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A Discrete-Time Two-Factor Model for Pricing Bonds and Interest Rate Derivatives under Random Volatility

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Abstract: This paper develops a discrete-time two-factor model of interest rates with analytical solutions for bonds and many interest rate derivatives when the volatility of the short rate follows a GARCH process that can be correlated with the level of the short rate itself. Besides bond and bond futures, the model yields analytical solutions for prices of European options on discount bonds (and futures) as well as other interest rate derivatives such as caps, floors, average rate options, yield curve options, etc. The advantage of our discrete-time model over continuous-time stochastic volatility models is that volatility is an observable function of the history of the spot rate and is easily (and exactly) filtered from the discrete observations of a chosen short rate/bond prices. Another advantage of our discrete-time model is that for derivatives like average rate options, the average rate can be exactly computed because, in practice, the payoff at maturity is based on the average of rates that can be observed only at discrete time intervals.

Calibrating our two-factor model to the treasury yield curve (eight different maturities) for a few randomly chosen intervals in the period 1990–96, we find that the two-factor version does not improve (statistically and economically) upon the nested one-factor model (which is a discrete-time version of the Vasicek 1977 model) in terms of pricing the cross section of spot bonds. This occurs although the one-factor model is rejected in favor of the two-factor model in explaining the time-series properties of the short rate. However, the implied volatilities from the Black model (a one-factor model) for options on discount bonds exhibit a smirk if option prices are generated by our model using the parameter estimates obtained as above. Thus, our results indicate that the effects of random volatility of the short rate are manifested mostly in bond option prices rather than in bond prices.

JEL classification: G12, G13

Key words: GARCH, volatility, bonds, options

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1 Introduction

Extending the work of Vasicek (1977), Cox, Ingersoll and Ross (1985, henceforth CIR) and Heath-Jarrow and Morton (1992, henceforth HJM), researchers have developed a plethora of models for the term structure of interest rates that often involves pricing bonds and/or interest rate derivatives. One class of models often takes the short rate to be the state variable and then prices bonds and bond options by deriving the risk-neutral dynamics of the short rate. Since the short rate is not a traded asset, a parameter related to the interest rate risk premium appears in the formulae. On the other hand, the HJM approach takes the entire yield curve (or equivalently the set of forward rates or the bond prices) to be the state variable and derives arbitrage-free prices of option on bonds such that any functional specification for the risk premium is not required to compute option prices because the various bonds are traded assets. Also the HJM approach is able to match the existing term structure by default as the term structure is itself the state variable.

It is well known that the dynamics of the term structure cannot be captured by one factor (see Litterman and Scheinkman (1991)). In fact, Dybvig (1997) emphasizes the existence of a second factor related to the volatility of interest rates that may not have any major impact on the prices of the spot bonds, but may be very important for bond options. Andersen and Lund (1996) also find that a factor which helps explain the curvature of the yield curve and is, in fact, closely related to the volatility of the short rate. Unlike the HJM approach, the short rate based approaches can easily accommodate a second non-traded state variable such as volatility and maintain analytical as well as numerical tractability. In fact, Longstaff and Schwartz (1992, henceforth LS) and Chen and Scott (1992, 1994, henceforth CS) develop continuous-time two-factor models along the lines of CIR (1985) that can incorporate random volatility in the evolution of the short rate and offer analytical solutions for bonds and other interest rate derivatives.

Although the continuous-time models offer valuable insights, they are hard to implement for pricing and hedging bonds and bond options with volatility as the second factor. This is simply because volatility is unobservable in these continuous-time stochastic volatility models and it is not possible to filter a continuous volatility variable from discrete observations of interest rates or bond prices.[1] An unobservable volatility implies that the spot volatility at time t which is needed to price bonds and bond options is not known at the time such a price needs to be calculated and one is constrained to use the spot volatility inferred from a previous period that does not necessarily reflect the current information in the term structure.[2]. The non-observability of volatility also precludes the use of the information in the time series of interest rate/bond prices for parameter estimation; instead one is constrained to use the information only in a cross-section of bond/option prices.

While GARCH volatility processes have been very popular to describe the dynamics of volatility in equity and currency markets (see Bollerslev et al. (1992)), Brenner, Harjes and Kroner (1996), Koedjik et al. (1994) and others have shown that certain GARCH processes can also capture the volatility dynamics of interest rates; also it is possible to allow correlation between interest rates and interest rate volatility in the GARCH models. As shown in Foster and Neslson (1994), GARCH models are closely related to the continuous time stochastic volatility models because discrete-time GARCH processes converge to continuous-time stochastic volatility processes as

[1]One could possibly use an extended Kalman filter (an approximation to the regular Kalman filter) to get the spot volatility in continuous time models as in Melino and Turnbull (1990). Aside from being computationally intensive, the volatility obtained therein is only an approximation and options prices can be very sensitive to errors in volatility. The same is true of the efficient methods of moments of Gallant and Tauchen (1996).

[2]One can calibrate the continuous-time stochastic volatility models to market data on options/bonds and therefore get an estimate of the volatility at an earlier time, say $t-1$, which is different from the volatility at time t

the time/trading interval shrinks. However, a discrete-time volatility model based on GARCH has the advantage that volatility is an observable function of the history of interest rates/bond prices. Until now though there did not exist explicit solutions for bond prices and prices of various interest rate derivatives under a GARCH process.[3]

This paper develops a discrete time two-factor model of interest rates with analytical solutions for bonds and other interest rate derivatives in which the second factor is a random volatility following a GARCH process while the first factor is the mean reverting short rate and the two factors are correlated. Besides bond and bond futures, the model generates explicit analytical solutions for prices of options on discount bonds, discount bond futures as well as other interest rate derivatives such as caps, floors, average rate options etc. The advantage of this model besides its analytical tractability is that unlike continuous-time stochastic volatility models, volatility is easily estimated from the discrete observations of the short rate. Thus the spot volatility that is input to the options formula is known and can be updated on the basis of the most current information. Therefore, in terms of parameter estimation, our model enables the simultaneous use of the implied information in the cross-section of bond/bond-option prices and the historical information in the evolution of the interest rate. Moreover, for derivatives like average rate options, our discrete-time model has the practical advantage that the average rate can be explictly computed because in practice, the payoff at maturity is based on the average of rates observed at discrete time intervals and not any continuously observed rate as assumed in continuous time models.

Calibrating our model to the yield curve (eight different maturities) for a few randomly chosen two week intervals in the period 1990-1996, we find that the two-factor version does not improve (statistically and economically) upon the nested one factor model (which is a discrete-time version of the Vasicek model) in terms of pricing

[3]Duan (1996) resorts to a computationally intensive Monte Carlo procedure for valuing bonds and bond options under GARCH.

the cross-section of spot bonds. This occurs despite the fact that the one-factor is rejected in favor of the two factor model in explaining the time series behavior of a chosen short rate (computed from the three month T-bill yields). Also this particular conclusion is robust to the length of the interval used to sample the term structure. However, given option prices (on discount bonds) from our two-factor model, the implied volatilities from the Black model (which is a one-factor model and widely used in the market place) exhibit the smirk/skew feature. These option prices are generated not by conjectured parameter estimates, but actual parameter estimates obtained by calibrating the model to the observed yield curve.

2 Model

We assume that the short rate, r_t, which is the interest rate applicable on a loan at time t, that is to be repaid at time, $t + \triangle$, follows the following process over time steps of length $\triangle$.

$$r_{t+\triangle} = \mu_0 + \mu_1 r_t + \lambda\, h_{t+\triangle} + \sqrt{h_{t+\triangle}} z_{t+\triangle} \tag{1}$$

$$h_{t+\triangle} = \omega + \beta\, h_t + \alpha(z_t - \gamma\sqrt{h_t})^2 \tag{2}$$

where z_t is a standard normal and $h_{t+\triangle}$ is the variance of $r_{t+\triangle} - r_t$, conditional on the information at time t. The dynamics of h_t is similar to the asymmetric GARCH processes, namely, NGARCH and VGARCH that were used by Engle and Ng (1993. A non-zero γ allows correlation between the level of interest rates and the conditional variance. Note that $h_{t+\triangle}$ is known as of time t given the history of r_t until t. In particular, in the above system of equations, we have a mean reverting short rate with GARCH volatility and the level of the short rate is itself a function of the volatility. Ignoring the λ term in (1) , it can be roughly thought of as a discrete-time counterpart of the Vasicek (1977) model, augmented with GARCH volatility. In fact, if we restrict h_t to be constant, we do have the discrete-time counterpart of the continuous-time Vasicek model. Note that we can easily drop λ from (1) and still generate all the results of this paper. Similarly we can include r_t in the RHS of (2) to reflect that the level of the variance depends on the level of interest rates and still get the analytical solutions.

Following Heston and Nandi (1999) (who model the convergence to continuous-time along the lines of Foster and Nelson (1994)) it is easy show that the continuous-time limit of (1) and (2) as $\triangle$ shrinks to zero is

$$
\begin{aligned}
dr_t &= (\mu_0 + \mu_1^* r_t + \lambda\, v_t)\, dt + \sqrt{v_t} dW_t \qquad (3)\\
dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t \qquad (4)
\end{aligned}
$$

where, $\mu_1^* = \mu_1 - 1$, $\alpha(\triangle) = \frac{1}{4}\sigma^2\triangle^2$, $\beta(\triangle) = 0$, $\omega(\triangle) = (\kappa\theta - \frac{1}{4}\sigma^2)\triangle^2$, and $\gamma(\triangle) = \frac{2}{\sigma\triangle} - \frac{\kappa}{\sigma}$ and $\lambda(\triangle) = \lambda$ and W_t is a Wiener process. Note that unlike the continuous-time limit of many GARCH processes, the same Wiener process drives the short rate and its volatility. It can be verified that the continuous-time models lends itself to an affine structure in that the logarithm of a bond yield is affine in r_t and v_t. As a result, one can work out analytical solutions for the continuous-time model also. However, this paper will concentrate only on the discrete-time model as parameter estimation and therefore model implementaion is much easier under the discrete-time setup.

From now on, working with the discrete-time model, we will set $\triangle = 1$. Let the time t price of a discount bond that matures at T be $P(t,T)$. As is the case, we will work under the risk-neutral distribution to evaluate bond and bond option prices. It can be shown (see the appendix) that under the risk-neutral distribution, z_t is replaced by z_t^q, such that $z_t^q = z_t - \eta\sqrt{h_{t+1}}$ is a standard normal under the risk-neutral distribution, where η is a constant. In other words, the risk neutral dynamics of r_t and h_t are given by

$$
r_{t+1} = \mu_0 + \mu_1 r_t + \lambda^*\, h_{t+1} + \sqrt{h_{t+1}} z_{t+1}^q \qquad (5)
$$

$$h_{t+1} = \omega + \beta h_t + \alpha(z_t^q - \gamma^* \sqrt{h_t})^2 \tag{6}$$

where $\lambda^* = \lambda + \eta$, $\gamma^* = \gamma - \eta$. Consider the price of a zero coupon bond at t that expires at t+2 i.e., $P(t, t+2)$. It is true by simple definition that,

$$\begin{aligned} P(t, t+2) &= \exp\left(-r_t E_t^q(\exp(-r_{t+1}))\right) \\ &= \exp(-\left(\mu_0 + (\mu_1 + 1)r_t + (\lambda^* + 0.5)h_{t+1}\right)) \end{aligned} \tag{7}$$

where $E_t^q()$ is the conditional expectation under the risk-neutral distribution.

Thus the yield of the two period bond is $\frac{1}{2}\left(\mu_0 + (\mu_1 + 1)r_t + (\lambda^* + 0.5)h_{t+1}\right)$. Note that the bond yield is affine in the state variables r_t and h_{t+1}. In particular, we can similarly calculate the yield on a three period bond by using iterated conditional expectation and show that the three period yield is also affine in the state variables. The affine nature of the bond yield in the state variables, r_t and h_{t+1} imply that we can write the price of the a bond with T-t periods to maturity as

$$P(t, T) = \exp(A(t, T) + B(t, T)r_t + C(t, T)h_{t+1}) \tag{8}$$

as in Cox-Ingersoll-Ross (1985), Heston (1990), Duffie and Kan (1996) and many others. As r_t and h_{t+1} are known as of time t, it remains to solve for the coefficients $A(t, T)$, $B(t, T)$ and $C(t, T)$ in terms of the model parameters. These coefficients can be solved recursively from a boundary condition. For the boundary condition, consider the price of a bond (that will mature at T) at $T - 2$.

$$P(T-2, T) = \exp\left(A(T-2, T) + B(T-2, T)r_{T-2} + C(T-2, T)h_{T-1}\right) \tag{9}$$

But, following (7), it is also true that $P(T-2, T) = \exp(-\left(\mu_0 + (\mu_1 + 1)r_{T-2} + (\lambda^* + 0.5)h_{T-1}\right))$. Equating the two expressions for $P(T-2, T)$ and matching coefficients on the state variables and the constant, we get the following boundary conditions,

$$A(T-2,T) = -\mu_0 \tag{10}$$

$$B(T-2,T) = -(\mu_1+1) \tag{11}$$

$$C(T-2,T) = -(\lambda^*+0.5) \tag{12}$$

Now we will derive the backward recursion formula for the coefficients $A(t,T)$, $B(t,T)$ and $C(t,T)$ given the above boundary conditions. Since the expected one-period appreciation in the price of a bond under the risk-neutral distribution is the short rate,

$$\begin{aligned} P(t,T) &= \exp(-r_t)E_t^q\left[P(t+1,T)\right] \\ &= \exp(-r_t)E_t^q\left[\exp(A(t+1,T)+B(t+1,T)r_{t+1}\right. \\ &+ \left. C(t+1,T)h_{t+2})\right] \end{aligned} \tag{13}$$

Substituting the risk-neutralized dynamics for r_{t+1} and h_{t+1}, and doing the algebra as shown in the appendix, we get that,

$$\begin{aligned} A(t,T) &= A(t+1,T)+\mu_0 B(t+1,T)+\omega C(t+1,T) \\ &- \frac{1}{2}\log(1-2\alpha C(t+1,T)) \end{aligned} \tag{14}$$

$$B(t,T) = \mu_1 B(t+1,T)-1 \tag{15}$$

$$\begin{aligned} C(t,T) &= \lambda^* B(t+1,T)+\beta C(t+1,T) \\ &+ \alpha C(t+1,T)\left[\frac{\frac{B(t+1,T)^2}{2\alpha C(t+1,T)}-2\gamma^* B(t+1,T)+\gamma^{*2}}{1-2\alpha C(t+1,T)}\right] \end{aligned} \tag{16}$$

In other words, starting from the boundary conditions, (10), (11) and (12), we have to do a simple backward recursion using (14), (15) and (16) to find the coefficients $A(t,T)$, $B(t,T)$ and $C(t,T)$ and once we have these, then $P(t,T)$ is given by (8). The next section describes how one can calculate bond futures prices in this model.

3 Bond Futures Prices

Let $F(t, \tau, T)$ be the time t price of a futures contract on a discount bond such that the futures contract expires at τ and the discount bond expires at T, where $T > \tau$. As with the spot bond, let the futures price be given by[4]

$$F(t, \tau, T) = \exp(A_f(t; \tau, T) + B_f(t; \tau, T) r_t + C_f(t; \tau, T) h_{t+1}) \tag{17}$$

As with the bond price, the coefficients, $A_f(t; \tau, T)$, $B_f(t; \tau, T)$ and $C_f(t; \tau, T)$ are derived recursively from a boundary condition. To derive the boundary condition, note that futures price equal the spot price at the maturity of the futures contract. Therefore, $F(\tau, \tau, T) = P(\tau, T)$. This implies the following:

$$\begin{aligned} A_f(\tau; \tau, T) &= A(\tau, T) & (18) \\ B_f(\tau; \tau, T) &= B(\tau, T) & (19) \\ C_f(\tau; \tau, T) &= C(\tau, T) & (20) \end{aligned}$$

where $A(\tau, T)$, $B(\tau, T)$ and $C(\tau, T)$ are known from the recursions needed to calculate the price of the discount bond that matures at T.

It is a well known result that futures prices are martingales under a martingale measure (see Cox, Ingersoll and Ross (1981)). Thus

$$F(t; \tau, T) = E_t^q \left[F(t+1; \tau, T) \right] \tag{21}$$

Substituting the guess for the futures price in the above equation,

$$\begin{aligned} \exp(A_f(t; \tau, T) + B_f(t; \tau, T) r_t + C_f(t; \tau, T) h_{t+1}) &= E_t^q [\exp(A_f(t+1; \tau, T) \\ &+ B_f(t+1; \tau, T) r_{t+1} \end{aligned}$$

[4]The fact that we can write the bond futures price in this form is simply due to the fact that like spot bonds, prices of bond futures are also affine in the state variables.

$$+ \quad C_f(t+1;\tau,T)h_{t+2})] \quad (22)$$

Substituting the dynamics of r_{t+1}, h_{t+2} in the above, taking the relevant expectation and matching the coefficients, we get

$$\begin{aligned} A_f(t;\tau,T) &= A_f(t+1;\tau,T) + \mu_0 B_f(t+1;\tau,T) + \omega C_f(t+1;\tau,T) \\ &- \frac{1}{2}\log(1-2\alpha C_f(t+1;\tau,T)) \quad (23) \\ B_f(t;\tau,T) &= \mu_1 B_f(t+1;\tau,T) \quad (24) \\ C_f(t;\tau,T) &= \lambda^* B_f(t+1;\tau,T) + \beta C_f(t+1;\tau,T) \\ &+ \alpha C_f(t+1;\tau,T)\left[\frac{\frac{B_f(t+1;\tau,T)^2}{2\alpha C_f(t+1;\tau,T)} - 2\gamma^* B_f(t+1;\tau,T) + \gamma^{*2}}{1-2\alpha C_f(t+1;\tau,T)}\right] \quad (25) \end{aligned}$$

Given the boundary conditions in (18), (19) and (20), (23), (24) and (25) will give us the required coefficients that are needed to calculate $F(t,\tau,T)$. This completes our calculation of the bond futures price.[5]

4 Bond Options

4.1 Option on Discount Bond

Now we will calculate the price of an European option on a discount bond. Although we do not have explicit representations for the prices of American options on puts, one could compute the early exercise premium from the continuous-time analog of the nested one-factor model (the Vasicek model) along the lines of Carr, Jarrow and Myneni (1992), Huang, Subrahmanyam and Yu (1996), Ju (1998) and others by using the price of the discount bond that matures at the same time as the option as the numeraire. This early exercise can be added to the European value to obtain an

[5] Note that we can readily adapt our formula to calculate futures prices in the way they are sometimes quoted in the market.

approximate American value.

Let the time to expiration of the option be τ and that of the underlying bond be T and $T > \tau$. Let the price of a call option (at time t) on the discount bond be $C_o(t, \tau, T)$. The payoff from the option at maturity is $C_o(\tau, \tau, T) = \max[P(\tau, T) - K, 0]$. As in Merton (1973), instead of the money market account, we choose the τ maturity bond as the numeraire and assume that it is traded. Deflated by the numeraire, all asset prices are martingales under the martingale measure, which we shall refer to as the forward measure (as in Jamshidian (1989), Brace, Gatarek and Musiela (1997) and others). Hence, $\frac{C_o(t,\tau,T)}{P(t,\tau)} = E_t^F(\frac{C_o(\tau,\tau,T)}{P(\tau,\tau)})$ where $E_t^F()$ is the conditional expectation under the forward measure. Noting that $P(\tau, \tau) = 1$, we have that

$$\frac{C_o(t, \tau, T)}{P(t, \tau)} = E_t^F\left(\max[\exp(A(\tau, T) + B(\tau, T)r_\tau + C(\tau, T)h_{\tau+1}) - K, 0]\right) \tag{26}$$

Let $x \equiv A(\tau, T) + B(\tau, T)r_\tau + C(\tau, T)h_{\tau+1}$. Thus, in order to calculate the option price, we have to know the conditional density of x under the forward measure which in turn is known if we know its corresponding characteristic function or equivalently the moment generating function. Let $f(\phi) = E_t^F(\exp(\phi x))$ denote the conditional moment generating function of x at time t. $f(\phi)$ also depends on the state variables and the parameters of the model; however, they are being suppressed for notational convenience. We shall guess the following affine functional form for $f(\phi)$[6],

$$f(\phi) = \exp(A_1(t; \phi, \tau) + B_1(t; \phi, \tau)r_t + C_1(t; \phi, \tau)h_{t+1}) \tag{27}$$

The coefficients $A_1(t; \phi, \tau)$, $B_1(t; \phi, \tau)$ and $C_1(t; \phi, \tau)$ can be obtained from a boundary

[6]The fact that the moment generating function is affine in the state variables as bond yields were verified to be affine in the date variables in a previous section.

condition using a recursive procedure. The boundary conditions are

$$A_1(\tau;\phi,\tau) = 0 \tag{28}$$

$$B_1(\tau;\phi,\tau) = \phi B(\tau,T) \tag{29}$$

$$C_1(\tau;\phi,\tau) = \phi C(\tau,T) \tag{30}$$

where $A(\tau,T)$, $B(\tau,T)$ and $C(\tau,T)$ are known from the recursions needed to calculate the price of the discount bond that expires at T, $P(t,T)$. Let $A^*(t+1;\phi,\tau) \equiv A_1(t+1;\phi,\tau) + A(t+1;\tau)$, $B^*(t+1;\tau) \equiv B_1(t+1;\phi,\tau) + B(t+1;\tau)$, $C^*(t+1;\tau) \equiv C_1(t+1;\phi,\tau) + C(t+1;\tau)$ where $A(t+1;\tau), B(t+1;\tau)$ and $C(t+1;\tau)$ are already gotten from the recursion needed to compute the price of the discount bond that expires at τ, $P(t,\tau)$. The recursion required for calculating $f(\phi)$ from the above boundary conditions are than given as follows (note that they are very similar to the recursions for bond price). For brevity of notation, we are suppressing T from the notations in the following recursion (derivation is in the appendix):

$$\begin{aligned} A_1(t;\phi,\tau) &= -A(t;\tau) + A^*(t+1;\phi,\tau) + \mu_0 B^*(t+1;\phi,\tau) + \omega C^*(t+1;\phi,\tau) \\ &- \frac{1}{2}\log(1-2\alpha C^*(t+1;\phi,\tau)) \end{aligned} \tag{31}$$

$$B_1(t;\phi,\tau) = -(B(t;\tau)+1) + \mu_1 B^*(t+1;\phi,\tau) \tag{32}$$

$$\begin{aligned} C_1(t;\phi,\tau) &= -C(t;\tau) + \lambda^* B^*(t+1;\phi,\tau) + \beta C^*(t+1;\phi,\tau) \\ &+ \alpha C^*(t+1;\phi,\tau)\left[\frac{\frac{B^*(t+1;\phi,\tau)^2}{2\alpha C^*(t+1;\phi,\tau)} - 2\gamma^* B^*(t+1;\phi,\tau) + \gamma^{*2}}{1-2\alpha C^*(t+1;\phi,\tau)}\right] \end{aligned} \tag{33}$$

At time t, having gotten $A_1(\phi,t;\tau)$, $B_1(\phi,t;\tau)$ and $C_1(\phi,t;\tau)$, we know the conditional moment generating function of x at time t. If $f(\phi)$ is the moment generating function, $f(i\phi)$ is the characteristic function. Inverting the characteristic function as in Heston and Nandi (1999), one gets the price of the call option to be

$$
\begin{aligned}
C_o(t;\tau,T) &= \frac{1}{2}P(t,T) + P(t,\tau)\frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\log(K)}f(i\phi+1)}{(i\phi)}\right]d\phi \\
&- KP(t,\tau)\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\log(K)}f(i\phi)}{(i\phi)}\right]d\phi\right) \qquad (34)
\end{aligned}
$$

The derivation of (34) is shown in the appendix. By a simple rearrangement, we can also write the above equation in the typical Black-Scholes format as $C_o(t;\tau,T) = P(t,T)M_1() - KP(t,\tau)M_2()$ where $M_1()$ and $M_2()$ are two probability distribution functions. The two univariate integrals converge very fast and are very easy to integrate numerically. In particular, the two univariate integrals can be combined and evaluated as a single univariate integral in fractions of a second using any good integration routine. We used the "Romberg" integration routine of Press et al. (1992) to produce option prices, for example the prices of options on bond futures (discussed next) that generate the smile shown in Figure 2.

Options on coupon bonds cannot be evaluated through a straightforward analytical procedure unlike in the one-factor model (see Jamshidian (1979)). While one can write down a formula for these options and evaluate them quasi-analytically as in Chen and Scott (1992), the numerical accuracy of such a procedure is an open issue that cannot be addressed within the scope of this paper.

4.2 Options on Bond Futures

This section shows how one can calculate the price of an European option on a discount bond futures under our model. Let $F(t,T1,T2)$ denote the current futures price for a contract on a discount bond that expires at $T2$; the futures contract expires at $T1$. Suppose a call option is traded on the futures contract and the option expires at τ. Let $\tau < T1 < T2$. It can be shown that the price of the option, $C_o(t;\tau,T1,T2)$

is given by

$$
\begin{aligned}
C_o(t;\tau,T1,T2) &= P(t,\tau)\left(\frac{1}{2}F(t;T1,T2)+\frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\log(K)}f(i\phi+1)}{(i\phi)}\right]d\phi\right. \\
&- \left. K\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\log(K)}f(i\phi)}{(i\phi)}\right]d\phi\right)\right) \qquad (35)
\end{aligned}
$$

As in the preceding section for options on discount bonds, $f(i\phi)$ is the characteristic function and the corresponding moment generating function is

$$
f(\phi)=\exp(A_1(t;\phi,\tau)+B_1(t;\phi,\tau)r_t+C_1(t;\phi,\tau)h_{t+1}) \qquad (36)
$$

$f(\phi)$ can be calculated recursively exactly as with options on discount bonds, but with a different boundary condition as given below:

$$
A_1(\tau;\phi,\tau)=\phi A_f(\tau,T1,T2) \qquad (37)
$$

$$
B_1(\tau;\phi,\tau)=\phi B_f(\tau,T1,T2) \qquad (38)
$$

$$
C_1(\tau;\phi,\tau)=\phi C_f(\tau,T1,T2) \qquad (39)
$$

Note that $A_f(\tau,T1,T2)$, $B_f(\tau,T1,T2)$ and $C_f(\tau,T1,T2)$ are known from the recursions needed to calculate the futures price, namely the price of a .

5 Caps (Floors), Average Rate Options

Prices of other types of interest rate options such as caps and floors can be calculated in the same way as above by noting that these are portfolios of options on discount bonds (see Hull (1997) for the analogies).

5.1 Average Rate Option

The price of an Asian option on interst rate or an option on the average rate from some time t until some later time T can be computed by taking advantge of the affine structure for bond prices as in the continuous-time models of Bakshi and Madan (1998) and Ju (1998). Our model has a distinct advantage over continuous time models in this regard because in practice the average rate can only be calculated from the interest rates that are observed at discrete intervals of time. Using a discrete-time averaging in a continuous time averaging formula is problematic as it can induce biases, the magnitudes of which are not known. However, our model being set in discrete time has the exact average and therefore can be readily implemented in parctice without any biases.

Let the payoff from the call option at time, T be $\max(\frac{1}{T-1}\sum_{u=0}^{u=T-1} r_u - K, 0)$. Let $M(t,T-1) = \sum_{u=t}^{u=T-1} r_u$ and $a(t-1) = \sum_{u=0}^{u=t-1} r_u$. Thus the price of the call option at time t, $C_o(t,T)$ is

$$C_o(t,T) = \frac{1}{T-1} P(t,T)(E_t^F\left(\max[a(t-1) + M(t,T-1) - K(T-1), 0]\right) \tag{40}$$

where $E_t^F()$ denotes the expectation under the forward measure i.e. where the numeraire is the price of a discount bond that expires at T. Let $x \equiv a(t-1)+M(t,T-1)$. Let $f(i\phi) = E_t^F(\exp(i\phi x))$ be the characteristic function of x. It can be shown that

$$f(i\phi) = \exp\left(i\phi a(t-1) + A_1(t;\phi,T-1) + B_1(t;\phi,T-1)r_t + C_1(t;\phi,T-1)h_{t+1}\right) \tag{41}$$

Note that $a(t-1)$ and h_{t+1} are known as of time t. As with bond options, the the coefficients $A_1(t;\phi,T-1)$, $B_1(t;\phi,T-1)$ and $C_1(t;\phi,T-1)$ can be obtained from a boundary condition using a recursive procedure. The boundary conditions are (see the appendix):

$$A_1(T-2;\phi,T-1) = -A(T-2,T) + (i\phi - 1)\mu_0 \tag{42}$$

$$
\begin{aligned}
B_1(T-2;\phi,T) &= -B(T-2,T) + (i\phi - 1)(1+\mu_1) & (43)\\
C_1(T-2;\phi,T-1) &= -C(T-2,T) + (i\phi - 1)\left[\lambda^* + \frac{1}{2}(i\phi - 1)\right] & (44)
\end{aligned}
$$

Note that $A(t-2,T)$, $B(T-2,T)$ and $C(T-2,T)$ can be obtained from the recursions for computing the price of the zero-coupon bond expiring at time T. In fact, the recursions for $A_1(t,T)$ and $C_1(t,T)$ are functionally equivalent to the recursions for the option on the zero-coupon bond i.e., (31) and (33). However, the recursion for $B_1(t,T)$ is slightly different and is given as

$$
B_1(t;\phi,T-1) = \mu_1(B(t+1,T-1) + B_1(t+1;\phi,T-1)) - B(t,T) \tag{45}
$$

where $B(t,T)$ and $B(t+1,T)$ come from the recursion needed to compute the price of a zero-coupon bond expiring at T and is known as of time t (note that $B(t,T)$ is independent of ϕ). Inverting the characteristic function one gets the price of the call option to be (derivation of (46) is shown in the appendix),

$$
\begin{aligned}
C_o(t;\tau,T) &= \frac{1}{T-1}P(t,T)\left[\left(\frac{1}{2}E_t^F(M(t,T)) + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi K^*}f_\phi(i\phi)}{-\phi}\right]d\phi\right)\right.\\
&\left. - \quad K^*\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi K^*}f(i\phi)}{(i\phi)}\right]d\phi\right)\right] \qquad (46)
\end{aligned}
$$

where, $K^* = (T-1)K - a(t-1)$ and $f_\phi(i\phi) = \frac{\partial f(i\phi)}{\partial \phi}$. At this stage we do not know $f_\phi(i\phi)$ and $E_t^F(M(t,T))$. However, both of these can be explicitly computed given the model parameters.

Note that

$$
f_\phi(i\phi) = f(i\phi)\left[a(t-1) + \frac{\partial A_1(t;\phi,T-1)}{\partial\phi} + \frac{\partial B_1(t;\phi,T-1)}{\partial\phi}r_t + \frac{\partial C_1(t;\phi,T-1)}{\partial\phi}h_{t+1}\right] \tag{47}
$$

and the recursions for each of the partial derivatives can be calculated from the recursions for $A_1(t;\phi,T-1)$, $B_1(t;\phi,T-1)$ and $C_1(t;\phi,T-1)$ and using the following boundary conditions:

$$
\begin{aligned}
\frac{\partial A_1(T-2;\phi,T-1)}{\partial \phi} &= -\mu_0 && (48)\\
\frac{\partial B_1(T-2;\phi,T-1)}{\partial \phi} &= 1+\mu_1 && (49)\\
\frac{\partial C_1(T-2;\phi,T-1)}{\partial \phi} &= \lambda^* + \phi - 1 && (50)
\end{aligned}
$$

For example, the recursion for $\frac{\partial B_1(t;\phi,T-1)}{\partial \phi}$ is

$$
\frac{\partial B_1(t;\phi,T-1)}{\partial \phi} = \mu_1 \frac{\partial B_1(t+1;\phi,T-1)}{\partial \phi} \tag{51}
$$

Now we will show how to calculate $E_t^F[M(t,T-1)]$. Note that

$$
E_t^F[M(t,T-1)] = \frac{1}{P(t,T)} E_t^Q\left[M(t,T-1)\exp(-M(t,T-1)\right] \tag{52}
$$

Let $g(\phi) = E_t^Q\left[\exp(\phi M(t,T-1))\right]$. Then it follows that

$$
\frac{\partial g(\phi)}{\partial \phi} = E_t^Q\left[M(t,T-1)\exp(\phi(M(t,T-1))\right] \tag{53}
$$

From (52) and (53), it directy folows that $E_t^F[M(t,T-1)] = \frac{1}{P(t,T)}\frac{\partial g(\phi)}{\partial \phi}|_{\phi=-1}$. Note that $g(\phi)$ can be explicitly calculated as follows: $g(\phi) = \exp(A_3(t,T-1) + B_3(t,T-1)r_t + C_3(t,T-1)h_{t+1})$, where the coefficients $A_3()$, B_3 and C_3 can be computed using the recursions for options on the zero-coupon bond i.e. (31), (32) and (33), but with the following boundary conditions:

$$
\begin{aligned}
A_3(T-2,T-1) &= \phi\mu_0 && (54)\\
B_3(T-2,T-1) &= \phi(1+\mu_1) && (55)\\
C_3(T-2,T-1) &= \phi(\lambda^* + \frac{1}{2}\phi) && (56)
\end{aligned}
$$

Note that having obtained $g(\phi)$ one can obtain $\frac{\partial g(\phi)}{\partial \phi}$ through straight differentitaion in exactly the same way as $\frac{\partial f(i\phi)}{\partial \phi}$ was derived.

6 Estimation

Having developed analytical formulae for bonds and bond derivatives, we do an exploratory investigation of how well the model fits the observed term structure and whether strike price biases (such as the smile/smirk in implied volatilities in some interest rate derivatives market) from a one-factor model such as the Black model can be accounted for in our framework after we have estimated the model parameters. It is to be noted that the purpose of this section is not to undertake a thorough empirical/econometric investigation of the theoretical model because model building and not model testing is the focus of this paper. Nevertheless, calibrating the model to the actual term structure should generate some basic insights about the basic functionality of the model in the context of the real world that hopefully will inspire more extensive empirical research in the future.

For the zero coupon bond prices of various maturities, we use a modification of the Fisher, Nychka and Zervos (1995) method of constructing zero coupon yield curves constructed from the daily CRSP bond file data due to Waggoner (1996)(see Bliss (1997) for the details).[7] The criterion function for parameter estimation is the minimization of the sum of squared errors between model and market zero coupon bond prices. In fact, for a random sample of two weeks of consecutive data in the period spanning 1990-1996, we minimize the following criterion function:

$$\min_{\mu_0,\mu_1,\omega,\alpha,\beta,\gamma,\lambda,\eta} \sum_{t=1}^{T} \sum_{i=1}^{N_t} \left(P_{i,t} - M_{i,t}\right)^2 \qquad (57)$$

where $P_{i,t}$ and $M_{i,t}$ are respectively the model and market prices and for bond i on

[7]We thank Daniel Waggoner for constructing the zero coupon yield curves.

date t. Note that in computing the above criterion function, one needs to know h_{t+1} which is the conditional variance of the $r_{t+1} - r_t$, at time , t. However, at each t, h_{t+1} is known from the history of the short rate until time t. For the short rate, we use the continuously compounded overnight rate implicit in the three month T-bill prices. The cross-section of maturities (in days) used are: 90,180,270,360,730,1095,1460,3650.

As an example, the parameter estimates obtained by fitting the model to market prices for two weeks starting on January 03, 1994 are as follows: $\mu_0 = 2.13 \times 10^{-8}$, $\mu_1 = 0.999$, $\lambda = -3.36$, $\nu = 15.58$, $\omega = 1.44 \times 10^{-11}$, $\beta = 0.256$, $\alpha = 1.093 \times 10^{-11}$, $\gamma = 12.68$. Table 1 shows the market and model yields for the various maturities on January 03, 1994. Now, if we fit the nested one-factor model (the discrete time version of the Vasicek model),[8] to the same sample, we find that the average absolute pricing error is only a little higher (around two basis points) than the two-factor model. A likelihood ratio test of testing the nested one-factor model against the two-factor model cannot reject the one-factor model. In other words the two-factor model is only a trivial improvement if fitting the entire yield curve over a given period of time is used as the criterion function.[9] This is in sharp contrast to checking the model directly on the time series of a short rate (computed from the three month T-bill yields) where the one-factor version is overwhelmingly rejected.[10] In other words caution is warranted in drawing conclusions on the adequacy of an interest rate model in being able to describe the entire yield by checking only the dynamics of a prescribed short rate.

Having found that a random volatility does not add much to a simple one-factor model in terms of explaining the cross-section of bond prices, we now turn to options

[8] The discount bond prices for the one-factor or the constant volatility model are easily derived by noting that $z_t^* = z_t - \eta$, instead of $z_t^* = z_t - \eta h_{t+1}$ as in the two-factor model.

[9] We have repeated this estimation exercise for other randomly chosen two week periods between 1990 and 1996 and the conclusions regarding the mispricing remain essentially unchanged, although the parameter estimates are somewhat different depending on the period.

[10] Checking directly the short rate is performing a straightforward maximum likelihood estimation on a time series of the short rate as described by (1) and (2). The results of such an estimation on a daily time series of the short rate are available from the authors upon request.

markets where volatility is expected to be of much more importance. Using the parameters estimated from the cross-section of bond prices, in other words from the previous yield curve calibration, we generate prices of options on discount bond of different strike prices under our two-factor model. Then using the Black (1976) model, which is a widely used one-factor model for pricing these options in the market place, we back out the implied volatility of different strike prices. It is found that the implied volatilities decrease as the strike price increases. In other words, the implied volatilities display a smirk/skew across strikes for these options (see Figure 1). Thus our results suggest that volatility as a second factor is important primarily for bond options, rather than for spot bonds or bond futures.

7 Conclusion

We developed a discrete time two-factor model of interest rates, in which the second factor is a time varying volatility following an asymmetric GARCH process; the model can be used to price spot bonds, bond futures and different and bond options using easily computable analytical solutions. Unlike continuous-time two-factor models with random volatility, volatility in our model is observable on the basis of the history of interest rates. This makes the empirical estimation of the model very straightforward and tractable unlike the estimation of continuous time stochastic volatility model of interest rates. Moreover for a class of interest rate Asian options called average rate options, unlike continuous time models, our model can be readily implemented without any bias. This is because in practice one can only calculate the average of interest rates observed at disceret intervals and not the average from continuous sampling.

We find that the second factor does not matter so much for the cross-section of spot bonds as it does for bond options. While we cannot reject the nested one-factor version of the model for the entire cross-section of spot bonds, on the basis of parameters estimated from the yield curve, our model produces the smirk/skew in implied volatilities under the one-factor Black model for options on discount bonds. Future research can be directed towards calibrating our model directly on bond options data and checking its out-of-sample pricing performance against alternative models, such as the various HJM specifications for forward rate volatilities.

Appendix

In this appendix we show how one obtains the risk-neutralized representation of the interest rate process in (1) and (2). Also it is shown how one obtains (14),(15) and (16), the recursions needed to calculate the bond price and the price for options on discount bonds and discount bond futures.

In our interest rate environment, it is not possible to derive a risk-neutral representation solely on the basis of arbitrage arguments. This is simply because we have a discrete-time model in which the short rate at the next instant can take an infinite number of values and therefore spanning by a finite number of discount bonds is not possible. Instead has to specify a process for the state price density (pricing kernel) or the marginal utility of consumption as in Constantinides (1992) and others and then perform the change of measure or compute prices under the statistical/data generating measure. Let M_t denote the state price density process (proportional to the marginal rate of substitution in a representative agent economy) and $X_t = \log(M_t)$. Note that standard asset pricing theory implies that a cash flow V_{t+1} can be valued as of time t as

$$V_t \quad = \quad E_t\left(V_{t+1}\frac{M_{t+1}}{M_t}\right) \tag{58}$$

Let X_t follow the process,

$$X_{t+1} - X_t = \eta\sqrt{h_{t+1}}z_{t+1} - \frac{1}{2}\eta^2 h_{t+1} - r_t \tag{59}$$

Note that the state price density process, $M_t = \exp(X_t)$, specified through the dynamics of X_t in (59) is a valid state price density as it is positive and values a non-random cash flow of \$1 at $t+1$ as earning the risk free rate from t to $t+1$. Although we have exogenously specified the state price density given the interest rate process, this state price density can be derived endogenously from a production economy with agents having logarithmic utility (see Constantinides (1992) and also Campbell, Lo and McKinlay (1997)). Given (59), a cash flow of V_{t+1} at $t+1$ can be valued at t as,

$$\begin{aligned} V_t &= E_t\left(V_{t+1}\exp\left(X_{t+1}-X_t\right)\right) \\ &= E_t\left(V_{t+1}\exp\left(\eta\sqrt{h_{t+1}}z_{t+1}-\frac{1}{2}\eta^2 h_{t+1}-r_t\right)\right) \end{aligned} \tag{60}$$

Writing out the above expectation in the integral form, taking the expectation with respect to the distribution of z_{t+1} and letting $z \equiv z_{t+1}$,

$$\begin{aligned} V_t &= \exp(-r_t)\int_{-\infty}^{\infty} V_{t+1}\exp(\eta\sqrt{h_{t+1}}z-\frac{1}{2}\eta^2 h_{t+1})\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2)dz \\ &= \exp(-r_t)\int_{-\infty}^{\infty} V_{t+1}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(z-\eta\sqrt{h_{t+1}}\right)^2\right)dz \end{aligned} \tag{61}$$

In the above equation, let $z^q_{t+1} = z_{t+1}-\eta\sqrt{h_{t+1}}$. Given this, we can write (61) as,

$$\begin{aligned} V_t &= \exp(-r_t)\int_{-\infty}^{\infty} V_{t+1}\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}\left(z^q\right)^2)dz^q \\ &= \exp(-r_t)E^q_t(V_{t+1}) \end{aligned} \tag{62}$$

where E^q is the expectation corresponding to the random variable z^q. Note that the under the distribution generated by z^q, the expected appreciation in the risky asset is the risk free rate and therefore E^q is the expectation with respect to the risk-neutral distribution. In particular, it can be shown that z^q is distributed as a standard normal under the risk-neutral distribution.[11] Thus, under the risk-neutral distribution, the dynamics of r_t is given by,

$$r_{t+1} = \mu_0+\mu_1 r_t+\lambda^* h_{t+1}+\sqrt{h_{t+1}}z^q_{t+1} \tag{63}$$

$$h_{t+1} = \omega+\beta\, h_t+\alpha(z^q_t-\gamma^*\sqrt{h_t})^2, \tag{64}$$

[11] It suffices to note that if $f(x)$ is the probability density corresponding to the data generating/statistical measure, then one can define another valid probability density $f^q(x)$ such that the Radon-Nikodym process, $\frac{f^q(x)}{f(x)} = \exp(\eta\sqrt{h_{t+1}}z_{t+1}-\frac{1}{2}\eta^2 h_{t+1})$. The interested reader is referred to Karatzas and Shreve (1988) for greater details regarding this type of calculation.

where $\lambda^* = \lambda + \eta$ and $\gamma^* = \gamma - \eta$ and $z_t^q = z_t - \eta\sqrt{h_{t+1}}$.

Now we will show how one derives (14),(15) and (16). Substituting the dynamics of r_{t+1} and h_{t+1} in (13),

$$\begin{aligned}
\exp(A(t,T) + B(t,T)r_t + C(t,T)h_{t+1}) &= \exp(-r_t + A(t+1,T) \\
&+ B(t+1,T)(\mu_0 + \mu_1 r_t + \lambda^* h_{t+1}) \\
&+ C(t+1,T)(\omega + \beta h_{t+1}). \\
& E_t^* \Big[\exp(B(t+1,T)\sqrt{h_{t+1}} z_{t+1} \\
&+ \alpha C(t+1,T)(z_{t+1}^* - \gamma^* \sqrt{h_{t+1}})^2\Big] \qquad (65)
\end{aligned}$$

Completing the square in the portion to which the expectation applies, using the fact that for a standard normal z^*, $E(a(z+b)^2) = \exp(-\frac{1}{2}\log(1-2a) + \frac{ab^2}{1-2a})$ and matching the coefficients on r_t, h_{t+1} and the constant, we get (14), (15) and (16).

The next thing is to show how one arrives at (31), (32) and (33). Recall that $x = A(\tau,T) + B(\tau,T)r_\tau + C(\tau,T)h_{\tau+1}$. $f(\phi) = E_t^F(\exp(\phi x))$ is the conditional moment generating function of x under the forward measure. Following (1997), the Radon-Nikodym derivative process, L_t to go from the risk-neutral to the forward measure is $L_t = \frac{P(t,\tau)}{\exp(\sum_{u=0}^{t-1} r_u)P(0,\tau)}$. Given this, it follows from Karatzas and Shreve (1988, Lemma 3.5.3) that E_t^F and E_t^q are related as follows for any random variable, Y that is measurable with respect to the information set at time t+1:

$$E_t^F[Y] = E_t^q\left[Y \exp(-r_t)\frac{P(t+1,\tau)}{P(t,\tau)}\right] \qquad (66)$$

where $P(t,\tau)$ and $P(t+1,\tau)$ are the time t and $t+1$ prices of a discount bond maturing at τ, when the option matures.

Now, we will show how one arrives at (34). If the τ period zero-coupon bond is chosen to be the numeraire, the call option price is given by

$$C_o(t,\tau,T) = P(t,\tau)E_t^F(\max[\exp(x) - K, 0]) \tag{67}$$

where $x = A(\tau,T) + B(\tau,T)r_\tau + C(\tau,T)h_{\tau+1}$. Let $g(x)$ be the conditional density function of x. In terms of integrals

$$C_o(t,\tau,T) = P(t,\tau)\left(\int_{\log(K)}^{\infty} exp(x)g(x)dx - K\int_{\log(K)}^{\infty} g(x)dx\right) \tag{68}$$

Define a new density $g^*(x)$ such that $g^*(x) = \frac{g(x)\exp(x)P(t,\tau)}{P(t,T)}$. Note that it is a valid probability density because it is non-negative and $\int_{-\infty}^{\infty} g^*(x)dx = 1$ because under the numeraire, deflated asset prices are martingales, i.e., $E_t\left(\frac{P(\tau,T)}{P(\tau,\tau)}\right) = \frac{P(t,T)}{P(t,\tau)}$. Now, the moment generating function of $g^*(x)$ is

$$\begin{aligned}\int_{-\infty}^{\infty} \exp(\phi x)g^*(x)dx &= \frac{P(t,\tau)}{P(t,T)}\int_{-\infty}^{\infty} \exp((\phi+1)x)g(x)dx \\ &= \frac{f(\phi+1)P(t,\tau)}{P(t,T)}\end{aligned} \tag{69}$$

This implies that

$$C_o(t,\tau,T) = P(t,T)\int_{\log(K)}^{\infty} g^*(x)dx - KP(t,\tau)\int_{\log(K)}^{\infty} g(x)dx \tag{70}$$

Now using the inversion formulae to recover the distribution functions from their corresponding characteristic functions as in Heston (1993) and others we get (34).

The options on bond futures formula is derived exactly like the options on discount bonds, except noting that futures prices are martingales under the new numeraire and hence one can define a density $g^*(x)$ such that $g^*(x) = \frac{\exp(x)g(x)}{F(t,\tau,T)}$ where x is such that $F(\tau,\tau,T) = \exp(x)$.

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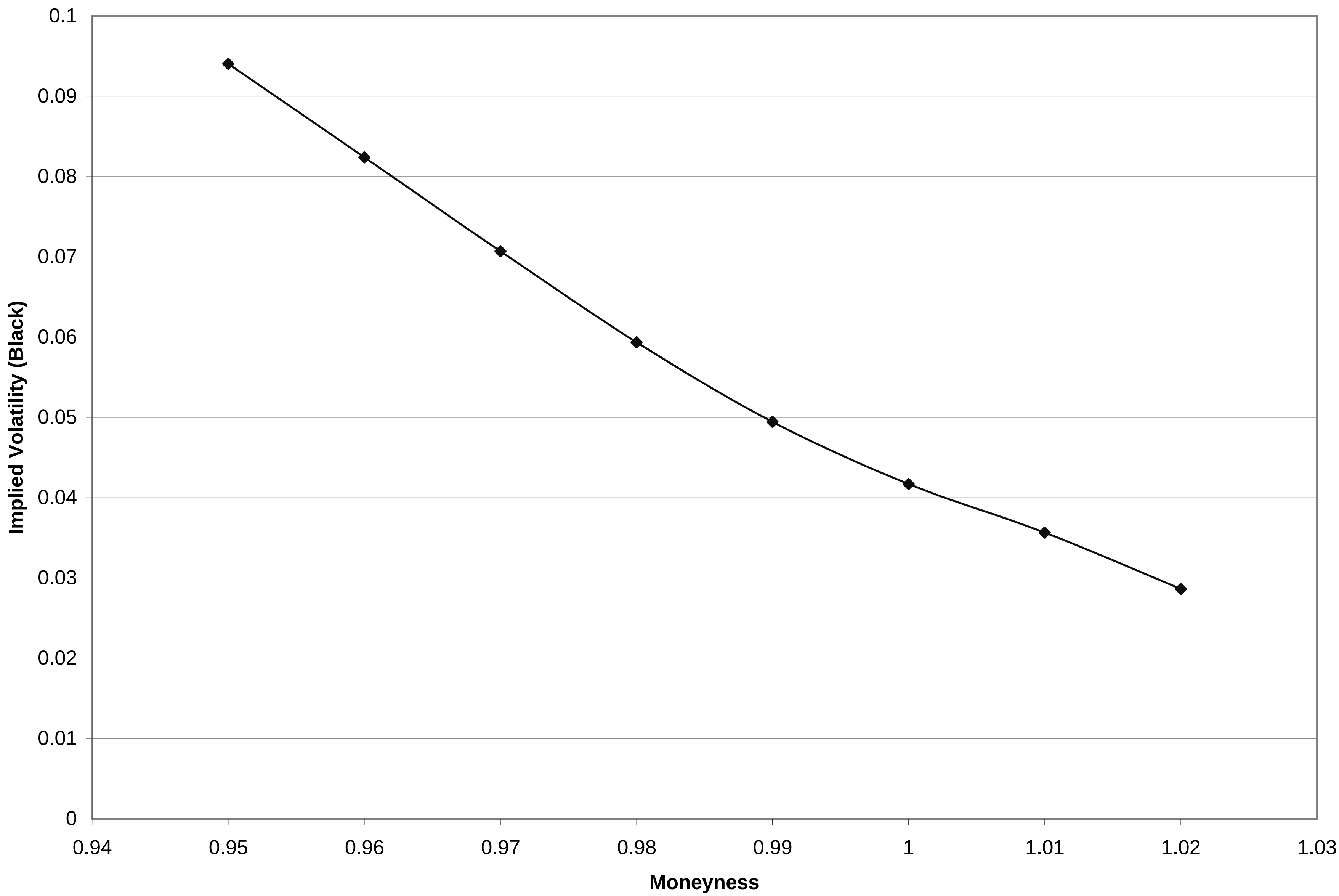
Table 1

This table shows the market and model (two-factor) zero-coupon yields in percentages for eight different bills, notes and bonds as of January 03, 1994. The market yields are computed from the daily CRSP bond file using a procedure of Waggoner (1996) that is a modification of Fisher, Nychka and Zervos (1995).

Maturity (in days)	Market Yield	Model Yield
90	3.13	3.23
180	3.3	3.36
270	3.46	3.49
360	3.67	3.62
730	4.26	4.11
1095	4.65	4.53
1460	5.04	4.9
3650	6.09	5.95

Figure 1



Shows the implied volatilties from the Black model for call options on discount bonds. The bond and the option expire in 500 and 100 days respectively. Moneyness is the ratio of strike price to the price of the discount bond.