Preference-Free Option Pricing with Path-Dependent Volatility: A Closed-Form Approach

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Abstract: This paper shows how one can obtain a continuous-time preference-free option pricing model with a path-dependent volatility as the limit of a discrete-time GARCH model. In particular, the continuous-time model is the limit of a discrete-time GARCH model of Heston and Nandi (1997) that allows asymmetry between returns and volatility. For the continuous-time model, one can directly compute closed-form solutions for option prices using the formula of Heston (1993). Toward that purpose, we present the necessary mappings, based on Foster and Nelson (1994), such that one can approximate (arbitrarily closely) the parameters of the continuous-time model on the basis of the parameters of the discrete-time GARCH model. The discrete-time GARCH parameters can be estimated easily just by observing the history of asset prices.

Unlike most option pricing models that are based on the absence of arbitrage alone, a parameter related to the expected return/risk premium of the asset does appear in the continuous-time option formula. However, given other parameters, option prices are not at all sensitive to the risk premium parameter, which is often imprecisely estimated.

JEL classification: G13

Key words: volatility, path-dependent, options, closed-form

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Introduction:

Pricing options and other derivatives by appealing only to the absence of arbitrage is often preferred by many because such pricing does not depend on the preferences of any representative investor. Instead, one just needs the much weaker assumption that an investor prefers more to less. Continuous time stochastic volatility models developed by Heston (1993) and extended in Bates (1996 a, b), Bakshi, Cao and Chen (1997), Scott (1997) yield closed-form solutions for European option prices. However, these models cannot price options by the absence of arbitrage alone because volatility is driven by a Wiener process that is imperfectly correlated with the Wiener process driving the asset return and volatility is not a traded asset. Equivalently, the risk neutral measure in these models is not unique and one cannot form an instantaneous risk-free portfolio by trading in the asset and one option only.

Heston and Nandi (1998) have developed an option pricing model based on an asymmetric GARCH process that offers closed-form solutions for option prices. It is shown here that the continuous-time limit of that particular GARCH model is a diffusion model in which the spot asset and its variance are driven by two Wiener processes that are perfectly correlated. In other words, from a distributional point of view, the same Wiener process drives both spot asset and the variance. As a result, although the variance is path dependent and the price process of the spot asset is non-Markovian, a unique risk-neutral measure exists and it is possible to price options by the absence of arbitrage only. Thus our setup retains the flexibility of pricing options by the absence of arbitrage alone, but accords a richer and more realistic dynamics to the volatility process than the Black-Scholes-Merton model or those based on the implied binomial trees of Dupire (1994), Derman and Kani (1994), Rubinstein (1994) whose empirical drawbacks were elaborately pointed out by Dumas, Fleming and Whaley (1998).

Further, in the diffusion setup, the asset price is conditionally lognormal and the variance follows a square root process. As a result, one can directly compute closed-form solutions for option prices using the results of Heston (1993). Towards that purpose, we present the necessary mappings, based on Foster and Nelson (1994), such that one can approximate (arbitrarily closely) the parameters of the continuous time model on the basis of the parameters of the discrete-time GARCH model. The discrete-time GARCH parameters can be easily estimated just by observing the history of asset prices.

Unlike in typical preference-free option pricing models, the option price in our model is a function of the mean asset return. Thus, from a purely theoretical point of view, it is not sufficient to know the current price of the spot asset to calculate the option price. One also needs to know the expected return of the spot asset. This stands in sharp contrast to the typical preference-free framework where the expected asset return is redundant in the option formula. A similar result has also been noted in Kallsen and Taqqu (1998), except that they do not offer any analytical solution. Although the expected asset return enters the option pricing formula (through the drift of the risk-neutral volatility process), option prices are not at all sensitive to the mean return. Thus form a practical point of view, the expected rate of return, although a parameter is basically inconsequential and one can in effect, option prices by setting the expected return to zero.

The results of this paper also show that the distinction between preference-free and preference-dependent option pricing under time varying volatility depends on whether trading is assumed to take place continuously or at discrete time intervals. What one eventually wants is an option pricing model that adequately captures the dynamics of the spot price and volatility and yields option prices that are close to those observed in the market.

Section 1 describes the discrete-time GARCH model, section 2 derives the continuous-time limit and provides the necessary mapping between the discrete-time parameters and continuous-time parameters and section 3 concludes.

1. Discrete-time Model

The discrete time GARCH model is as follows:

$$\log(S(t)) = \log(S(t-\Delta)) + r + \lambda h(t) + \sqrt{h(t)}z(t), \tag{1}$$
$$h(t+\Delta) = \omega + \beta h(t) + \alpha(z(t)-\gamma\sqrt{h(t)})^2$$

where S is the asset price, r is the continuously compounded interest rate for the time interval Δ and z(t) is a standard normal disturbance. h(t) is the conditional variance of the log return between t - Δ and t and is known from the information set at time t - Δ. The conditional variance in equation (1), although functionally different from the existing GARCH models, in fact is similar to the NGARCH and VGARCH models of Engle and Ng (1993). The conditional variance h(t) appears in the mean as a return premium. This allows the average spot return to depend on the level of risk.[1] α_1 determines the kurtosis of the distribution and α_1 being zero implies a deterministic time varying variance. The γ_1 parameter results in asymmetric influence of shocks; a large negative shock, z(t) raises the variance more than a large positive z(t). As the α and β parameters approach zero, the GARCH model is equivalent to the Black-Scholes model observed at discrete intervals.

[1] The functional form of this risk premium, $\lambda h(t)$, prevents arbitrage by ensuring that the spot asset earns the riskless interest rate when the variance equals zero.

2. Continuous-time Limit:

In the discrete-time model, the conditional mean and variance of h(t) as well as the covariance with the spot returns are,

$$E_{t-\Delta}\,[h(t+\Delta)] \;=\; \omega + \alpha_1 + (\beta_1+\alpha_1\gamma_1{}^2)h(t). \qquad (2)$$

$$Var_{t-\Delta}[h(t+\Delta)] \;=\; \alpha_1{}^2(2+4\gamma_1{}^2h(t))$$

$$Cov_{\,t-\Delta}[h(t+\Delta),\log(S(t))] \;=\; -2\,\alpha_1\gamma_1h(t).$$

There are various ways to approach a continuous-time limit as the time interval Δ shrinks. Since h(t) is the variance of the spot return over time interval Δ, it should converge to zero. To measure the variance per unit of time we define $v(t) = h(t)/\Delta$ and v(t) has a well defined continuous time limit. The stochastic process v(t) follows the dynamics

$$v(t+\Delta) \;=\; \omega_V + \beta_V\, v(t) + \alpha_V(z(t)-\gamma_V\sqrt{v(t)})^2, \qquad (3)$$

where

$$\omega_V \;=\; \omega/\Delta,\;\; \beta_V = \beta_1,\; \alpha_V = \alpha_1/\Delta,\; \gamma_V = \gamma_1\sqrt{\Delta}.$$

Let $\alpha_1(\Delta) = ¼\sigma^2\Delta^2$, $\beta_1(\Delta) = 0$, $\omega(\Delta) = (\kappa\theta-¼\sigma^2)\Delta^2$, $\gamma_1(\Delta) = \frac{2}{\sigma\Delta}-\frac{\kappa}{\sigma}$, and $\lambda(\Delta) = \lambda$. Then,

$$E_{t-\Delta}[v(t+\Delta)-v(t)] \;=\; \kappa(\theta-v(t))\Delta + ¼\kappa^2v(t)\Delta^2, \qquad (4)$$

$$Var_{t-\Delta}[v(t+\Delta)] \;=\; \sigma^2v(t)\Delta + \left(\frac{\sigma^4}{8}-\sigma^2\kappa v(t)+\frac{\sigma^2\kappa^2}{4}v(t)\,\Delta\right)\Delta^2. \qquad (5)$$

(Note that α_1, β_1, ω, γ_1 as defined above are not α_V, β_V, ω_V, and γ_V corresponding to the v(t) process). The correlation between the variance process and the continuously compounded stock return is

$$\mathrm{Corr}_{t-\Delta}[v(t+\Delta),\log(S(t))] = \frac{-\mathrm{sign}(\gamma_v)\sqrt{2\gamma_v{}^2 v(t)}}{\sqrt{1+2\gamma_v{}^2 v(t)}}. \tag{6}$$

As the time interval Δ shrinks the skewness parameter, $\gamma_v(\Delta)$ approaches positive or negative infinity. Consequently the correlation in equation (A5) approaches 1 (or negative 1) in the limit.

The variance process, v(t) has a continuous-time diffusion limit following Foster and Nelson (1994). As the observation interval Δ shrinks, v(t) converges weakly to the square-root process of Feller (1951), Cox, Ingersoll Ross (1985), and Heston (1993)

$$d\log(S) = (r+\lambda v)dt + \sqrt{v}dz \tag{7}$$

$$dv = \kappa(\theta-v)dt + \sigma\sqrt{v}dz,$$

where z(t) is a Wiener process. Note that the same Wiener process drives both the spot asset and the variance. This limiting behavior of this GARCH process is very different from those of other GARCH processes such as GARCH 1-1 or most of the other asymmetric GARCH processes in which two different Wiener processes drive the spot assets and the variance. Also, while the above shows that the asset returns and variance processes under the data generating measure converge to well-defined continuous-time limits, one still needs to verify that the discrete risk-neutral processes converge to appropriate continuous-time limits if the discrete-time GARCH option prices are to converge to their continuous-time limits.

As shown in Proposition 1 of Heston and Nandi (1998), in the risk-neutral distribution, γ is replaced by γ+λ+½ and λ is replaced by –1/2. Therefore, the risk-neutral parameter for the v(t) process is

$$\gamma_v^*(\Delta) = \gamma_1^*(\Delta)\sqrt{\Delta} = \frac{2}{\sigma\sqrt{\Delta}} - (\frac{\kappa}{\sigma}-\lambda-½)\sqrt{\Delta}. \qquad (8)$$

Consequently the risk-neutral process has a different mean

$$E_{t-\Delta}^*[v(t+\Delta)-v(t)] = [\kappa(\theta-v(t))-\sigma(\lambda+½)v(t)]\Delta + ¼(\kappa+\sigma(\lambda+½))^2 v(t)\Delta^2 \qquad (9)$$

Again following Foster and Nelson (1994), it follows that the continuous-time risk-neutral processes are

$$d\log(S) = (r-v/2)dt + \sqrt{v}dz^*, \qquad (10)$$

$$dv = (\kappa(\theta-v)-\sigma(\lambda+½)v)dt + \sigma\sqrt{v}dz^*,$$

where $z(t)^*$ is a Wiener process under the risk-neutral measure. As with the data generating measure, the same Wiener process drives both asset returns and variance under the risk-neutral measure. Note that the risk-neutral processes are equivalent to the risk-neutral processes of Heston (1993) with the two Wiener processes therein being perfectly correlated. Consequently, the discrete-time GARCH option prices converge to the continuous-time option prices of Heston (1993) as Δ shrinks. Figure 1 shows how the discrete time GARCH model converges to the continuous time model as the number of time intervals, Δ increases. The parameters used for an at-the-money option with a spot asset price, S= \$100, strike price, K = \$100 with 0.5 years to maturity are, $\kappa + \sigma(\lambda + ½) = 2$, $\kappa\theta = 0.02$, $\rho = -1$, $\sigma = 0.1$, $v = 0.01$. Given these parameters, one can directly use Heston (1993) formula to compute the price of an option.

As the two Wiener processes are perfectly correlated, one can price options by appealing only to the absence of arbitrage using the hedging arguments of Black and Scholes (1973) and Merton (1973) or equivalently by showing the existence of a unique risk-neutral process as per Cox and

Ross (1976), Harrison and Kreps (1979), Harrison and Pliska (1981). Note however, the parameter related to the asset risk premium, λ appears in option prices unlike in the Black-Scholes-Merton setup. Although one can create an instantaneously risk-free portfolio by trading in the underlying asset and a single option, one has to hedge both the changes in the spot asset and the change in volatility. Volatility, in turn is a function of past asset prices and therefore a function of λ, even under the risk-neutral distribution. In other words, λ appears in the option pricing formula simply due to the path dependence of the volatility function and the fact that volatility is not traded; this type of result has also been noted in Kallsen and Taqqu (1998).

Also this is a continuous-time model of the type suggested by Dumas, Fleming, and Whaley (1998) in which variance is a function of the path of the spot asset price. The above variance process overcomes the very restrictive spot variance assumption of the implied binomial tree models of Derman and Kani (1994), Dupire (1994) and Rubinstein (1994), but still permits pricing only by the absence of arbitrage and also admits a closed-form solution.

3. Conclusion

We have shown how to compute preference free option prices using closed-form solutions when volatility of the asset price is path dependent and trading takes in continuous time. The particular option pricing model is a continuous time limit of the discrete GARCH option pricing model of Heston and Nandi (1998). The discrete-time model also admits closed-form solutions and has been shown to dominate, *out-of-sample*, the very flexibly parametrized ad hoc Black-Scholes model of Dumas, Fleming and Whaley (1998) with a strike and maturity specific implied volatility for each option.

However, with discrete trading, the GARCH option price is not necessarily preference free, although its continuous time limit is. Thus the distinction between the preference free and preference dependent option pricing models with time varying volatility is essentially an artifact of the assumption behind the frequency of trading. What one really needs is an option pricing model that yields prices closer to those observed in the market than other competing models for better pricing and risk management.

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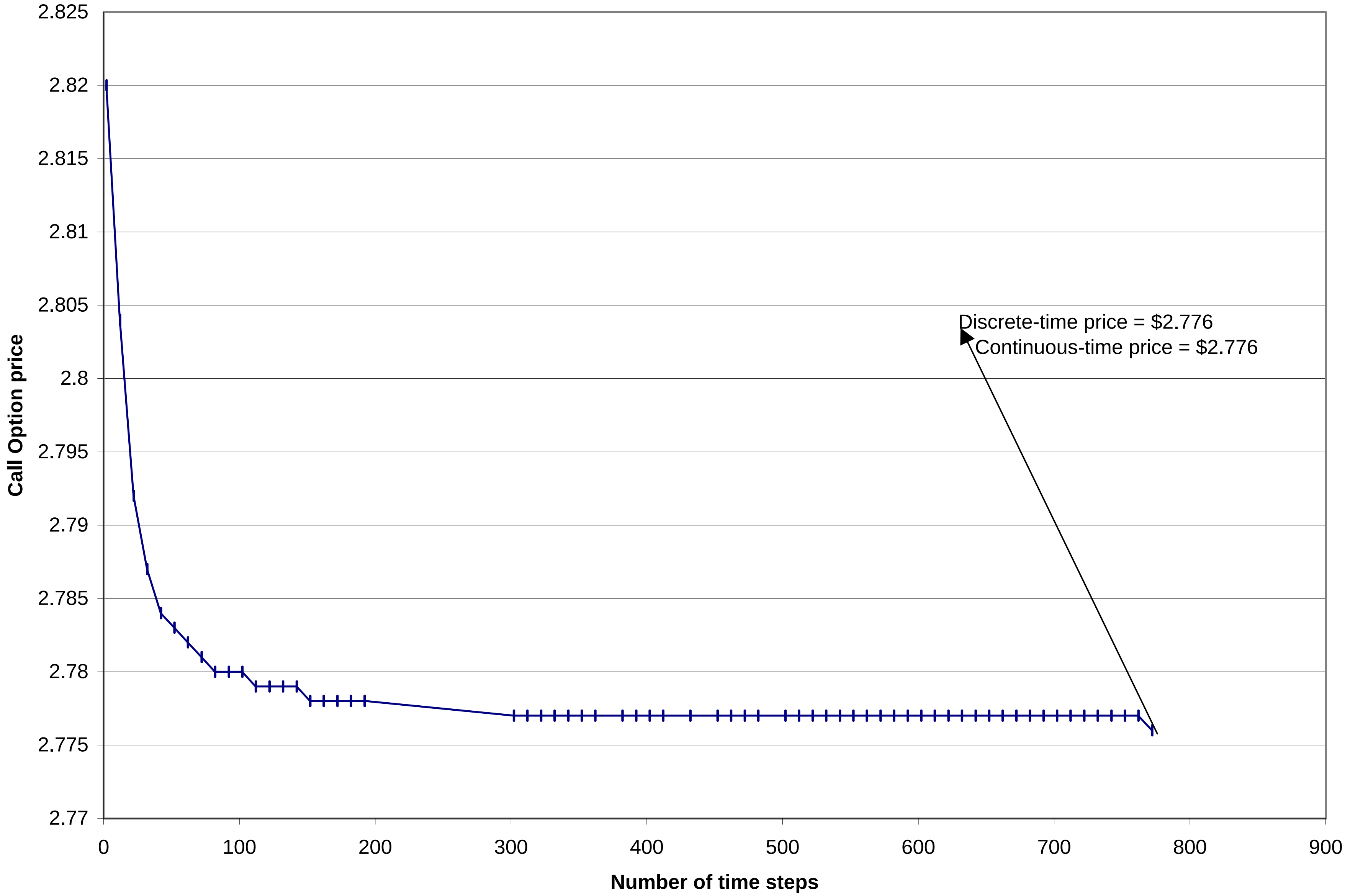
Figure 1



Shows how the discrete-time GARCH prices converge to the continuous-time option prices with an increase in the number of time/trading intervals.