used to adjust the principal. The technique of building synthetic bonds using existing matched-maturity instruments limits the callable bonds that may be studied to a small fraction of those outstanding. Most studies of the Treasury paradox, Carayannopoulos (1995), Jordan, Jordan, and Kuipers (1996) and this study excepted, have used a variant of this technique.

Both Longstaff and Edelson et al. report persistence in the occurrences of negative option value bonds. Their results suggest that random measurement errors alone are not sufficient to account for the existence of negative option values.

Jordan, Jordan, and Jorgensen (JJJ) (1995) examine seven callable bonds over various periods, depending on the method of constructing synthetic bonds, ending in April 1993. They use monthly data and bid-asked prices to account for the spread. The longest sample begins November 1984, though most observations are post-1990. JJJ use a variety of methods to construct the synthetic bonds: (1) a matched-coupon non-callable bond maturing on a call date, (2) a combination of a non-callable coupon bond and a coupon STRIPS ("striplets"), and (3) two matched-maturity non-callable coupon bonds ("triplets").

JJJ treat each embedded option as a separate observation. Thus, a not-currently-callable callable bond with a 5-year call period is considered to provide 10 independent put value option observations, with associated non-callable synthetic bonds maturing on each of the feasible call dates, and one call option value observation with the associated non-callable synthetic bond having the same maturity as the callable bond.[11] Because JJJ usually report only the fraction of "options" mispriced, it is impossible to determine the fraction of bonds which are mispriced. Therefore, their results are difficult compared to Longstaff or Edelson et al.

JJJ find that only a small fraction of embedded options are mispriced; 3.24% with the matched-coupon method, 2.26% with the triplets method and "near maturity" methods. However, when looking only at embedded call options using the triplets method, the percentage negative increases to 28.9% for mid-market prices and 7.23% when the spread is accounted for. This analysis, however, only covers two callable bonds.

JJJ conclude that previous finding of significant incidence of negative option values is due, in part, to tax effects in the pricing of STRIPS and that actual negative option values,

[11] This creates a distortion in the number of "mispricings" reported by permitting a single callable bond to contain both fairly and irrationally priced options. Since these options are not independent, exercising a call option early extinguishes the "remaining" put and call options. We prefer to define mispricing in terms of bonds: if any of the embedded options has a negative value, the bond is mispriced.

when they occur, are too small to give rise to trading opportunities; that is they find no anomaly. However, the tax argument relies implicitly on the unlikely absence of a tax-exempt entity which can ignore the tax effects and arbitrage the pricing errors.

Carayannopoulos (1995) examines all 23 callable bonds outstanding over the period July 1989 through June 1993. Synthetic bonds are constructed entirely from coupon STRIPS. Carayannopoulos constructs both short and long synthetic bonds, tests for violations of the condition $B \leq \min\{S, L\}$, and finds violations in 28.6% of cases when mid-market prices are used and 17.8% when spreads are accounted for.

Jordan, Jordan, and Kuipers (JJK) (1996) re-examine Carayannopoulos' results using the same method, 22 callable bonds, and data for the period January 1990 through December 1994. JJK find 18.9% violations of $B \leq L$ (negative call options) when using spread adjusted comparisons (21.3% using bid prices only), and 4.0% (9.0%) violations of $B \leq S$ (negative put options). Depending on the overlap of bonds violating $B \leq L$ and $B \leq S$, the incidence of violations of $B \leq \min\{S, L\}$ could be anywhere from 18.9% to 22.9%, actually higher than Carayannopoulos' 17.8%. However, by counting the violations as independent outcomes and averaging them, JJK report a violation rate of 8.9% and thereby assert that violations are less frequent than reported by Carayannopoulos. Both Carayannopoulos and JJK find that the incidence of violations are related to the premium/discount status of the callable bond.

To summarize, past studies have found varying frequencies of mispriced callable bonds. Only Jordan, Jordan and Jorgensen have dismissed the phenomenon, but their method of counting violations leaves their conclusion in doubt. Because, in part, of their reliance on STRIPS to construct synthetic bonds, none of these studies has covered a very long time period.

3.3 Long-Term History of the Treasury Paradox

In contrast to these studies, our sample covers the period 1926 through 1995. Unlike Longstaff, Edelson et al., and Jordan, Jordan and Jorgensen, our methodology does not depend on finding like-maturity non-callable bonds from which to construct synthetic non-callables. Unlike Carayannopoulos and Jordan, Jordan, and Kuipers, we do not rely exclusively on the existence of STRIPS prices either. We construct synthetic non-callable bonds using two different estimates of the term structure of interest rates estimated from the full range of non-callable bills, notes and bonds available on each observation date. For the post-1985 period, we use STRIPS prices as well. For the post-1985 period the results of our analysis are broadly in agreement regardless of what term structure is used—differences in the incidence of violations are small. Because the Fama-Bliss term structures cover the longest sample period, those results are reported here.

Rather than constructing the non-callable equivalent and comparing its price to that of the callable bond, we use the term structures and our interest-rate model to estimate implied volatilities. A finding of a negative option value is not interest-rate model-specific because a violation of inequality (2) depends only on values of S and L, which come from the term structure rather than a dynamic interest-rate model. Therefore, when an implied volatility of zero results in the bond being over-priced relative to the observed market price, it is equivalent to the option value being negative. In addition, when an implausibly high implied volatility is insufficient to lower the fitted price to the observed value, we conclude that the callable bond is irrationally underpriced. Thus, unlike previous studies, we are able to identify both over- and under-priced callable bonds. When checking for negative option values, the bid price was used to compute the implied volatility, thus biasing against finding a violation. When looking for "huge" violations, the asked price was used. The Fama-Bliss term structure estimation used mid-market prices, the Nelson-Siegel-Bliss estimation method incorporated the spread, and for STRIPS the mid-market prices were used.

Using term structures of interest rates estimated via the Fama-Bliss method, we determine which callable bonds are overpriced ("negative" option values), rationally priced ("good" implied volatilities), and irrationally underpriced ("huge" implied volatilities). Figure 3 plots the annual frequency of these outcomes. It is apparent that mispricing of callable bonds was prevalent in one form or another prior to 1992. The irrationally underpriced bonds occurred mainly between 1935 and 1950, periods during which interest rates were low and showed little variability. The degree of actual mispricing is not usually large. In fact, very small price changes, on the order of the fitted price errors for out-of-sample non-callable bonds observed in Bliss (1996), would have been sufficient to obtain reasonable implied volatilities.

Our results indicate that there is considerable time variation in the frequency of callable bonds with negative option-values, a point not previously noted. The frequency of negative option value callable bonds has declined in recent years. This provides an explanation for the differential results reported in previous studies. Leaving aside the method of counting violations, the differences between results reported by different authors is evidently related to the sample periods used. Later studies included more recent data when violations were rare. Looking at sub-periods used in previous studies, we find:

Previous Study	Period Covered	Percentage of Violations	Our Findings for Same Period
Edelson, Fehr, and Mason	Mar'85–Sep'91	56.4	38.1
Longstaff	Jun'89–Sep'90	61.5	25.2
Carayannopoulos	Jul'89–Jun'93	17.8	20.4
Jordan, Jordan, and Kuipers	Jan'90–Dec'94	18.9	14.3

Our sample of bonds in each period is not always identical, which accounts in part for differences. For instance, for the five bonds used by Longstaff, we find 40% violations in the period of his study, suggesting that his results are due in part to sample selection. However, overall our results confirm the time dependence of the differences found in other studies.

Using term structures of interest rates estimated via the Fama-Bliss method, we next examine the classification of callable bonds' implied volatilities into "negative," "good," and "huge" in the 1985 through 1995 period in Table 2. The results are broken down by maturity, the moneyness of the callable bond's forward price, and by sensitivity of the option value (and hence callable bond value, V) to volatility, or vega $\equiv \partial V/\partial\sigma$. The rationale for examining an option's vega lies in vega's relation to the informativeness of the estimated implied volatility. A low-vega option will require large changes in implied volatility to accommodate small changes in option value, and thus will be particularly sensitive to measurement errors in market prices. A high-vega option, on the other hand, will be more robust to measurement errors. More reliance may, therefore, be placed on implied volatilities with higher associated vegas.

An option's vega depends on its moneyness and time-to-expiration T and is non-monotonic in both.[12] Thus, in considering the classification of callable bonds' implied volatilities into "negative," "good," and "huge," we look both at the interest-rate model-specific vegas and the interest-rate model-independent characteristics of moneyness and time-to-maturity.

Table 2 demonstrates that the "good" volatilities are overwhelmingly medium- and high-valued vegas, and hence informative. In contrast, the "negative" and "huge" implied volatilities occur predominantly in low-vega bonds, where the volatility estimates, if they were possible, could not contribute meaningfully to our later analysis. Further, the "huge" implied volatilities are overwhelmingly long-dated in-the-money-forward bonds. These results are also not term structure estimation method-specific.

These pricing errors may well be a consequence of bonds with low vegas and low option time-values being more sensitive to market imperfections such as tax-clientele or tax-timing option effects. With low vega and low option time-value, small pricing errors will cause the bond to exhibit "negative" or "huge" implied volatilities. Recalling that the prices of non-callable U. S. Treasury bonds also contain pricing "errors" (in that they cannot all be fitted by a unique term structure of interest rates), the prevalence of low-vega bonds in the "negative" and "huge" classifications is consistent with callable bonds' pricing errors displaying behavior similar to that of their non-callable counterparts.[13]

Our analysis of the data underlying Figure 3 and Table 2 indicates that the "huge" volatilities may arise from (at least) three possible sources. First, short maturity bonds combined with a fixed-interval (six months) binomial lattice produce an insufficiently dense distribution of future bond prices. Second, long-dated callables are priced under an interest-rate distribution that has sparse density at reasonable interest rates; the remaining few

[12]To see this, consider the well-known case of a call option on stock. For such securities, vega peaks at-the-money-forward, per Cox-Rubinstein (1985, p. 225), and declines as the price of the underlying asset moves away from the strike price. As a function of time-to-maturity T, vega starts at zero for $T = 0$, when a stock option's value is $\max\{S - K, 0\}$ and thus not dependent on σ, rises to a maximum, and then falls to zero as $T \to \infty$, when the option's value reduces to the σ-insensitive S.

[13]Bliss (1996) has shown that the fitted price errors of out-of-sample non-callable bonds increase with maturity.

nodes at the center of the tree have insufficient density to produce meaningful changes in the bonds' present values as σ changes. The consequence of both these phenomena is to produce extremely low vegas, with bond values that are insensitive to changes in volatility. Finally, we observe a number of intermediate-maturity, low-coupon bonds, primarily in the 1950s, whose prices appear unreasonably low and cannot be attributed to numerical problems. Their (reasonable) vegas indicate a price sensitivity to volatility changes, but even "large" volatilities are unable to explain the large option value.

For the purpose of this study, these results are fortuitous. We are not interested in estimating implied volatilities for options far away-from-the-money. Indeed, it is most difficult to do so: the option's vega (defined as $\partial V/\partial\sigma$, at the σ where model value equals market price) will be extremely low, rendering the implied volatility estimate unreliable. Thus, we will not be concerned with having to discard those observations for which the implied volatility could not in any case have been reliably estimated.

4 The Optimality of the Treasury's Call Policy

We begin by reviewing the past literature on when a callable bond should be called and point out the fallacies of these recommended rules. Next we develop the optimal call rule based on option theory and the institutional framework in which calls take place. Lastly, we analyze the optimality of the Treasury's past call policy by examining the decisions made at every event date when the Treasury could have called a bond in the past.[14]

The pertinent institutional details are that the indenture provisions for (most) callable U.S. Treasury securities call for a 120-day notification period prior to a call. Most callable Treasury bonds are call protected until five years prior to maturity, after which the actual call can take place only on a coupon payment date, i.e., twice per annum.[15]

[14] Bühler and Schultze (1993) investigated the German government's call policy and concluded that rational call opportunities were frequently missed. Not surprisingly, they also document the presence of implied negative option values in the German bond market.

[15] Bliss and Ronn (1996) note exceptions to these norms, mostly in the first half of this century. They also discuss in intuitive terms the rules developed rigorously below.

4.1 Previously Recommended Call Rules

Vu (1986), studying callable corporate bonds, argued that callable bonds should be called when their market price first exceeds the call price. We will refer to this as the "naive rule" for reasons which will be made clear shortly. Vu asserts that this will minimize the value of the embedded non-callable bond and thus maximize the value of equity. Longstaff (1992) adopts this rule to the Treasury context and asserts that callable U.S. Treasury bonds should be called when their price exceeds par (the call price for Treasury bonds). Longstaff then notes:

> "One particularly surprising result ... is that the callable bond price is often substantially above the call price of 100 at the time the Treasury calls the bond. On average ... 61 cents (based on a par value of \$100) ... (p. 584)"

Longstaff also notes bonds trading above par that are not called. Addressing the call notice period, Longstaff concludes that "[n]evertheless the call notice period cannot account for an average premium of 61 cents over par." Jordan, Jordan, and Jorgensen (1995) also assert that calling a U.S. Treasury bond as soon as it reaches par is optimal (p. 160).

One obvious short-coming of the naive rule is that it ignores the effect of the notice period. A call cannot be executed immediately. Edelson, Fehr, and Mason (1993) modify the naive rule to calling when the callable bond price exceeds the present value of par plus the next coupon, in effect "call when $B \geq S$."[16] Jordan, Jordan, and Kuipers (1996, footnote 2) and Jordan and Jorgensen (1996, p. 11) assert a similarly motivated, though perhaps somewhat imprecise, rule that the Treasury should call when the price of the callable bond exceeds the price of a 4-month Treasury Bill on the notification date.

Unfortunately, the naive rule, in both its original and modified forms, is wrong for two key reasons:

1. The lack of informativeness of the callable bond's current market price, B;

2. It ignores the time-value component of the option value.

[16] Note that it is quite possible for $100 < B < S$. In this case, the naive rule would say "call" and the modified rule "do not call." Similarly it is possible, when the term structure is downward sloping, for $S \leq B < 100$ with the opposite results.

The current market price of the callable bond is of little use in deciding what to do. If the bond is marginally in- or out-of-the-money, the Treasury has relatively little, and imprecise, information on which to act. Further, as we have seen, a single bond may be mispriced, due to minor market frictions or the possible impact of taxation.[17] The current market price also embeds the market's expectations of what the issuer will do. Using these expectations as a basis for deciding what should be done leads to a circularity of reasoning.

4.2 The Optimal Call Decision Rule

The correct decision rule is to compare the present value of the bond if it is called (par plus the final coupon payment) with the expected value if it is not called (the next coupon and the expected value of the remaining callable bond at that future date). The former may be easily valued using today's term structure. The latter requires an interest-rate and volatility model. The naive rules contain neither.

In order to base the call/no-call decision on the wide array of information available in the prices of traded bonds—i.e., in the fair value, V, of the bond—we estimate the term structure of interest rates from the prices of non-callable securities, and then use the callable bond's coupon rate to assess the threshold volatility, the volatility at which the call option time-value would have eroded to zero, and thus the bond should be called (or purchased, if $V < S$). Setting the callable bond price to just below S, we solve for the implied volatility at that price. This threshold volatility is denoted σ_T. We show that when σ_T is "large" in the sense defined below, the bond should be called.

Consider now a bond that is approaching the end of its call protection period. Since Treasury securities typically pay interest mid-month, and given our *end*-of-month price observations, we constrain the Treasury to decide whether or not to call no later than 4.5 months prior to a coupon payment date. Clearly, any decision at 5.5 or 6.5 months prior to

[17] An analogy from equity markets may be useful: European-style options, as they near their expiration dates, may sell at a slight discount to their intrinsic value due to the tax implications of exercising a call option. Tax effects in bond markets have been documented by Schaefer (1982), Ronn (1987), Bliss (1996), and Jordan and Jordan (1991) and hence constitute potential "non-PV" effects in bond prices.

the next coupon payment would be suboptimal. At 4.5 months prior to the next call date, the Treasury should give call notification if both of the following two conditions are satisfied:

1. <u>On the notification date, the option is in-the-money forward.</u>

   This will be true if the forward price of the long bond L as of the next call date, observed on the notification date, is greater than par, or equivalently the long bond L is trading at a premium to the short bond S. Recall that L was previously defined to be an otherwise-equivalent non-callable bond that matures on the callable bond's final maturity date. Let $S_{4.5\,m}$ be an otherwise-equivalent bond that matures in 4.5 months. This first necessary condition specifies that if

$$\mathrm{FP}_{4.5\,m} \equiv \frac{L - (S_{4.5\,m} - 100\,\mathrm{PV}_{4.5\,m})}{\mathrm{PV}_{4.5\,m}} > 100 \Longleftrightarrow L > S_{4.5\,m}, \tag{3}$$

   then the option's intrinsic value is positive.[18]

2. <u>The time-value of the option has eroded to zero.</u>

   The value of the currently callable bond at a notification date 4.5 months prior to the next coupon date is

$$V_0(\sigma) = \min\left\{\frac{V^u_{4.5\,m}(\sigma) + V^d_{4.5\,m}(\sigma)}{2}\,\mathrm{PV}_{4.5\,m},\ S_{4.5\,m}\right\},$$

   where

   $V_0(\sigma) \equiv$ is today's value of the callable bond conditional on interest-rate volatility being equal to σ;

   $V^s_{4.5\,m}(\sigma) \equiv$ the value of the callable bond 4.5 months hence, if not called today, in the up- ($s = u$) and down-states ($s = d$), for the same level of volatility σ, next period.

[18] The quantity $(S_{4.5\,m} - 100\,\mathrm{PV}_{4.5\,m})$ strips the present value of the next coupon payment, which must be paid regardless, from the long bond value, leaving present value of the coupon and principal payments due during the remaining period when the bond is callable. To arrive at the forward price, this quantity is compounded to the first call date by dividing by $\mathrm{PV}_{4.5\,m}$.

The first term in the min operator is the value of the callable bond if the option is not exercised; the second term is the value if it is. The difference,

$$\frac{1}{2}\left(V^u_{4.5\,m}(\sigma)+V^d_{4.5\,m}(\sigma)\right)\,\mathrm{PV}_{4.5\,m}-\,S_{4.5\,m},$$

represents the time-value remaining in the option. When this difference is zero, and $V_0(\sigma)=S_{4.5\,m}$, the time-value is completely eroded.

Now, define the threshold volatility, σ_T, implicitly by the relation

$$V_0(\sigma_T)\equiv\min\left\{\frac{V^u_{4.5\,m}(\sigma_T)+V^d_{4.5\,m}(\sigma_T)}{2}\,\mathrm{PV}_{4.5\,m},\,S_{4.5\,m}\right\}=\lim_{\epsilon\to 0}S_{4.5\,m}-\epsilon, \tag{4}$$

for a positive, but arbitrarily small, ϵ (e.g., $\epsilon=5$ cents).[19] Note that σ_T is defined only on notification dates, i.e., 4.5 months prior to the next call date/coupon payment date.

The threshold volatility, σ_T, represents the maximum volatility consistent with an extinguished time-value. If this maximum volatility is large relative to "normal" market volatilities, we can conclude that there is no time-value remaining in the call option, and the Treasury should call the bond if it is in-the-money forward.[20] On the other hand, if the threshold volatility is small, there must be time-value remaining in the option at normal levels of volatility, and the Treasury should refrain from calling the bond. Note that σ_T depends only on the interest-rate model and the prices of non-callable bonds needed to generate the term structure—it does not depend on the observed prices of callable instruments.

Suppose that option is in-the-money (forward) and that σ_T has a low, but positive value. Then the true σ is likely to be greater than σ_T and $V(\sigma)<V_0(\sigma_T)=S_{4.5\,m}-\epsilon$. In this case the bond's value is less than the value at which it should be called, and calling is thus suboptimal since it increases the value of the liability.

[19] If $V(\sigma=0)<S_{4.5\,m}$, the threshold volatility condition (4) cannot be satisfied (the threshold volatility is not defined), and the bond should not be called.

[20] Alternatively, the Treasury should purchase the bond in the marketplace if its current price is less than $S_{4.5\,m}$. For example, the 7s of May 1993–1998 could have been acquired at 101.76 rather than called at the (effective) price of $S=102.26$.

If, on the other hand, for a very large σ (we use $\sigma = 100\%$), it is still true that $V_0(\sigma) > S_{4.5\,m} - \epsilon$, we conclude that σ_T is arbitrarily large and cease our search. Clearly, for such large σ_T's, the true σ cannot exceed σ_T, and for the true σ $V_0(\sigma) > S_{4.5\,m} - \epsilon$. To minimize the Treasury's liability, the bond should be called.

4.3 Analysis of Past Treasury Call Decisions

Given observed information on or beliefs regarding volatility, we are now in a position to examine the Treasury's call policy. We deem it optimal when the Treasury calls a bond if and only if both (1) $\mathrm{FP}_{4.5\,m} > 100$ *and* (2) σ_T exceeds normal values for that period are satisfied, and suboptimal whenever it clearly deviates from such a policy. Where the threshold volatility approximates normal market volatility within the normal ranges shown in Figure 2, we designate these cases as "ambiguous."

Table 3 documents the threshold volatilities for U.S. Treasury callable bonds for the time period for which C-STRIPS prices are available: 1985–1995. During this time frame, there were five bonds that ended their call protection period: the $7\frac{1}{2}$s of August 1988–1993, the 7s of May 1993–1998, the $8\frac{1}{2}$s of May 1994–1999, the $7\frac{7}{8}$s of February 1995–2000, and the $8\frac{3}{8}$s of August 1995–2000.

Several important conclusions can be drawn from Table 3:

1. The term structure estimates provided by the C-STRIPS data, and the Fama-Bliss and Nelson-Siegel-Bliss methods, are in broad agreement with respect to the present values of cash flows: i.e., $\mathrm{FP}_{4.5\,m}$, $S_{4.5\,m}$ and L. Further, all term structure estimates are in agreement with respect to cases when $L < S_{4.5\,m}$. Therefore, the usage of Fama-Bliss and Nelson-Siegel-Bliss term structures is appropriate when C-STRIPS prices are not available.

2. The three term structure estimates yield essentially similar estimates of the threshold volatility σ_T.

3. We conclude from Table 3 that, at least in recent years, the Treasury has called the bonds optimally. They did not call the bond when the forward price was at a discount,

and they did not call precipitously: the threshold volatility was at least in the 20% range when a call was triggered. Note, specifically, that the Treasury declined to call when, on 28 March 1991, the option was in-the-money forward, but the threshold volatility was a normal 7.5% to 10.3%. This bond, the $7\frac{1}{2}$s of August 1988–1993, provides the only recent clean test of the optimality of the call policy. In this case, the bonds were in-the-money forward in March 1991, but with relatively low threshold volatilities. When the Treasury did call the following September, both conditions were clearly met.

Table 4 presents evidence from the Fama-Bliss term structure estimates for the earlier part of the century, the years 1932 through 1971.[21] The table displays the call decisions for the 41 callable bonds that had moved beyond their call protection period in the four decades beginning in the 1930s. Several conclusions can be drawn from Table 4:

1. There were a total of 11 bonds with apparently suboptimal Treasury behavior. For the remaining 33 cases, the call decision was optimal, giving the Treasury the benefit of the doubt when the threshold volatility was in the normal range for the period.

2. A clearly irrational decision is calling a bond when the forward price is at a discount because that implies that the option is being exercised when it is out-of-the-money. There are only four instances when the Treasury has called a bond when the option was out-of-the-money. In each case the spot price was at a premium, but the forward price was at a discount, suggesting that the Treasury ignored the forward nature of the call.

3. On the other hand, for the most part the Treasury also did not wait unduly to call. Thus, we see in the remaining 40 bonds only seven bonds with high threshold volatilities and forward prices in excess of par unaccompanied by a call. These seven bonds involved 17 missed optimal call opportunities.

[21] There were no currently callable bonds, with maturities spanned by non-callable bonds, between November 1971 and February 1988.

4. In comparing the Flat Price column with $S_{4.5\,m}$, we observe that the market was reasonably efficient in anticipating the Treasury's call: in virtually all cases, $P \approx S_{4.5\,m}$ when a subsequent call notice was given.[22]

5. In 14 instances the naive rule recommended calling a bond, while the correct optimal rule gave an unambiguous "no-call" signal. In 9 of these cases the Treasury did not in fact call the bond. In cases of two of the bonds involved, the Treasury eventually called when it should not have, indicating that their earlier, and correct, forbearance may not have been due to application of the optimal call rule.

6. In a single case, the optimal rule unambiguously recommended calling an at- or below-par bond which was not callable under the naive rule. In this case, the bond was not called.

7. In the earlier part of the century, the Treasury issued callable bonds with other than a five-year terminal call period.

While we cannot justify each Treasury call decision, the overall Treasury call policy appears consistent with financial principles. This result may be a fortuitous consequence of a naive strategy. In most instances where calls were made, both the naive and optimal rules lead to the same decision.

Lastly, returning to Longstaff's "anomalous" observation of bonds trading above par at time of call or trading above par and not being called, this is explained by our optimal call rule. In Tables 3 and 4 we find 86 instances of bonds trading in excess of par on their call notification dates. The average premium was 1.02% and the maximum 4.67%. Of these, 46 were optimally callable (30 were actually called) with average premiums of 0.97% and a maximum premium of 4.67%. A few of these bonds were clearly overpriced ($P > S$), but most were within a few basis points of their rational prices. In 15 instances, it was optimal to defer the call even though the bond was trading above par. The average premium in these 15 cases was 0.82% and the maximum 3.31%. The remaining 25 cases involved bonds with

[22] A notable exception is the $3\frac{1}{2}$s of June 1947. When the call notice was given in 1935, $S_{4.5\,m} =$ 101.70, but the market price was quoted at an inflated 104.70.

threshold volatilities in the normal ranges. These ambiguous cases had average premia of 1.24% and a maximum premium of 4.25%. The 40 cases of premium bonds that should not clearly be called demonstrate that there is nothing inherently irrational in observing market prices of currently callable bonds in excess of par.

5 Summary

Using a lognormally distributed interest-rate model, numerically implemented using a binomial lattice, we have examined the valuation of callable U.S. Treasury securities from the late 1920s to the present and calculated the volatilities implicit in the pricing of these instruments. Post-1987, these implied volatilities are typically in the range of 6% to 18%.

We formally derive an optimal bond call policy, taking into account the required prior notification period of intent to call a Treasury bond. We do this by developing the concept of the "threshold volatility," which we use to determine when the option's time-value has been driven to zero. Applying this technique, we then examine the optimality of the Treasury's observed call policy and conclude on balance that this policy has been reasonably optimal.

We find that the prices of such securities frequently exhibit "irrationalities"— in the sense that they appear mispriced relative to similar non-callable securities—predominantly in cases where the option is away-from-the-money forward, precisely those cases wherein the estimate of implied volatility will be unreliable due to the bond's low vega. The conflicting estimates of the frequency of negative option values is shown to result from time-variation in the frequency of negative option value bonds together with the disparate sample periods used in previous studies. Another discrepancy arose from differences in the nature of the violations being measured and the means of counting them.

A second apparent form of "irrational" pricing, bonds trading above their call price, is shown to be, in fact, frequently rational when the deferred exercise element is properly accounted for. This resolves the anomaly noted in Longstaff (1992).

Unlike the negative option value determinations, determinations of whether to call or not are interest rate model-specific. Alternative models of interest rate movements, for instance a square-root process, might yield different estimates of implied volatilities and historical optimal call decisions. More sophisticated models might incorporate a term structure of volatility, though given the paucity of available callable bonds in most periods, fitting such highly parameterized models may prove difficult. Future research may investigate the degree to which call decisions are dependent on the interest-rate model assumed. Indications in this paper from robustness tests using different estimates of the term structure suggest that borderline cases will be relatively few.

Appendix A Descriptions of Term Structure Estimation

A.1 The Unsmoothed Fama-Bliss Method

The Unsmoothed Fama-Bliss method is a minor variant on the term structure estimation method employed in Fama and Bliss (1987). The underlying idea of the Fama-Bliss method is as follows:

- Assume the shortest maturity issue is a bill (as is always the case). We can then compute the value of the term structure at that maturity which would exactly price the bill.

- We assume that the term structure from maturity 0 to the maturity of the first issue is constant (flat).

- We then examine the next longest maturity issue. Taking the previously determined term structure as given, we compute the constant forward rate over the interval from the previously included maturity to this issue's maturity, which would result in the current issue being accurately priced.

- We continue to extended the term structure by adding longer and longer maturity issues, at each point computing the (constant) forward rate necessary to exactly price the new issue, holding fixed the previously determined term structure.

- The Unsmoothed Fama-Bliss method achieves a continuous term structure by linearly interpolating the discount rates between maturities.

A number of filter rules are iteratively applied during the process to exclude issues that produced large jumps or reversals of the fitted term structure at adjacent maturities—the logic being that, while most issues are correctly priced, data collection and transcription procedures occasionally introduce erroneous quotes that do not reflect useful information. This method necessarily prices the remaining included issues exactly. The reader is referred to the Fama and Bliss (1987) paper for the full details of the original method, including filter rules, parameters used, and the iterative selection procedure.

The Fama-Bliss method reduces quotations to a single value, the mean of the bid and asked quotes, as is the practice with most methods.

A.1.1 Differences from Previous Work

The application of the Fama-Bliss method in this paper differs from the original in several minor ways.

- The term structure beyond five years was computed and used.
- Rather than picking off selected values at fixed maturities, as was done in Fama and Bliss (1987), we use here the full information set produced by the Fama-Bliss estimation method with discount rates determined at each horizon corresponding to the maturity of a bond in the sample. This preserves all the information in the prices of the included issues. When pricing bonds with principal and coupon payments, which may occur at any point along the term structure, this additional detail is necessary.
- The original Fama-Bliss method forced in the longest available issue. This did not matter since the term structure was subsequently truncated at 5 years. In this paper, where we utilize the full term structure, we have chosen to let the longest maturity bond be included or excluded on the basis of the same filters used for other issues.

A.2 The Nelson-Siegel-Bliss Method

The Nelson-Siegel-Bliss method developed in Bliss (1996) introduces a new estimation method to fit a modified version of the approximating function developed by Nelson and Siegel (1987). The Nelson-Siegel-Bliss method brings together several desirable characteristics—accounting for bid/asked spreads, fitting the discount rate function directly to bond prices, and producing an asymptotically flat term structure[23]—using the following approximating function:

$$r(m) = \beta_0 + \beta_1 \left(\frac{1 - e^{-m/\tau_1}}{m/\tau_1} \right) + \beta_2 \left(\frac{1 - e^{-m/\tau_2}}{m/\tau_2} - e^{-m/\tau_2} \right). \tag{5}$$

[23] Livingston and Jain (1982) and Siegel and Nelson (1988) demonstrate that this property is appropriate if forward rates are finite.

The parameter vector $\boldsymbol{\phi} \equiv \{\boldsymbol{\beta_0}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2}, \boldsymbol{\tau_1}, \boldsymbol{\tau_2}\}$ is then estimated using the following non-linear, constrained optimization, estimation procedure:

$$\min_{\boldsymbol{\phi}} \sum_{i=1}^{N_t} (w_i \epsilon_i)^2,$$

where

$$\epsilon_i = \begin{cases} P_i^A - \hat{P}_i & \text{if } \hat{P}_i > P_i^A \\ P_i^B - \hat{P}_i & \text{if } \hat{P}_i < P_i^B \\ 0 & \text{otherwise} \end{cases}$$

and the weights, w_i, are defined in terms of Macaulay duration, d_i, measured in days

$$w_i = \frac{1/d_i}{\sum_{j=1}^{N_t} 1/d_j}$$

subject to:

$$0 \leq r(m_{\text{min}})$$

$$0 \leq r(\infty)$$

and

$$\exp\left[-r(m_k)\, m_k\right] \geq \exp\left[-r(m_{k+1})\, m_{k+1}\right] \quad \forall m_k < m_{\text{max}}.$$

The constraints ensure that the discount function is non-increasing (non-negative forward rates) and that the short and long ends of the discount rate function are positive. Only the long-rate constraint proved binding, and then only rarely.

Appendix B Computing Implied Variances from Callable Bond Prices

B.1 Iterative Search Procedure

The term structure model used in this paper is the Black and Karasinski (1991) lognormal process with constant proportional variance, zero mean reversion, and time-varying drift, μ_t, calibrated to the current term structure:

$$dr = \mu_t \, r \, dt + \sigma \, r \, dz.$$

The process for finding the implied interest-rate volatility for a particular callable bond has three steps:

1. The term structure of interest rates is estimated from the prices of non-callable bonds using any of several estimation methods or from the prices of Treasury C-STRIPS.

2. A value for the short-interest-rate volatility is assumed. This volatility is then used to build a binomial tree of the future evolution of the short rate. This tree is constructed to match the term structure estimated in the first step. To enhance speed of execution the tree is constructed using the forward induction method developed by Jamshidian (1991). The nodes of the tree are placed at horizons corresponding to the cash flows of the particular bond being priced.

3. The fitted price for the callable bond is next determined by backward induction beginning with the tree nodes at the maturity of the bond. This fitted value is conditional upon the assumed σ, as well as the estimated term structure, bond cash flows, and call timing, which remain invariant.

A search is conducted over feasible values of σ, repeating (2) and (3) at each iteration, until a σ is found for which the fitted price is equal to the observed price for that bond.

B.2 Matching the Term Structure in a No-Arbitrage Recombining Interest-Rate Lattice

In Step (2) above the short-rate tree, r_{tj},[24] is constructed for a given value of σ to guarantee that three conditions are met:

At each node (t, j),

- The expected change in the natural logarithm of the short rate is the same regardless of the actual state j at time t (though it will vary for different time horizons, t).

- The variance of the change in the logarithm of the short rate is equal to σ regardless of the actual state j or time t.

And lastly:

- At each horizon t the "median" interest rate u_t is selected so the price of a zero-coupon bond maturing at the horizon will conform to the pre-specified discount function.

The construction of the tree guarantees that the first two conditions are met. These conditions also ensure that the process underlying the short rate is Markovian.

A generic node of the binomial tree looks like:

$$\underset{t}{|} \longleftarrow \Delta \longrightarrow \underset{t+1}{|}$$

$$\begin{tikzcd}
 & r_{t+1,j+1} \equiv u_{t+1}\, e^{(j+1)\,\sigma\sqrt{\Delta}} \\
r_{tj} \equiv u_t\, e^{j\,\sigma\sqrt{\Delta}} \arrow[ur, dash, "\frac{1}{2}"] \arrow[dr, dash, "\frac{1}{2}"'] & \\
 & r_{t+1,j-1} \equiv u_{t+1}\, e^{(j-1)\,\sigma\sqrt{\Delta}}
\end{tikzcd}$$

The expected change in the short rate and its variance are:[25]

$$E\left(\triangle \ln r_{t+1} \mid r_t\right) \;=\; \ln\left(\frac{u_{t+1}}{u_t}\right);$$

[24] The tree is constructed for $\{(t, j) \;:\; t = 0, 1, \ldots, T; j = t, t-2, \ldots, -t\}$, where t indexes the time horizons, j indexes the possible states at that horizon, and T corresponds to the maturity of the callable bond being priced.

[25] We are cognizant of the fact that the time index is more properly incremented by Δ than by unity. For example, the expectation in the text should be written $\ln\left(u_{t+\Delta}/u_t\right)$. We maintain the integer increment for simplicity of notation and understanding in this exposition.

$$\operatorname{Var}(\triangle \ln r_{t+1} \mid r_t) \quad = \quad \sigma^2 \Delta.$$

Notice that $E(\triangle \ln r_{t+1} \mid r_t)$ is independent of j and $\operatorname{Var}(\triangle \ln r_{t+1} \mid r_t)$ is independent of both j and t.

For valuing coupon bonds the normal interval of interest Δ corresponds to the semi-annual coupon payment dates.[26]

B.3 A Numerical Example

A simple numerical example will demonstrate the lattice, or "tree," approach to the valuation of callable bonds. Suppose $\Delta = 1$ year, and the present value factors have been calculated to be

$$\begin{aligned} \mathrm{PV}_1 &= 0.9561, \\ \mathrm{PV}_2 &= 0.9028. \end{aligned}$$

Thus, $r_1 = \ln(1/0.9561) = 4.489\%$. Note that the forward rate here is $-\ln(.9561/.9028) = 5.736\%$. Assume further that $\sigma = 9\%$.

Using a volatility of 9%, the interest-rate tree specifies that the short-term rate moves in a binomial fashion:

[26] We ignore slight differences in the length of time between coupons arising from variation in the number of days in a half year and the adjustment for coupon payment dates that fall on weekends.

$$
\begin{array}{ccccc}
\underline{\text{Today}} & & \underline{\text{Time 1}} & & \underline{\text{Time 2}} \\
 & & & & u_2\ e^{2\times 0.09} \\
 & & & \nearrow & \\
 & & u_1\ e^{0.09} & & \\
 & \nearrow & & \searrow & \\
4.489\% & & & & u_2 \\
 & \searrow & & \nearrow & \\
 & & u_1\ e^{-0.09} & & \\
 & & & \searrow & \\
 & & & & u_2\ e^{-2\times 0.09}
\end{array}
$$

We seek to find a parameter, designated u_1, such that the price of \$100 in two years' time conforms to the current yield curve's implied value of \$90.28. The price behavior of the two-year zero-coupon bond is:

$$
\begin{array}{ccccc}
\underline{\text{Today}} & & \underline{\text{Time 1}} & & \underline{\text{Time 2}} \\
 & & 100\exp\{-u_1 e^{0.09}\} & \rightarrow & 100 \\
P_2 & \begin{smallmatrix}\nearrow \\ \searrow\end{smallmatrix} & & & \\
 & & 100\exp\{-u_1 e^{-0.09}\} & \rightarrow & 100
\end{array}
$$

The no-arbitrage properties of the model imply that the price of the zero-coupon bond must equal the discounted expected value, using probabilities of 0.5,[27] of the cash flows next period. In other words, P_2 must satisfy

$$
P_2 = 100\ e^{-0.0459}\left(\frac{1}{2}\exp\{-u_1 e^{0.09}\} + \frac{1}{2}\exp\{-u_1 e^{-0.09}\}\right).
$$

[27]The choice of risk neutral probability of $\{0.5, 0.5\}$ is innocuous in this application. Any other valid pair of probabilities could be used, and the values of u_i's would adjust. The resulting tree would price securities identically to the tree using $\{0.5, 0.5\}$.

Since we know $P_2 = 90.28$ from the current yield curve, we can solve for $u_1 = 5.714\%$. (Note that this u_1 is approximately equal to the forward rate, 5.36%.) This procedure can then be repeated to derive the interest-rate tree out to 30 or more years.

Suppose we now seek to find the no-arbitrage value of a two-year, 6% bond callable at par in one year's time. The interest-rate tree is

$$
4.59\% \begin{array}{l} \nearrow \quad 5.714\% \times e^{0.09} = 6.252\% \\ \\ \searrow \quad 5.714\% \times e^{-0.09} = 5.222\% \end{array}
$$

The price behavior of the two-year callable 6% coupon bond is:

$$
\begin{array}{ccc}
\underline{\text{Today}} & \underline{\text{Year 1}} & \underline{\text{Year 2}} \\
 & 6 + \min\left\{106e^{-0.06252},\, 100\right\} = 6 + 99.58 \quad \rightarrow & 106 \\
B \quad \begin{array}{l}\nearrow \\ \searrow\end{array} & & \\
 & 6 + \min\left\{106e^{-0.05222},\, 100\right\} = 6 + 100 &
\end{array}
$$

The bond is called at Year 1 if rates decline to 5.222%. Hence, the current bond value B is given by

$$B = e^{-0.0459}\left(\frac{1}{2}105.58 + \frac{1}{2}106\right) = 101.04.$$

With the current term structure, the value of a non-callable two-year 6% bond is given by the present value of the cash flows:

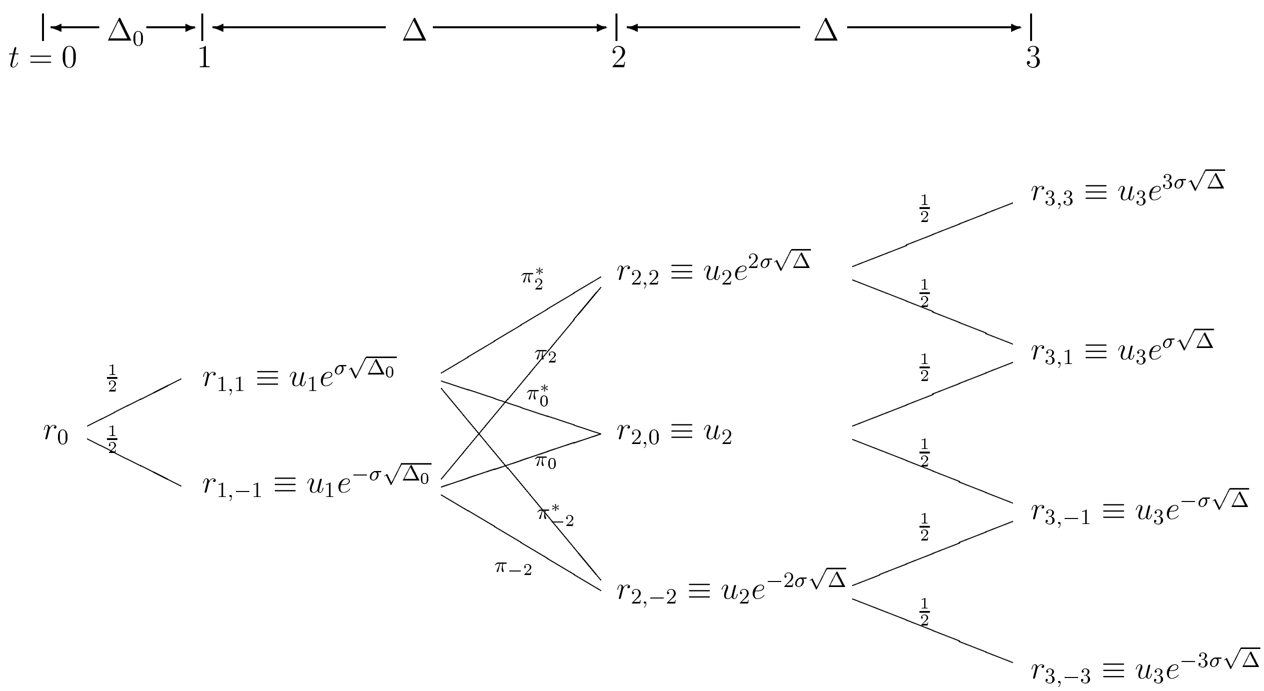
$$6 \times P_1 + 106 \times P_2 = 6 \times 0.9561 + 106 \times 0.9028 = 101.43.$$

Thus, the value of the one-year call is $101.43 - 101.04 = 0.39$.

B.4 Accounting for the Length of the First Time Interval in the Construction of the Interest-Rate Tree

The time to the first remaining coupon will rarely equal six months,[28] so we must construct the tree with a short first time interval, Δ_0.

Because of the short first interval, the two possible $t = 1$ interest rates are "too close together" and the tree must be adjusted to ensure that the variance of changes from $t = 1$ to $t = 2$ equals σ. This is done by using a trinomial tree over that interval as follows:



The binomial sections of the tree have only one degree of freedom, u_t, and this is used to match the term structure. Over the second interval we have five degrees of freedom, $\{u_2, \pi_2^*, \pi_2, \pi_0^*, \pi_0\}$, and five conditions to meet: that the expected changes in log-interest rate are identical whether we begin at $r_{1,1}$ or $r_{1,-1}$, that the variance of changes in the log-interest rate is likewise invariant and equal to σ, and finally that the tree through $t = 2$ correctly prices a $(\Delta_0 + \Delta)$-period zero-coupon bond.

[28] This arises because most Treasury notes and bonds mature on the 15th of the month and our quotes are all month-end quotes. For this reason $\frac{1}{2} \leq \Delta_0 \leq 5\frac{1}{2}$ months while $\Delta = 6$ months.

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Table 1a — Summary Statistics for Implied Volatilities
Data Period: January 1987 – December 1995

Estimation Method	Intermediate Maturities (5 – 20 Years) Mean Implied Vol	Standard Deviation	N (months)	Long Maturities (> 20 Years) Mean Implied Vol	Standard Deviation	N (months)
Nelson-Siegel-Bliss	9.15%	3.77%	105	17.82%	4.88%	93
Fama-Bliss	9.02%	3.93%	105	16.04%	3.65%	93
C-STRIPS	9.09%	4.08%	87	12.29%	3.65%	81

Table 1b — Correlation Matrices for Implied Volatilities
Data Period: 1987 – 1995

Estimation Method	Intermediate Maturities (5 – 20 Years) Nelson-Siegel-Bliss	Fama-Bliss	C-STRIPS	Long Maturities (> 20 Years) Nelson-Siegel-Bliss	Fama-Bliss	C-STRIPS
Nelson-Siegel-Bliss	1.000			1.000		
Fama-Bliss	0.970	1.000		0.661	1.000	
C-STRIPS	0.852	0.853	1.000	0.626	0.621	1.000

Notes:

1. Statistics are computed using the monthly vega-weighted implied volatilities,
2. Using observations with "good" implied volatilities and vega > 0.02.

Table 2 — Breakdown of Implied Volatility Estimates by Moneyness, Maturity, and Vega
Dates: January 1985 – December 1995

Implied Volatilities	Moneyness	Maturities Included: Short	Medium	Long	All	Vega	Frequency
Neg.	Out	0	236	23	259	Low	658
	At	5	281	121	407	Medium	102
	In	0	10	88	98	High	4
	Total	5	527	232	764		764
Good	Out	7	49	4	60	Low	78
	At	27	528	59	614	Medium	801
	In	0	612	796	1408	High	1203
	Total	34	1189	859	2082		2082
Huge	Out	0	0	0	0	Low	50
	At	3	0	0	3	Medium	2
	In	0	0	49	49	High	0
	Total	3	0	49	52		52

1. Fama-Bliss term structure estimates were used to estimate the implied volatilities and vegas.

2. "Implied Volatilities" refers to whether the implied volatility could be estimated.
   - "Huge" indicates that the implied volatility exceeded 100% and was deemed unreasonably large.
   - "Good" indicates that an implied volatility was found in the range [0,100%].
   - "Neg." indicates that the option value was negative and the implied volatility could not be computed.

3. "Moneyness" refers to the forward price of the equivalent non-callable bond as of the call date. For "at-the-money" bonds: $95 \leq \text{FP} \leq 105$. "In-" and "out-of-the-money" bonds have forward prices above and below that range respectively.

4. "Maturities" are measured as of the notification date. "Medium" maturities are defined by: 5 years $\leq T_m \leq$ 20 years.

5. "Vega" refers to $\partial V/\partial \sigma$ and is measured at the implied volatility for "good" implied volatilities, at zero volatility for negative option value bonds, and at 100% volatility for "huge" implied volatilities. Medium vegas are in the range [2%,10%].

Table 3 — Analysis of Threshold Volatilities
Dates: March 1988 – December 1995

			Term Structure Estimation Technique Used											
Quote Date	T_m	Flat Price	C-STRIPS				Fama-Bliss				Nelson-Siegel-Bliss			
			$\mathrm{FP}_{4.5\,m}$	$S_{4.5\,m}$	L	σ_T	$\mathrm{FP}_{4.5\,m}$	$S_{4.5\,m}$	L	σ_T	$\mathrm{FP}_{4.5\,m}$	$S_{4.5\,m}$	L	σ_T
$7\frac{1}{2}$s of Aug'88–93														
880331	5.37	97.16	96.40	100.38	96.87	Neg.	96.68	100.37	97.13	Neg.	96.60	100.49	97.17	Neg.
880930	4.88	95.38	95.25	99.88	95.26	Neg.	95.63	99.83	95.58	Neg.	95.53	99.89	95.55	Neg.
890331	4.37	92.38	93.08	99.26	92.58	Neg.	93.15	99.28	92.67	Neg.	93.29	99.34	92.85	Neg.
890929	3.88	96.75	97.55	99.69	97.32	Neg.	97.21	99.63	96.92	Neg.	97.29	99.67	97.04	Neg.
900330	3.38	96.41	96.51	99.76	96.38	Neg.	96.73	99.74	96.56	Neg.	96.78	99.77	96.64	Neg.
900928	2.88	98.25	98.04	99.95	98.05	Neg.	98.32	99.95	98.31	Neg.	98.17	99.97	98.19	Neg.
910328	2.38	100.50	100.02	100.60	100.62	7.5%	100.18	100.56	100.74	9.7%	100.18	100.56	100.73	10.3%
910930	1.88	101.41	101.85	100.79	102.61	66.2%	101.86	100.77	102.59	66.2%	101.90	100.79	102.66	67.5%
Called Feb'92			Call was optimal				Call was optimal				Call was optimal			
7s of May'93–98														
921231	5.37	100.87	102.37	101.37	103.72	20.3%	102.15	101.31	103.44	19.4%	102.37	101.38	103.72	19.8%
Called May'93			Call was optimal				Call was optimal				Call was optimal			
$8\frac{1}{2}$s of May'94-99														
931231	5.37	101.88	113.25	101.96	115.06	62.8%	113.26	101.88	114.98	63.1%	113.34	101.94	115.12	63.1%
Called May'94			Call was optimal				Call was optimal				Call was optimal			
$7\frac{7}{8}$s of Feb'1995-2000														
940930	5.38	100.84	101.67	100.97	102.61	11.1%	101.58	100.92	102.46	10.2%	101.55	100.99	102.51	9.8%
Called Feb'95			Call was optimal				Call was optimal				Call was optimal			
$8\frac{3}{8}$s of Aug'1995-2000														
950331	5.37	100.69	105.11	100.94	105.94	20.0%	104.84	100.86	105.59	19.1%	104.92	100.88	105.69	19.5%
Called Feb'96			Call was optimal				Call was optimal				Call was optimal			

Notes:

T_m ≡ Time to maturity, in years

$\mathrm{FP}_{4.5\,m}$ ≡ Forward price of L on next call date date

L ≡ Present value of otherwise equivalent non-callable bond to T_m

Flat Price ≡ Market price, excluding accrued interest

$S_{4.5\,m}$ ≡ Present value of otherwise equivalent non-callable bond which matures in 4.5 months

σ_T ≡ Threshold volatility; "Neg" means *negative* option value

Appendix B. describes the term structure estimation techniques,

Table 4 — Analysis of Threshold Volatilities
Dates: January 1932 – October 1971

Bond ID	Quote Date	T_m	Flat Price	Fama-Bliss Term Structure Estimates $FP_{4.5\,m}$	$S_{4.5\,m}$	L	σ_T	Optimal to Call?	When Called	Evaluation
$3\frac{1}{2}$s of Mar'30–32	290830	2.54	97.50	99.93	99.41	99.34	Negative	No		
	300228	2.05	99.56	99.90	99.97	99.88	Negative	No		
	300829	1.55	100.72	100.53	100.05	100.58	70.0%	Yes		
	310228	1.05	100.06	101.51	100.08	101.59	> 100%	Yes	Mar'31	Suboptimal (Late)
$3\frac{1}{2}$s of Sep'30–32	300228	2.55	99.56	99.90	99.95	99.85	Negative	No		
	300829	2.05	100.72	100.70	100.05	100.75	48.7%	Yes		
	310228	1.55	100.06	101.68	100.08	101.76	> 100%	Yes	Mar'31	Suboptimal (Late)
$3\frac{1}{2}$s of Dec'30–32	300529	2.55	100.37	100.40	100.55	100.95	15.0%	?		
	301129	2.05	101.03	100.87	100.01	100.88	58.1%	Yes		
	310529	1.55	101.53	102.13	100.12	102.25	> 100%	Yes		
	311130	1.04	100.03	100.89	100.06	100.95	> 100%	Yes	Dec'31	Suboptimal (Late)
4s of Nov'27–42	270430	15.54	100.03	109.77	100.24	109.83	11.2%	?		
	271031	15.04	99.81	110.74	100.04	110.77	17.3%	?	May'28	Optimal
$3\frac{3}{8}$s of Mar'41–43	401031	2.37	102.38	106.42	101.20	107.62	> 100%	Yes	Mar'41	Optimal

- "Optimal to Call?"
  - **"Yes"** indicates (1) the threshold volatility, σ_T, is higher than normal levels of volatility observed in the market, indicating that the time value of the option has eroded to zero, and (2) the forward price is in the money, i.e. $FP_{4.5\,m} > 100$.
  - **"No"** indicates either (1) σ_T is low relative to normal and therefore that the option time has value remaining; or that (2) $FP_{4.5\,m} < 100$.
  - **"?"** indicates σ_T is comparable to market levels and that calling or not is a matter of indifference.
- "Evaluation"
  - **"Optimal"** indicates either
    - the call was made at the first call date at which it was clearly indicated by the forward price and high threshold volatility, or
    - the bond was clearly not optimal call at each notification date and the call never occurred.
  - **"Suboptimal (Late)"** indicates that the call was rational but should have occurred at another, earlier date.
  - **"Irrational"** indicates that the bond was called either when the forward price was not in-the-money or when the time value remaining in the option was high, as indicated by a low threshold volatility.

Table 4 — Analysis of Threshold Volatilities (Cont'd)
Dates: January 1932 – October 1971

Bond ID	Quote Date	T_m	Flat Price	Fama-Bliss Term Structure Estimates $FP_{4.5\,m}$	$S_{4.5\,m}$	L	σ_T	Optimal to Call?	When Called	Evaluation
$3\frac{3}{8}$s of Jun'40–43	400131	3.37	102.00	109.02	101.21	110.22	> 100%	Yes	Jun'40	Optimal
$3\frac{1}{4}$s of Apr'44–46	431130	2.38	101.05	104.34	100.97	105.31	> 100%	Yes	Apr'44	Optimal
$3\frac{3}{8}$s of Jun'43–47	430130	4.38	101.06	108.53	101.15	109.67	> 100%	Yes	Jun'43	Optimal
$3\frac{1}{2}$s of Jun'32–47	320130	15.38	94.47	98.54	100.28	98.83	Negative	No		
	320730	14.88	101.13	103.42	101.19	104.61	7.7%	No		
	330131	14.37	103.31	106.80	101.29	108.09	17.3%	No		
	330731	13.87	102.63	107.35	101.24	108.59	16.4%	No		
	340131	13.37	101.31	104.56	101.13	105.69	8.4%	No		
	340731	12.87	103.94	108.78	101.29	110.07	26.4%	?		
	350131	12.37	104.25	111.81	101.24	113.05	34.4%	?	Jun'35	Optimal
4s of Jun'32–47	320130	15.38	95.00	104.26	100.46	104.68	4.8%	No		
	320730	14.88	100.22	109.20	101.38	110.56	16.4%	No		
	330131	14.37	101.47	112.56	101.47	114.03	26.4%	?		
	330731	13.87	101.00	112.93	101.43	114.35	25.3%	?	Jun'35	Optimal*
$2\frac{3}{4}$s of Sep'45–47	450430	2.38	100.94	103.28	100.77	104.04	> 100%	Yes	Sep'45	Optimal
3s of Jun'46–48	460131	2.37	101.00	104.29	100.83	105.11	> 100%	Yes	Jun'46	Optimal
$3\frac{1}{8}$s of Jun'46–49	460131	3.37	101.03	106.93	100.88	107.79	> 100%	Yes	Jun'46	Optimal
2s of Mar'48–50	471031	2.37	100.36	101.87	100.43	102.29	> 100%	Yes		
	480430	1.88	101.25	100.89	100.39	101.28	> 100%	Yes		
	481029	1.38	100.78	100.55	100.35	100.89	> 100%	Yes		
	490429	0.88	100.69	100.39	100.32	100.70	> 100%	Yes	**	Suboptimal (Late)
2s of Dec'48–50	480730	2.38	100.44	101.17	100.38	101.55	92.5%	Yes	Dec'48	Optimal

*: The 4s of Jun'32-47 were called two years after the last date for which we have price quotations.

**: The 2s of Mar'48–50 were first partially called in Mar'48.

Table 4 — Analysis of Threshold Volatilities (Cont'd)
Dates: January 1932 – October 1971

Bond ID	Quote Date	T_m	Flat Price	Fama-Bliss Term Structure Estimates $FP_{4.5\,m}$	$S_{4.5\,m}$	L	σ_T	Optimal to Call?	When Called	Evaluation
$2\frac{3}{4}$s of Mar'48–51	471031	3.37	100.69	105.00	100.71	105.69	> 100%	Yes	Mar'48	Optimal
2s of Jun'49–51	490131	2.37	100.31	101.30	100.31	101.60	97.5%	Yes	Jun'49	Optimal
2s of Sep'49–51	490429	2.38	100.30	101.47	100.32	101.79	> 100%	Yes	Sep'49	Optimal
2s of Dec'49–51	490729	2.38	100.44	101.52	100.40	101.91	> 100%	Yes	Dec'49	Optimal
2s of Mar'50–52	491031	2.37	100.36	101.64	100.35	101.99	> 100%	Yes	Mar'50	Optimal
2s of Sep'50–52	500428	2.38	100.36	101.17	100.34	101.50	90.0%	Yes	Sep'50	Optimal
$2\frac{1}{2}$s of Sep'50–52	500428	2.38	100.59	102.15	100.53	102.67	> 100%	Yes	Sep'50	Optimal
$3\frac{1}{8}$s of Dec'49–52	490729	3.38	100.92	105.48	100.82	106.28	> 100%	Yes	Dec'49	Optimal
2s of Sep'51–53	510430	2.38	100.00	101.51	100.21	101.72	> 100%	Yes		
	511031	1.87	100.09	100.10	100.17	100.26	25.0%	?		
	520430	1.38	100.19	100.18	100.17	100.35	75.0%	Yes		
	521031	0.87	100.02	100.03	100.09	100.12	> 100%	Yes	Never	Suboptimal (Late)
$2\frac{1}{4}$s of Dec'51-53	510731	2.37	100.44	100.64	100.31	100.94	39.4%	?	Dec'51	Optimal
$2\frac{1}{2}$s of Dec'49–53	490729	4.38	100.69	104.76	100.59	105.33	93.7%	Yes	Dec'49	Optimal
$2\frac{1}{2}$s of Mar'52–54	511031	2.37	100.28	101.07	100.35	101.41	62.5%	Yes	Mar'52	Optimal
2s of Jun'52-54	520131	2.37	100.00	100.01	100.13	100.14	11.2%	No		
	520731	1.87	99.84	99.88	100.07	99.95	Negative	No		
	530130	1.38	99.78	99.84	100.09	99.93	Negative	No		
	530731	0.87	99.75	99.76	99.98	99.74	Negative	No	Never	Optimal
$2\frac{3}{4}$s of Jun'51–54	510131	3.37	100.59	103.09	100.49	103.56	82.50%	Yes	Jun'51	Optimal

Table 4 — Analysis of Threshold Volatilities (Cont'd)
Dates: January 1932 – October 1971

Bond ID	Quote Date	T_m	Flat Price	Fama-Bliss Term Structure Estimates $FP_{4.5\,m}$	$S_{4.5\,m}$	L	σ_T	Optimal to Call?	When Called	Evaluation
2s of Dec'44–54	450131	9.87	101.28	110.36	100.41	110.75	59.1%	?		
	450731	9.37	102.80	110.38	100.45	110.80	66.2%	?		
	460131	8.87	104.67	110.74	100.46	111.17	89.4%	Yes		
	460731	8.37	103.78	109.34	100.42	109.74	81.2%	Yes		
	470131	7.87	103.00	106.77	100.44	107.19	53.0%	?		
	470731	7.37	102.84	106.85	100.47	107.30	63.1%	?		
	480130	6.88	101.13	103.79	100.37	104.15	35.2%	?		
	480730	6.38	101.19	104.10	100.38	104.46	41.2%	?		
	490131	5.87	101.56	103.94	100.31	104.23	45.9%	?		
	490729	5.38	102.33	103.45	100.40	103.84	51.6%	Yes		
	500131	4.87	101.84	103.01	100.33	103.33	51.6%	Yes		
	500731	4.37	101.48	101.96	100.31	102.27	43.1%	?		
	510131	3.87	100.62	100.99	100.21	101.19	26.2%	?		
	510731	3.37	100.09	100.16	100.21	100.37	8.1%	?		
	520131	2.87	99.97	99.96	100.13	100.09	0.0%	No		
	520731	2.37	99.80	99.80	100.07	99.87	Negative	No		
	530130	1.88	99.66	99.65	100.09	99.74	Negative	No		
	530731	1.37	99.47	99.60	99.98	99.59	Negative	No		
	540129	0.88	100.72	100.28	100.54	100.82	$> 100\%$	Yes	Never	Suboptimal (Late)
4s of Dec'44–54	440731	10.37	101.36	129.98	101.26	131.17	$> 100\%$	Yes	Dec'44	Optimal
2s of Jun'53–55	530130	2.38	100.13	99.52	100.09	99.61	Negative	No	Jun'53	Irrational
$2\frac{1}{4}$s of Jun'52–55	520131	3.37	100.17	100.58	100.23	100.80	21.2%	?		
	520731	2.87	100.09	100.28	100.16	100.45	16.2%	?		
	530130	2.38	99.97	100.01	100.18	100.19	6.2%	?		
	530731	1.87	99.69	99.77	100.07	99.84	Negative	No		
	540129	1.38	100.44	100.63	100.64	101.26	$> 100\%$	Yes	Jun'54	Optimal
3s of Sep'51–55	510430	4.38	100.68	106.90	100.59	107.45	$> 100\%$	Yes	Sep'51	Optimal

Table 4 — Analysis of Threshold Volatilities (Cont'd)
Dates: January 1932 – October 1971

Bond ID	Quote Date	T_m	Flat Price	Fama-Bliss Term Structure Estimates $FP_{4.5\,m}$	$S_{4.5\,m}$	L	σ_T	Optimal to Call?	When Called	Evaluation
2s of Dec'51–55	510731	4.37	100.03	100.19	100.21	100.40	5.6%	?		
	520131	3.87	99.91	99.70	100.13	99.84	Negative	No		
	520731	3.37	99.78	99.52	100.07	99.60	Negative	No		
	530130	2.88	99.38	99.45	100.09	99.54	Negative	No		
	530731	2.37	99.06	99.16	99.98	99.14	Negative	No		
	540129	1.88	100.69	100.52	100.54	101.07	72.5%	Yes		
	540730	1.38	100.69	101.05	100.67	101.72	> 100%	Yes	Dec'54	Suboptimal (Late)
$3\frac{3}{4}$s of Mar'46–56	451031	10.37	101.13	128.94	101.10	129.95	> 100%	Yes	Mar'46	Optimal
$2\frac{1}{4}$s of Jun'54–56	540129	2.38	100.47	101.15	100.64	101.79	77.5%	Yes	Jun'54	Optimal
$2\frac{1}{2}$s of Mar'56–58	551031	2.37	99.94	100.23	100.20	100.43	0.0%	No		
	560430	1.88	98.69	98.93	99.89	98.82	Negative	No		
	561031	1.37	98.97	99.29	99.82	99.12	Negative	No		
	570430	0.88	99.28	99.32	99.81	99.14	Negative	No	Never	Optimal
$2\frac{3}{8}$s of Mar'57–59	561031	2.37	97.81	98.17	99.78	97.96	Negative	No		
	570430	1.88	97.91	98.50	99.76	98.28	Negative	No		
	571031	1.37	98.00	98.78	99.46	98.25	Negative	No		
	580430	0.88	100.37	100.47	100.47	100.94	> 100%	Yes	Sep'58	Optimal
$2\frac{3}{4}$s of Sep'56–59	560430	3.38	100.19	99.22	99.98	99.20	Negative	No	Sep'56	Irrational
$2\frac{7}{8}$s of Mar'55–60	541029	5.38	100.84	103.95	100.76	104.70	47.3%	?	Mar'55	Optimal
$2\frac{3}{4}$s of Jun'58-63	580131	5.37	100.38	99.26	100.37	99.63	Negative	No	Jun'58	Irrational
$2\frac{3}{4}$s of Dec'60–65	600729	5.38	100.19	96.81	100.05	96.89	Negative	No		
	610131	4.87	100.47	95.86	100.09	95.99	Negative	No		
	610731	4.37	100.50	96.34	100.16	96.53	Negative	No		
	620131	3.87	100.34	95.91	99.97	95.93	Negative	No		
	620731	3.37	100.31	97.13	99.97	97.13	Negative	No	Dec'62	Irrational

Table 4 — Analysis of Threshold Volatilities (Cont'd)
Dates: January 1932 – October 1971

Bond ID	Quote Date	T_m	Flat Price	Fama-Bliss Term Structure Estimates $FP_{4.5\,m}$	$S_{4.5\,m}$	L	σ_T	Optimal to Call?	When Called	Evaluation
$2\frac{1}{2}$s of Sep'67–72	670428	5.38	91.00	90.19	99.46	89.79	Negative	No		
	671031	4.87	88.13	87.46	99.15	86.83	Negative	No		
	680430	4.38	88.88	88.46	98.84	87.54	Negative	No		
	681031	3.87	91.63	91.26	98.87	90.31	Negative	No		
	690430	3.38	90.56	90.73	98.64	89.58	Negative	No		
	691031	2.87	87.56	89.28	98.13	87.71	Negative	No		
	700430	2.38	88.72	90.17	98.29	88.71	Negative	No		
	701030	1.88	92.88	93.98	98.73	92.84	Negative	No		
	710430	1.38	96.81	97.46	99.42	96.92	Negative	No		
	711029	0.88	98.31	98.95	99.36	98.33	Negative	No	Never	Optimal

Figure 1a: Maturities of Available Callable Bond Quotations

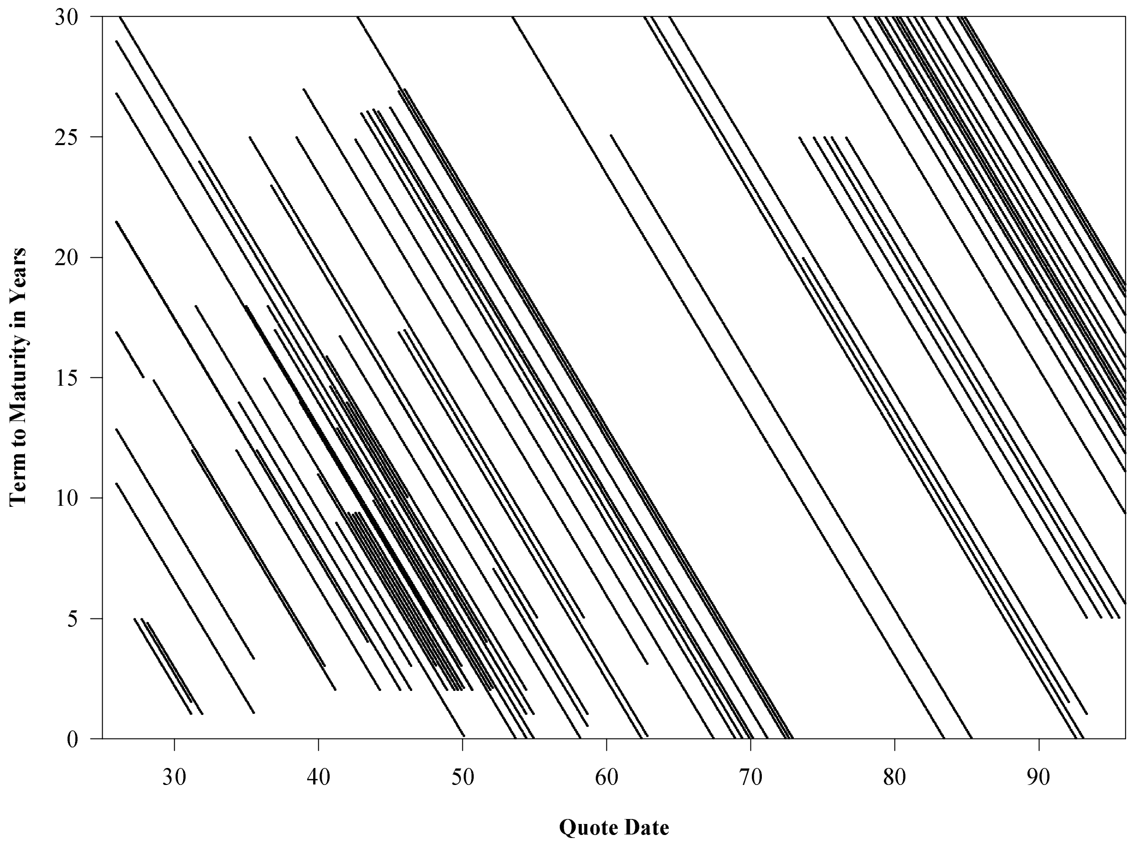

Figure 1b: Maturities of Usable Callable Bond Quotations

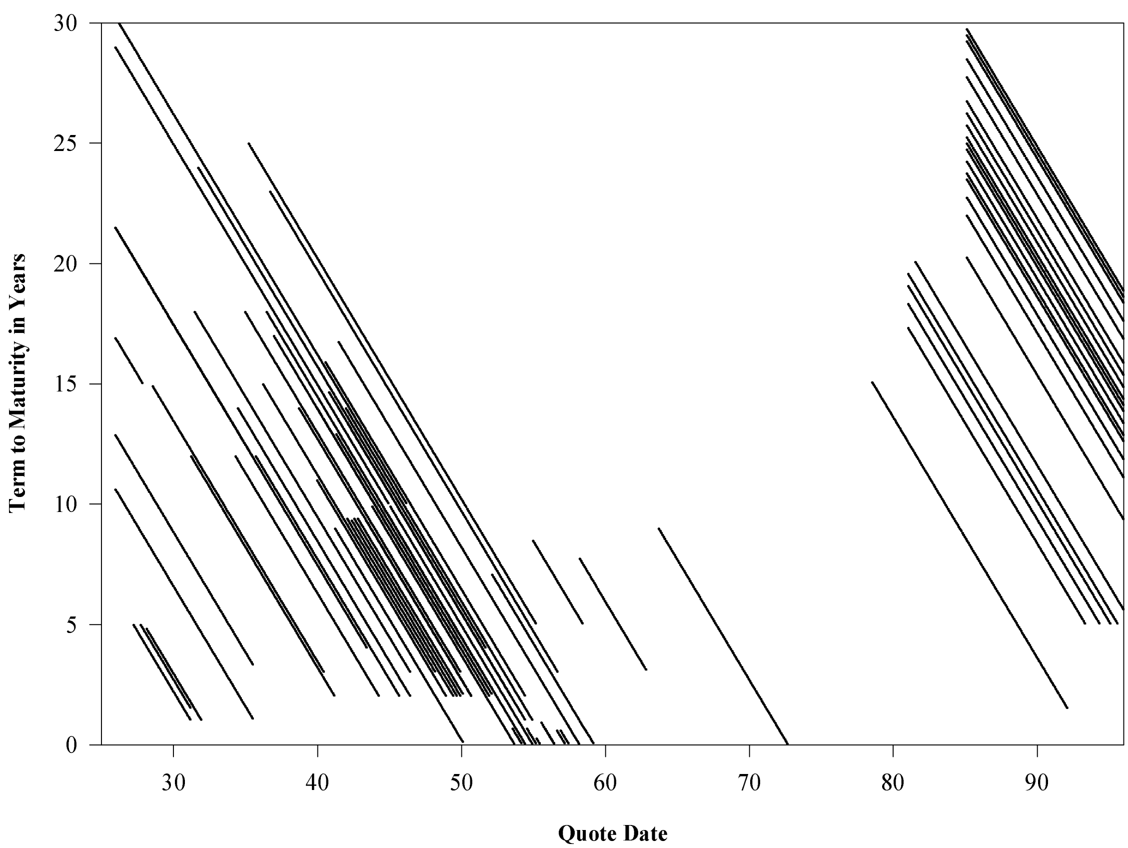

Figure 2a: Implied Volatilities, 1926 - 1955

Minimum Vega: 0.02; Minimum Oservations/Month: 1

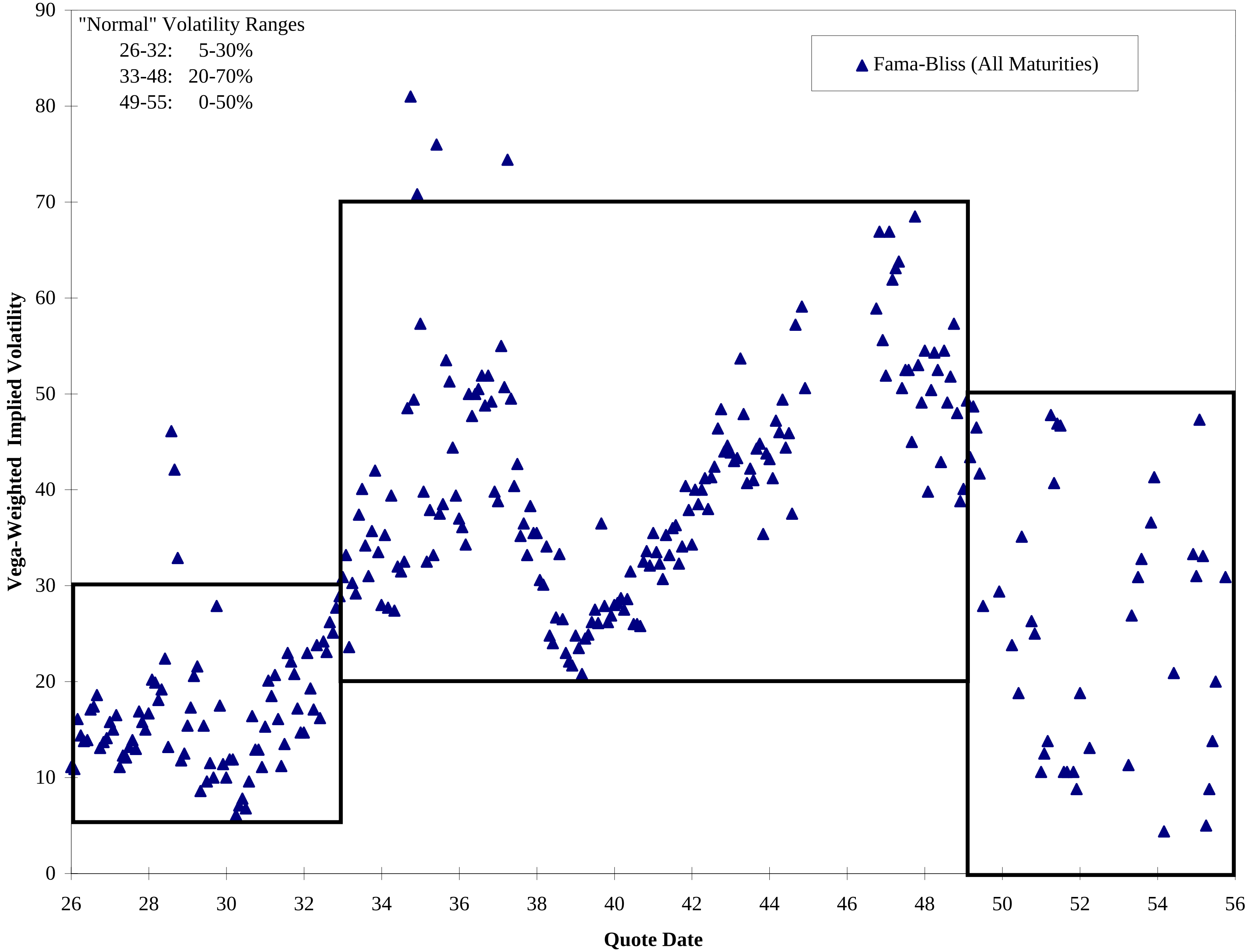

Figure 2b: Implied Volatilities, 1978 - 1995

Minimum Vega: 0.02; Minimum Observations/Month: 1

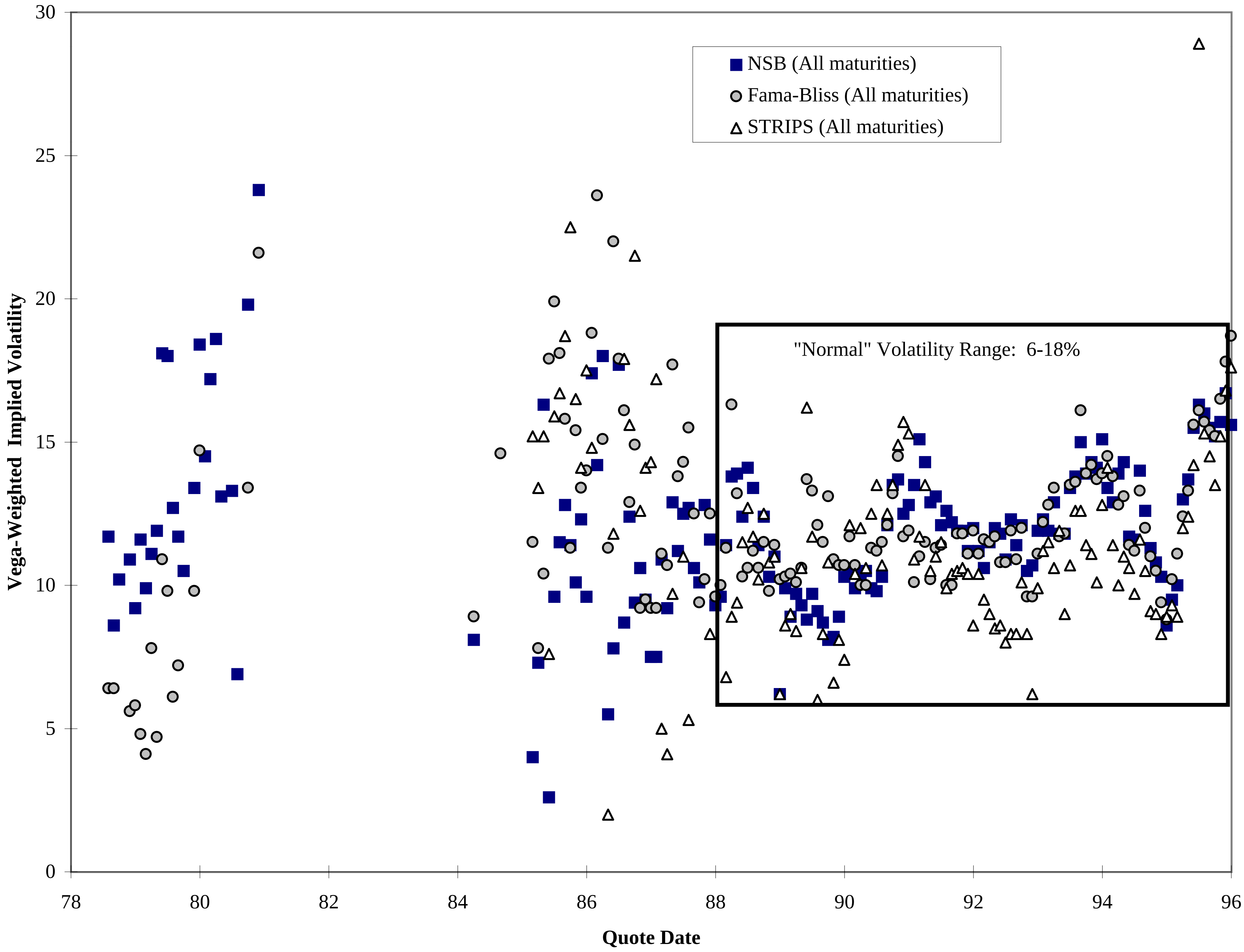

Figure 3: Classification of Callable Bonds by Implied Volatilities
(Using Fama-Bliss Term Structures)