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Stability of Steady States in a Model of Pleasant Monetarist Arithmetic

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Abstract: In this paper the authors study the stability properties of the alternative steady-state equilibria that arise in a neoclassical production model that delivers pleasant monetarist arithmetic. They show that if the government's monetary policy rule involves a fixed money supply growth rate, then "pleasant arithmetic'' steady states—steady states from which a permanent increase in the money growth and inflation rates is associated with a permanent decrease in the real interest rate and a permanent increase in the level of output—are dynamically stable.

JEL classification: E4, E5, E6

Key words: pleasant monetarist arithmetic, stability

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1 Introduction

The Keynesian tradition of the 60's and 70's saw the effects of monetary policy as powerful and long-lasting both in the short and the long-run. This Keynesian prediction was based on the assumption that economic agents were victims of persistent money illusion. This point was consistently made by Friedman and Phelps (e.g., Friedman (1968) and Phelps (1972)). They argued that analyses of the effects of policy changes that were based on the money illusion assumption were unsound and unconvincing. Their argument was eventually formalized in a forceful way by neoclassical economists such as Lucas (1972) and Sargent and Wallace (1976). They developed rational expectations models in which persistent money illusion was explicitly ruled out. In these models, permanent changes in the inflation rate induced by monetary policy had limited real effects in the short run and no real effects in the long run: that is, money was long-run superneutral. Most of the profession mistakenly took the Lucas and Sargent-Wallace's contributions as meaning that Neoclassical models would invariably deliver long-run superneutrality of money. This mistake helps explain the proliferation of monetary policy theoretical and empirical models with emphasis in the short-run.

Sargent and Wallace (1981) was one of the first Neoclassical models to introduce the foundation for an alternative mechanism to money illusion in which monetary policy could have long-run real effects. Key in the introduction of this analysis was the explicit study of intertemporal government budget policy. Although the seminal work of Sargent and Wallace (1981) open the door to the analysis of the long-run real effects of monetary policy, it delivers "unpleasant" monetarist arithmetic (UMA): a permanent monetary tightening produces higher inflation in the long-run (and in some instance even in the short-run) exactly the opposite of the inflation-economic activity trade-off suggested by the standard Keynesian model. Wallace (1984) uses a pure exchange model with reserve requirements and an endogenous, policy-dependent real rate of interest to study the effects of permanent changes in monetary policy across steady states. He uses the bonds-money ratio, instead of the money growth rate, as his policy variable. His model delivers steady state UMA. This is the first analysis of permanent policy changes across steady states, and the first to use the bonds-money ratio as a policy instrument. Since then a number of Neoclassical studies have emerged. These analyses study the long-run real effects of monetary policy that do not rely

on money illusion and are direct descendants of Sargent-Wallace (1981).

For example, Espinosa and Russell (1998a,b) use Wallace's (1984) model, and draw out some of the ultimate implications for inflation and the real interest rate from introducing the conditions under which Darby's (1984) observation hold. Darby points out that the real-rate of interest has been lower than the real rate of economic growth for long periods of time. Espinosa-Russell show that either pleasant (PMA) or UMA can conceivably occur in the long-run. Bhattacharya, Guzman and Smith (1998) use a model with linear storage and reserve requirements [as in Freeman (1987)] to show that steady state UMA can hold under some conditions *even* under Darby's scenario. Their policy instrument is the bonds-money ratio. Bhattacharya and Kudoh (2001) use a Diamond model with reserve requirements and find, like in Espinosa and Russell (1998a), either steady state PMA or steady state UMA can occur. They focus on UMA. Their policy instrument is bonds-money ratio. Their reserve requirement assumptions allow without additional intermediation assumptions that the long-run real rate of interest be lower than the real rate of growth in the economy, and they show that UMA can hold in that case.

A natural question is what do we know about the local dynamic properties of PMA and UMA equilibria? In this paper we show that in the long-run, whether the instrument is the money growth rate or the bonds-money ratio is of little relevance. From a short-run dynamics perspective matters can be significantly different. Bhattacharya and Kudoh show that under the fixed-saving preference assumptions of Wallace (1984), all the interesting steady states are dynamically stable. Under more general preference assumptions, none of the steady states are stable, but they are all dynamically approachable. Bhattacharya and Kudoh use these findings to defend the empirical plausibility of UMA. In this paper, we study the stability properties of Espinosa-Russell (1998b), a model that allows permanent changes in the money growth and inflation rates to have effects on the steady-state levels of real interest rates and output consistent with key aspects of the pre-1970s conventional wisdom; money growth and inflation rates increases associated with a real interest rate drop and output increases – PMA equilibrium.

We organize the bulk of our analysis around the properties of the government's seigniorage revenue function, which describes the dependence of total revenue from currency and bond seigniorage on the values of the real interest rate and the real rate of return on cur-

rency. This approach allows us to present more complete descriptions of the regions of the parameter space that deliver policy effects of different types. We show that if the government's monetary policy rule involves a fixed money supply growth rate, then a steady state is dynamically stable if and only if it is on the left (upward-sloping) side of the seigniorage revenue curve. This result is particularly interesting because these are the same steady states from which a permanent increase in the money growth and inflation rates produces a permanent decrease in the real interest rate and a permanent increase in the level of output. The next section of the paper lays out the model that provides the framework for our analysis.

2 The model

2.1 The market environment

At each discrete date $t \geq 1$ a positive number of identical two-period-lived households are born. This number N_t grows at an exogenous gross rate of $n \geq 1$ per period: $N_t = n^t N_0$, where $N_0 > 0$. Each two-period-lived household is endowed with a single unit of labor in the first period of its life and has no endowment of any kind in the second period. At date 1, there are N_0 "initial old" households who live for one period. These households are endowed only with bank deposits (see below).

Households derive satisfaction from consuming the single good in the first and/or second periods of their lives. Since households incur no disutility from providing labor, they supply their entire labor endowment at any positive real wage. Thus, the total supply of labor at any date is equal to the total population of young agents at that date.

The preferences of the households are assumed to have standard properties, plus the property that the fraction of their labor income they save is invariant to the rate(s) of return on the assets available to them. In particular, each young household born at date t saves an amount $s\ w_t$, where s represents the fraction of its income it devotes to saving and w_t represents the real wage rate at date t. We assume $s \in (0, 1)$.[1] Given our endowment assumptions, an intertemporal utility function that will generate this behavior is $u(c_1, c_2) =$

[1] This assumption follows Wallace (1984) and Espinosa and Russell (1998). It is adopted primarily for purposes of analytical tractability. There is, however, considerable empirical evidence that aggregate gross saving is relatively insensitive to changes in the real rate of return.

$\log c_1 + \frac{1}{1+\rho} \log c_2$, where $\rho > -1$ and c_1 and c_2 represent household consumption in the first and second life-periods, respectively. This function produces $s = \frac{1}{2+\rho}$. Household saving takes the form of consumption deposits at financial intermediaries ("banks"): see below.

At each date there are an indeterminate number of one-period competitive firms that produce the single good. The firms purchase labor from the households and rent physical capital from the financial intermediaries. The firms use the Cobb-Douglas constant-returns production technology $Y_t = F_t(K_t, L_t) = \lambda^{(1-\alpha)(t-1)} K_t^{\alpha} L_t^{1-\alpha}$ and must thus earn zero profits in equilibrium. Here Y_t represents date t output of the single good, which can be consumed at date t or stored and used as capital at date t+1. In addition, K_t represents the total stock of capital used in production at date t, L_t represents the total quantity of labor used at that date, $\alpha \in (0, 1)$, and $\lambda \geq 1$. If $\lambda > 1$ then there is exogenous technical progress at a gross rate of λ per period.

After consumption goods have been produced during a period they may be consumed or stored. Banks organized at date t purchase goods from young households. The banks may place these goods in storage, or they may use them to make consumption loans to young households or to purchase government liabilities (see below). The principal motive for storage is that goods that have been stored for one period can be used as physical capital, unit for unit. The banks may rent the capital to the firms at date t+1; afterwards, any goods that have not depreciated revert to being consumption goods and form part of the banks' gross revenue. Goods stored at date t depreciate at a net rate of δ from date t to t+1. This depreciation occurs after production takes place at date t+1, whether or not the particular goods in question have been used in production. For simplicity we set $\delta = 1$.[2]

Banks issue consumption deposits to young households at a gross real deposit rate R_t^d.[3] In a perfect foresight competitive equilibrium, arbitrage requires that the gross consumption lending rate R_t satisfies $R_t = 1 + r_{t+1} - \delta$, where r_{t+1} is the rental rate on capital at date $t+1$ and δ is the <u>net</u> rate of depreciation. Since we have assumed $\delta = 1$, zero profits for the banks requires $R_t = r_{t+1}$. As all young households are identical, equilibrium consumption lending may be taken to be zero.

[2] This assumption is not unreasonable in a two-period model: if we think of a period as thirty years, an annual depreciation rate of 0.1 corresponds to a per-period rate of almost 0.96.

[3] The budget contraints of a household born at date $t \geq 1$ are $c_{1t} + s_t \leq w_t$ and $c_{2,t+1} \leq R_t^d s_t$. The budget constraint of an initial old household is $c_{21} = d_0$ where d_0 represents the real value of its maturing bank deposits (see below).

We assume that financial intermediation consumes resources when it involves physical capital or household consumption loans. At date $t+1$ the banks incur a non-negative proportional cost c on each unit of the consumption good that they stored or lent at date t. Government bonds, in contrast, are intermediated costlessly.[4] As a result, if the banks are to earn zero profits and the government is to succeed in selling bonds then we must have

$$R_t^b = R_t - c\,, \tag{1}$$

where R_t is the gross real interest rate on loans to the firms and R_t^b is the gross real interest rate on government bonds (see below).

The goods price of a unit of fiat currency at date t is denoted p_t. The price level at date t (the money price of a unit of goods) is $P_t = 1/p_t$. We will confine ourselves to the study of "monetary" equilibria, which are equilibria in which $p_t > 0$ for all $t \geq 1$. The gross real rate of return on fiat currency is $R_t^m \equiv p_{t+1}/p_t \equiv 1/\Pi_t$, so that Π_t is the gross inflation rate. The nominal price of a unit-face-value nominal bond is P_t^b, so $R_t^{nom} = 1/P_t^b$ is the nominal interest rate on government bonds. Arbitrage in the markets for currency and government bonds requires

$$R_t^b = \frac{R_t^m}{P_t^b}\,, \tag{2}$$

where R_t^b the real interest rate on the bonds. Thus, $P_t^b = R_t^m/R_t^b$ and $R_t^{nom} = R_t^b/R_t^m$. If bonds and capital are to coexist with currency then we must have $P_t^b \leq 1$, so that either

$$\begin{aligned} R_t^m &= R_t^b \quad \text{or} \\ R_t^m &< R_t^b. \end{aligned} \tag{3}$$

We assume there are "initial banks" endowed with the initial capital stock K_1, H_0 nominal units of fiat currency, and $\mathcal{B}_0$ nominal units of one-period government bonds (payable in fiat currency) which are due at date 1. The initial banks distribute the gross revenue from these assets to the initial old in the form of maturing deposits, so that the total real value of the maturing deposits is $D_0 = R_0 K_1 + p_1(H_0 + \mathcal{B}_0)$, where $R_0 = \alpha k_1^{\alpha-1}$ and $k_1 = K_1/N_1$. Each initial old household is endowed with a maturing deposit whose value is $d_0 = D_0/N_0$.

[4] We think of c as a proxy for the information and diversification costs that are associated with the existence of default risk on private liabilities, and also, perhaps, for the "equity premium" on undiversifiable risk.

At each date $t \geq 1$ the government may issue new bonds and/or additional units of fiat currency. The government must finance a fixed real deficit $G_t > 0$ by a combination of currency and bond seigniorage. The government's budget constraint at dates $t \geq 1$ is

$$G_t = p_t \left[(H_t - H_{t-1}) + (P_t^b \mathcal{B}_t - \mathcal{B}_{t-1}) \right] . \tag{4}$$

Here H_t represents the nominal stock of fiat currency at date t, so that $M_t \equiv p_t H_t$ is the real value of the stock of currency, and $\mathcal{B}_t$ represents the nominal face value of the stock of bonds, so that $\mathrm{B}_t \equiv p_t \mathcal{B}_t$ is the real face value of the bond stock and $B_t \equiv P_t^b \mathrm{B}_t$ is the corresponding real present value. Thus, $B_t \equiv p_t \, P_t^b \mathcal{B}_t$ for $t \geq 1$. We assume that the central bank manages the nominal stock of fiat currency by setting

$$H_t = z \, H_{t-1} \tag{5}$$

for some fixed value z, starting at $t = 1$.

In a competitive equilibrium, the rental rate r_t must be equal to the marginal product of capital, so we have $r_t = \lambda^{(1-\alpha)(t-1)} \alpha k_t^{\alpha-1}$, where $k_t \equiv K_t / L_t$. Financial market arbitrage ensures $r_t = R_{t-1}$ (see above), so $k_t = \lambda^{t-1} \left(\frac{R_{t-1}}{\alpha} \right)^{\frac{1}{\alpha-1}}$.

In equilibrium, aggregate private credit demand at date t must equal firms' aggregate demand for capital at date $t+1$. In addition, labor market equilibrium requires $L_t = N_t$. Thus, aggregate *private* credit demand at date t is given by

$$K_{t+1}(R_t) \equiv (\lambda n)^t N_1 \left(\frac{R_t}{\alpha} \right)^{\frac{1}{\alpha-1}} . \tag{6}$$

In a competitive equilibrium, the real wage rate is given by $w_t = \lambda^{(1-\alpha)(t-1)} (1-\alpha) \, k_t^{\alpha}$. Using the expression for k_t, we have $w_t = \lambda^{t-1} (1-\alpha) \left(\frac{R_{t-1}}{\alpha} \right)^{\frac{\alpha}{\alpha-1}}$. It follows that the aggregate savings function (that is, aggregate net credit supply by the young) is

$$S_t(R_{t-1}) \equiv (\lambda n)^{t-1} N_1 \, s \left(\frac{R_{t-1}}{\alpha} \right)^{\frac{\alpha}{\alpha-1}} . \tag{7}$$

Note that $S_t(\cdot)$ is a function of R_{t-1}, which determines young households' income at date t, but not of R_t^d, which is the gross real rate of return they will receive on assets acquired at date t (see above). In addition, since $Y_t = L_t \, \lambda^{(1-\alpha)(t-1)} \, k_t^{\alpha}$, real output at each date t can be written

$$Y_t(R_{t-1}) = (\lambda n)^{t-1} N_1 \left(\frac{R_{t-1}}{\alpha} \right)^{\frac{\alpha}{\alpha-1}} .$$

Credit market clearing requires

$$S_t(R_{t-1}) - K_{t+1}(R_t) = M_t + B_t\,; \tag{8}$$

that is, the aggregate excess supply of private credit at each date must be equal to the aggregate government demand for credit, which is the aggregate real market value of the outstanding fiat currency and the newly-issued government bonds, at the same date.

The government requires each bank to hold a quantity of fiat currency no smaller than a fraction θ of its nominal deposit liabilities. If $P_t^b < 1 \Leftrightarrow R_t^m < R_t^b$ then the reserve requirement is binding and fiat currency is held only to satisfy it. We will confine ourselves to studying equilibria of this type. In these equilibria real fiat currency balances M_t satisfy

$$M_t = \theta\, S_t(R_{t-1})\,. \tag{9}$$

Since banks must earn zero profits in equilibrium, the gross real deposit interest rate R_t^d must satisfy[5]

$$R_t^d = (1-\theta)\, R_t^b + \theta\, R_t^m\,.$$

The assumption that binding reserve requirements are the only source of currency demand is not essential for obtaining the results we report. We think of the reserve demand in our model as a proxy for base money demand from all sources, including both bank reserves and currency held by the public. Empirical studies of the demand for base money indicate that it is relatively insensitive to changes in the nominal interest rate and roughly proportional to the level of real income. So our reserve-requirements-only assumption, under which real currency demand is completely insensitive to the nominal interest rate and exactly proportional to the level of labor income, seems reasonable as a first approximation. Augmenting our model to include an explicit source of household currency demand does not change its basic features.[6]

But the additional complexity prevents us from obtaining the relatively clean analytical results we present in this paper.

[5] If a bank that is organized at date t purchases and stores k_{t+1} units of the consumption good, purchases bonds with a market value of b_t and acquires real fiat currency balances of m_t, then its profits at date $t+1$ are $R_t\, k_{t+1} + R_t^b b_t + R_t^m m_t - c\, k_{t+1} - R_t^d \Lambda_t$, where $\Lambda_t = k_{t+1} + b_t + m_t$.

[6] Alternative versions of this example in which binding reserve requirements are supplemented by household money demand deriving from (1) transactions costs or (2) random relocation [as in Bencivenga and Smith (1991)] are available from the authors.

2.2 Equilibria

Given $K_1 > 0$, $H_0 > 0$, $\mathcal{B}_0 \geq 0$, $\theta \in (0,1)$, $z > 1$ and $\{G_t\}_{t=1}^{\infty}$ strictly positive, a *binding competitive equilibrium from date 1* consists of strictly positive sequences $\{R_t\}_{t=1}^{\infty}$, $\{R_t^b\}_{t=1}^{\infty}$, $\{p_t\}_{t=1}^{\infty}$, $\{P_t^b\}_{t=1}^{\infty}$, $\{H_t\}_{t=1}^{\infty}$ and a nonnegative sequence $\{\mathcal{B}_t\}_{t=1}^{\infty}$ that satisfy conditions (1)-(5) and (8)-(9), given definitions (6)-(7) as well as $R_t^m \equiv p_{t+1}/p_t$, $M_t \equiv p_t H_t$ and $B_t \equiv p_t\, P_t^b \mathcal{B}_t$ (see above).

In most of the paper we will focus on studying *binding steady states.* In these steady states the rate-of-return variables R_t, R_t^b, R_t^m, and P_t^b will be date-invariant. The nominal values of all goods-aggregate variables will grow at a gross rate of $\Psi\Pi$ per period, where $\Pi = 1/R^m$, and the real values of these variables will grow at a gross rate of Ψ per period. The government budget deficit must be assumed to grow at the same rate. The real wage rate w_t and the capital-labor ratio k_t which will grow at a gross rate of λ per period.

A binding steady state is not an equilibrium from date 1 unless the values of K_1 and $\mathcal{B}_0$ happen to be consistent with achievement of the steady state starting from date 1. Nevertheless, the conditions that characterize a binding steady state can be expressed in terms of the situation at date 1.

In a steady state we have $M_{t+1} \equiv \Psi M_t$, where $M_{t+1} \equiv p_{t+1} H_{t+1}$ and $M_t \equiv p_t H_t$, so

$$z = \frac{H_{t+1}}{H_t} = \frac{M_{t+1}/p_{t+1}}{M_t/p_t} = \frac{\Psi}{R^m} \;\Leftrightarrow\; R^m = \frac{\Psi}{z} \;\Leftrightarrow\; \Pi = \frac{z}{\Psi}\,.$$

Defining the steady-state capital-demand (or private asset-supply) function

$$K_2(R) \equiv \Psi N_1 \left(\frac{R}{\alpha}\right)^{\frac{1}{\alpha-1}}. \tag{10}$$

Note that this function has a 2-subscript because the capital goods that are employed at date 2 must be acquired by households at date 1. In addition, define the steady-state saving (or asset demand) function

$$S_1(R) \equiv N_1\, s\,(1-\alpha) \left(\frac{R}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}. \tag{11}$$

Given $\theta \in (0,1)$, $z > 1$ and $\{G_t\}_{t=1}^{\infty}$ with $G_1 > 0$ and $G_{t+1} = \Psi G_t$, a binding steady state consists of positive values R, R^b, R^m, M_1 and a non-negative value B_1 that satisfy the

following conditions, given definitions (10) and (11):

$$R^m = \frac{\Psi}{z}\,, \tag{12}$$

$$R^b = R - c\,, \tag{13}$$

$$R_m < R_b, \tag{14}$$

$$M_1 = \theta\, S_1(R)\,, \tag{15}$$

$$S_1(R) - K_2(R) = M_1 + B_1, \tag{16}$$

$$\Psi G_1 = (\Psi - R^m)\, M_1 + (\Psi - R^b)\, B_1. \tag{17}$$

We can define $Y_1(R) = N_1 \left(\frac{R}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$, with $Y_1'(R) = \frac{N_1}{\alpha-1}\left(\frac{R}{\alpha}\right)^{\frac{1}{\alpha-1}} < 0$. Thus, the steady-state level of output at any particular date is a strictly decreasing function of the steady-state real interest rate. However, total output grows at gross rate of Ψ in any steady state.

2.3 Supplementary assumptions

Henceforth we will drop the date subscripts on functions and variables. We will also make three important supplementary assumptions:

I. *There is a* laissez faire *($\theta = G = 0$) steady state with a low real interest rate.* The values of α, s and c are consistent with the existence of an $\underline{R} \in (0, \Psi + c)$ such that $S(\underline{R}) - K(\underline{R}) = 0$.

II. *There is no nonbinding steady state.* The equation $[\Psi - (R - c)]\ \ [S(R) - K(R)] = \Psi G$ has no real solutions on $R \in (\underline{R}, \Psi + c)$.

III. *Intermediation costs are not too high.* The values of α, s and c satisfy

$$1 + \frac{c}{\Psi} < \frac{1}{s\,(1-\alpha)}\,.$$

Lemma 1 *For $R \in (0, \Psi + c]$, [1] $S'(R) > K'(R)$ and [2] $S''(R) < K''(R)$.*

Proof. [1] Equations (10) and (11) imply

$$\frac{K'(R)}{S'(R)} = \frac{1}{s\,(1-\alpha)}\,\frac{\Psi}{R}\,. \tag{18}$$

Assumption III gives us

$$\frac{1}{s\,(1-\alpha)} > \frac{\Psi + c}{\Psi},$$

and it follows immediately that $K'(R)/S'(R) > 1$ on $(0, \Psi + c]$. Since both $K'(R)$ and $S'(R)$ are negative, we have $K'(R) < S'(R)$ on $(0, \Psi + c]$.

[2] Equations (10) and (11) also imply

$$\frac{K''(R)}{S''(R)} = \frac{2-\alpha}{s\,(1-\alpha)}\frac{\Psi}{R}\,.$$

Since $\alpha \in (0,1)$ implies $2 - \alpha > 1$ we have $K''(R)/S''(R) > 1$ on $(0, \Psi + c]$ (see above). Here $K''(R)$ and $S''(R)$ are positive, so we have $K''(R) > S''(R)$ on $(0, \Psi + c]$.

The Lemma implies that the aggregate outside-asset demand function $S(R) - K(R)$ is strictly increasing on $(0, \Psi + c]$. Consequently, Assumption I holds if and only if there exists a *laissez faire* ($\theta = G = 0$) steady state in which unbacked government liabilities are valued, which is to say iff $S(\Psi + c) - K(\Psi + c) > 0$. Equations (10) and (11) can be used to show that a necessary and sufficient condition for this to be the case is

$$s\left(1 + \frac{c}{\Psi}\right) > \frac{\alpha}{1-\alpha}\,; \tag{19}$$

they also imply

$$\underline{R} = \frac{\alpha}{1-\alpha}\frac{\Psi}{s}\,. \tag{20}$$

Assumption I also implies that for at least some positive values of θ the equation $(1-\theta)\,S(R) - K(R) = 0$, which characterizes binding steady states in which there are no government bonds, has a unique solution $\underline{R}_\theta \in (\underline{R}, \Psi + c)$. It is readily seen that

$$\underline{R}_\theta = \frac{\underline{R}}{1-\theta}\,. \tag{21}$$

When Assumption I holds, the least upper bound of the set of values of θ that produce $\underline{R}_\theta < \Psi + c$ is

$$\theta_{\max} \equiv 1 - \frac{\alpha}{s\,(1-\alpha)}\frac{1}{1+\frac{c}{\Psi}}\,. \tag{22}$$

For the moment we will adopt $\theta < \theta_{\max}$ as a provisional Assumption IV (call it Assumption IV-p). But we will strengthen this assumption slightly below.

Assumption II implies that the government cannot finance its deficit without imposing a binding reserve requirement. Assumption III is a technical assumption that is used in

the proof of Lemma 1 and holds for virtually any plausible (which is to say, relatively low) values of c, given that Assumption I holds. Assumptions I and III collectively imply $\alpha < (1-\alpha)\, s\, \left(1 + \frac{c}{\Psi}\right) < 1$.

2.4 The seigniorage revenue function

Equations (15) and (16) imply

$$B = (1-\theta)\, S(R) - K(R) \equiv B(R)\,, \tag{23}$$

This equation can be combined with equations (13), (15) and (17) to produce the seigniorage revenue function

$$\Gamma(R, R^m; \theta) \equiv (\Psi - R^m)\, \theta\, S(R) + [\Psi - (R-c)]\; B(R)\,. \tag{24}$$

This function describes the dependence of the level of government seigniorage revenue on the value of the gross real interest rate R, the real currency return rate R^m, and the required reserve ratio θ.[7] The revenue from currency seigniorage is $(\Psi - R^m)\, \theta\, S(R)$, while $[\Psi - (R-c)]\; B(R)$ is the revenue from bond seigniorage.

Given θ, a binding steady state can be characterized as values of R and R^m such that $\underline{R}_\theta < R < \Psi + c$, $0 \leq R^m < R$, and $\Gamma(R, R^m; \theta) = \Psi G$. For purposes of exposition, we will treat the seigniorage revenue function as a function of R, with domain $[\underline{R}_\theta, \Psi + c]$, and we will refer to a plot of the function against R as the "seigniorage revenue curve."[8] In this section, we will conduct policy experiments in which the central bank holds the reserve ratio fixed and uses a permanent change in the money growth rate to produce a permanent change in the inflation rate. In terms of the symbolic arguments of our seigniorage revenue function, these are experiments in which θ remains constant while R^m rises or falls. The change in R^m will shift the seigniorage revenue curve, and we will use the properties of the curve to determine the effects of these changes on the real interest rate.

The slope of the seigniorage revenue curve is

$$\frac{\partial \Gamma}{\partial R} = (\Psi - R^m)\, \theta\, S'(R) + [\Psi - (R-c)]\; B'(R) - B(R)\;. \tag{25}$$

[7] Note that this actually returns the product of total seigniorage revenue and the gross real growth rate Ψ.

[8] However, in a binding steady state we must have $R_m < R^d = R - c$, and if $R^m > \underline{R}_\theta$ then there are points on the seigniorage Laffer curve to the left of R^m. These points are not potential equilibrium points.

Lemma 2 *For $R \in (\underline{R}_\theta, \Psi + c]$, $B'(R) > 0$ and $B(R) > 0$.*

Proof. Since $K'(R) < 0$, $B'(R) > 0 \Leftrightarrow (1-\theta)\, S'(R)/K'(R) < 1$, which is $(1-\theta)\, s\,(1-\alpha)\, \frac{R}{\Psi} < 1$ (see the proof of Lemma 1, part [1]). Assumption III gives us $s\,(1-\alpha) < \Psi/(\Psi + c)$, so sufficient is $(1-\theta)\, R\, /(\Psi + c) < 1$, which holds for any $R \in (0, \Psi + c]$ when $\theta > 0$. We know that $B(\underline{R}_\theta) = 0$, and Assumption IV-p gives us $\underline{R}_\theta < \Psi + c$. Thus, $B'(R) > 0$ on $R \in (0, \Psi + c]$ implies $B(R) > 0$ on $.R \in (\underline{R}_\theta, \Psi + c]$.

Proposition 1 is proved in the appendix and it establishes three important facts about the seigniorage revenue curve:

Proposition 1 *The function $\Gamma(R, R^m; \theta)$ is [1] strictly concave in R on $[\underline{R}_\theta, \Psi + c]$, [2] positive and decreasing in R at $R = \Psi + c$, [3] positive and increasing in R at $R = \underline{R}_\theta$ if*

$$\theta < \gamma\, \theta_{\max},$$

where

$$\gamma \equiv \frac{1 + \frac{c}{\Psi}}{\left(1 + \frac{c}{\Psi}\right) + \frac{\alpha}{1-\alpha}}\,.$$

Henceforth we will add

IV. $\theta < \gamma\, \theta_{\max}$

to our list of supplementary assumptions. Note that $0 < \gamma < 1$ and that γ is increasing in c with a least upper bound of unity. Also, if $c = 0$ then $\gamma = 1 - \alpha$. Assumption IV is sufficient, but not necessary, for part [3] of Proposition 1 to hold.[9]

Assumption IV ensures that for given values of θ and R^m, the seigniorage revenue curve is upward-sloping at its left endpoint (see Figure 1). Under this assumption, the curve associated with each admissible vector (θ, R^m) is a positive-valued, downward-opening paraboloid on $(\underline{R}_\theta, \Psi + c]$, and thus has a unique peak on the interior of this domain. As we shall see, both the dynamic stability properties of a steady state and the results of comparative statics experiments beginning from it depend critically on whether the steady state in question is on the left (upward-sloping) or the right (downward-sloping) side of this curve.

[9] The sufficiency of Assumption IV for part [3] of Proposition 1 is established under the very conservative assumption that $R^m = 0$. As R^m increases the required condition becomes weaker, and it can be shown that if $R_m \geq \underline{R}_\theta$ then part [3] holds for all values of θ consistent with Assumption I. (See the proof of Proposition 1.)

For values θ and R^m with $R^m < \underline{R}_\theta$, the portion of the seigniorage revenue curve that is relevant to our analysis begins at domain value $\underline{R}_\theta$ and range value $\Gamma_1 \equiv (\Psi - R^m)\,\theta\,S(\underline{R}_\theta)$; it ends at domain value $\Psi + c$ and range value $\Gamma_2 \equiv (\Psi - R^m)\,\theta\,S(\Psi + c)$. If $R^m > \underline{R}_\theta$ then the lowest relevant domain value is R^m and the corresponding range value is $\Gamma_1 \equiv (\Psi - R^m)\,\theta\,S(R^m)$. Note that $\Gamma_1 > \Gamma_2$. Let R_{peak} and Γ_{peak} denote the domain and range values associated with the peak of the seigniorage revenue curve. We shall assume, for expository purposes, that $R^m < R_{\text{peak}}$, so that the relevant portion of the curve has two sides. Given θ and R^m, a binding steady state will exist only if $\Psi G \in (\Gamma_2, \overline{\Gamma}_{\text{peak}}]$. If $\Psi G \in (\Gamma_1, \overline{\Gamma}_{\text{peak}})$ then there will be two values of R that support binding steady states. One of these will be a relatively low value of the left side of the seigniorage revenue curve and the other will be a relatively high value on the right side of the curve. If, on the other hand, $\Psi G \in (\overline{\Gamma}_2, \Gamma_1]$, then there will be only one steady state and the associated R-value will be on the right side of the curve. We shall also assume, for expository purposes, that $\Psi G \in (\Gamma_1, \overline{\Gamma}_{\text{peak}})$.

2.5 Stability of steady states

Equation (4) can be rewritten

$$G_t = M_t - R^m_{t-1} M_{t-1} + B_t - R^b_{t-1}\,B_{t-1}.$$

We know from equation (5) that $M_t = p_t\,z\,H_{t-1} = z\,R^m_{t-1} M_{t-1} \Leftrightarrow R^m_{t-1} M_{t-1} = M_t/z$. So we can write

$$G_t = M_t\,(1 - \frac{1}{z}) + B_t - (R_{t-1} - c)\,B_{t-1}. \tag{26}$$

Equations (8) and (9) give us $B_t = (1-\theta)\,S(R_{t-1}) - K(R_t)$. Substituting this equation and equation (8) into equation (25) produces

$$G_t = \theta\,S_t(R_{t-1})\,(1-\frac{1}{z}) + [(1-\theta)\,S_t(R_{t-1}) - K_{t+1}(R_t)] - (R_{t-1} - c)\,\left[(1-\theta)\,S_{t-1}(R_{t-2}) - K_t(R_{t-1})\right].$$

We know $S_t(\cdot) = \Psi S_{t-1}(\cdot)$ and $K_t(\cdot) = \Psi K_{t-1}(\cdot)$, and if we are interested in steady states we must assume $G_t = \Psi G_{t-1}$. So we are free to define $S(\cdot) \equiv S_1(\cdot)$, $K(\cdot) \equiv K_1(\cdot)$ and $G \equiv G_1$ and write

$$\Psi G = \Psi\left[(1 - \frac{\theta}{z})\,S(R_{t-1}) - K(R_t)\right] - (R_{t-1} - c)\,\left[(1-\theta)\,S(R_{t-2}) - K(R_{t-1})\right]. \tag{27}$$

This is the implicit second-order difference equation that characterizes binding competitive equilibrium paths for R_t. We can use this equation to study the local stability properties of binding steady states. We find that

Theorem 1 *A binding steady state is locally stable if and only if it is on the left (upward-sloping) side of the seigniorage revenue curve.*

Proof. [See Azariadis (1993), chs. 1,6.] The univariate second-order difference equation for R_t is $\Psi G = \Psi\left[(1-\frac{\theta}{z})\,S(R_{t-1}) - K(R_t)\right] - (R_{t-1}-c)\,\left[(1-\theta)\,S(R_{t-2}) - K(R_{t-1})\right]$. We define $Z_t \equiv R_{t-1}$, so that $Z_{t-1} = R_{t-2}$. This gives us the bivariate second-order equation $\Psi G = \Psi\left[(1-\frac{\theta}{z})\,S(R_{t-1}) - K(R_t)\right] - (R_{t-1}-c)\,\left[(1-\theta)\,S(Z_{t-1}) - K(R_{t-1})\right]$.Total differentiation with respect to R_t and R_{t-1} produces $0 = \Psi(1-\frac{\theta}{z})\,S'(R_{t-1})\,dR_{t-1} - \Psi K'(R_t)\,dR_t - \left[(1-\theta)\,S(Z_{t-1}) - K(R_{t-1})\right]\,dR_{t-1} + (R_{t-1}-c)\,K'(R_{t-1})\,dR_{t-1}$and thus

$$\frac{\partial R_t}{\partial R_{t-1}} = \frac{\Psi(1-\frac{\theta}{z})\,S'(R_{t-1}) - \left[(1-\theta)\,S(Z_{t-1}) - K(R_{t-1})\right] + (R_{t-1}-c)\,K'(R_{t-1})}{\Psi K'(R_t)}.$$

Total differentiation with respect to R_t and Z_{t-1} produces $0 = -\Psi K'(R_t)\,dR_t - (R_{t-1} - c)\,(1-\theta)\,S'(Z_{t-1})\,dZ_{t-1}$and thus

$$\frac{\partial R_t}{\partial Z_{t-1}} = \frac{-(R_{t-1}-c)\,(1-\theta)\,S'(Z_{t-1})}{\Psi K'(R_t)}.$$

Since $Z_t = R_{t-1}$, we have $\partial Z_t/\partial R_{t-1} = 1$ and $\partial Z_t/\partial Z_{t-1} = 0$. So the Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial R_t}{\partial R_{t-1}} & \frac{\partial R_t}{\partial Z_{t-1}} \\ \frac{\partial Z_t}{\partial R_{t-1}} & \frac{\partial Z_t}{\partial Z_{t-1}} \end{pmatrix} = \begin{pmatrix} \frac{\partial R_t}{\partial R_{t-1}} & \frac{\partial R_t}{\partial Z_{t-1}} \\ 1 & 0 \end{pmatrix}.$$

Define $\overline{J} = J\mid_{(\overline{R},\overline{R})}$. To determine the stability properties of binding steady states, we calculate the eigenvalues of $\overline{J}$ and investigate their dependence on possible steady state real interest rates $\overline{R}$.

We begin by constructing the characteristic equation $p(\lambda) = \left|\overline{J} - \lambda I\right| = \lambda^2 - (\operatorname{tr}\overline{J})\,\lambda + \det\overline{J}$. The solutions of the equation $p(\lambda) = 0$ are the eigenvalues of the Jacobian matrix. We have

$$\operatorname{tr}\overline{J} \equiv \overline{T} = \frac{\partial R_t}{\partial R_{t-1}}\mid_{(\overline{R},\overline{R})} = \frac{\Psi(1-\frac{\theta}{z})\,S'(\overline{R}) + (\overline{R}-c)\,K'(\overline{R}) - B(\overline{R})}{\Psi K'(\overline{R})},$$

and

$$\det\, \overline{J} \equiv \overline{D} = -\frac{\partial R_t}{\partial Z_{t-1}}\,|_{(\overline{R},\overline{R})} = \frac{(\overline{R}-c)\,(1-\theta)\,S'(\overline{R})}{\Psi K'(\overline{R})}.$$

The seigniorage revenue curve is $\Gamma(R, R^m) = (\Psi - R^m)\,\theta\,S'(R) + [\Psi - (R-c)]\;B'(R) - B(R)$ and its slope is $\partial\Gamma/\partial R = (\Psi - R^m)\,\theta\,S'(R) + [\Psi - (R-c)]\;B'(R) - B(R)$, which, given $R^m = \Psi/z$, is

$$\frac{\partial\Gamma}{\partial R} \equiv \Gamma'(R) = \Psi\left[(1-\frac{\theta}{z})\,S'(R) - K'(R)\right] - (R-c)\,B'(R) - B(R)\,.$$

Define $\overline{\Gamma}' \equiv \Gamma'(\overline{R})$. We have $\overline{\Gamma}' = \Psi K'(\overline{R})\,\overline{T} - (\overline{R}-c)\,K'(\overline{R}) - \Psi K'(\overline{R}) - (R-c)\,B'(R) = \Psi K'(\overline{R})\,\overline{T} - (\overline{R}-c)\,(1-\theta)\,S'(\overline{R}) - \Psi K'(\overline{R})$, so

$$\overline{T} = \frac{\overline{\Gamma}' + (\overline{R}-c)\,(1-\theta)\,S'(\overline{R}) + \Psi K'(\overline{R})}{\Psi K'(\overline{R})} = (1+\overline{D}) + \frac{\overline{\Gamma}'}{\Psi K'(\overline{R})}\,.$$

The eigenvalues of the Jacobian matrix solve $\lambda^2 - \overline{T}\,\lambda + \overline{D} = 0$, so they are

$$\{\lambda\} = \frac{\overline{T} \pm \sqrt{\overline{T}^{\,2} - 4\overline{D}}}{2} = \frac{1}{2}\left[(1+\overline{D}) + \frac{\overline{\Gamma}'}{\Psi K'(\overline{R})} \pm \sqrt{\left\{(1+\overline{D}) + \frac{\overline{\Gamma}'}{\Psi K'(\overline{R})}\right\}^2 - 4\,\overline{D}}\right].$$

Notice that if $\overline{\Gamma}' = 0$, so that the steady state is at the peak of the seigniorage revenue curve, then we have

$$\{\lambda\} = \frac{1}{2}\left[(1+\overline{D}) \pm \sqrt{(1+\overline{D})^2 - 4\,\overline{D}}\right] = \frac{1}{2}\left[(1+\overline{D}) \pm (1-\overline{D})\right]\,,$$

so in this case

$$\begin{aligned}\lambda_1 &= \frac{1}{2}\left[(1+\overline{D}) + (1-\overline{D})\right] = 1\\ \lambda_2 &= \frac{1}{2}\left[(1+\overline{D}) - (1-\overline{D})\right] = \overline{D}\,.\end{aligned}$$

Define

$$\overline{x} \equiv \frac{\Gamma'(\overline{R})}{-\Psi K'(\overline{R})}.$$

Note that $\overline{x}$ and $\Gamma'(\overline{R})$ have the same sign. In the general case we have

$$\begin{aligned}\lambda_1 &= \frac{1}{2}\left[\left\{(1+\overline{D}) - \overline{x}\right\} + \sqrt{\left\{(1+\overline{D}) - \overline{x}\right\}^2 - 4\overline{D}}\right]\\ \lambda_2 &= \frac{1}{2}\left[\left\{(1+\overline{D}) - \overline{x}\right\} - \sqrt{\left\{(1+\overline{D}) - \overline{x}\right\}^2 - 4\overline{D}}\right]\,.\end{aligned}$$

Lemma 3 $0 < \overline{D} < 1$.

Proof. We know $\overline{D} > 0$ because $S'(\overline{R}) < 0$ and $K'(\overline{R}) < 0$. Since $S'(R)/K'(R) = \frac{R}{\Psi} s(1-\alpha)$, we have

$$\overline{D} = \frac{\overline{R}(\overline{R}-c)(1-\theta)}{\Psi^2} s(1-\alpha).$$

[Note that

$$\frac{S'(\overline{R})}{K'(\overline{R})} = \frac{\Psi}{(\overline{R}-c)(1-\theta)}\overline{D}\ .]$$

We will show that $\overline{D} < \frac{\overline{R}-c}{\Psi}$, which is sufficient for $\overline{D} < 1$. We need

$$\frac{\overline{R}(1-\theta)}{\Psi} s(1-\alpha) < 1 \Leftrightarrow \overline{R} < \frac{\Psi}{(1-\theta)s(1-\alpha)}.$$

We know $\overline{R} < \Psi + c$, so sufficient is

$$\Psi + c < \frac{\Psi}{(1-\theta)s(1-\alpha)} \Leftrightarrow (1-\theta)s(1-\alpha) < \frac{\Psi}{\Psi+c}.$$

And we know from Assumption IV that

$$1 + \frac{c}{\Psi} < \frac{1}{s(1-\alpha)} \Leftrightarrow s(1-\alpha) < \frac{\Psi}{\Psi+c}. \quad \square$$

Theorem 1 establishes that equilibria on the left side of the seigniorage revenue curve are dynamically stable. Comparative statics experiments involving these equilibria – "PMA" experiments – can be interpreted as the consequences of interaction between an active central bank (the Fed) and a passive budgetary authority (the Treasury) that issues the quantity of debt necessary to reconcile the monetary policy decision of the Fed with the fiscal policy decisions that produced the government's budget deficit. The initial values of K_1 and $\mathcal{B}_0$ support a steady state from date 1, given initial values of θ and z. So we can think of the economy as starting out in that steady state. However, the Fed unexpectedly decides, at date 1, to reduce z by the amount necessary to move the real interest rate to a higher long-run target level. This decrease in the currency growth rate produces a shortage of revenue from currency seigniorage, relative to the steady state. The Treasury responds by increasing its borrowing in order to cover the shortfall, and the ratio of the government debt to output rises. As the real bond stock rises from period to period, however, the amount of new borrowing necessary to cover the revenue shortage gradually falls. The Treasury

ultimately finds that it is able to cover the deficit by borrowing an amount that keeps the debt-output ratio constant at a new, higher level, and thus supports a new steady state. But there is no need for the Treasury to make any active decision to change the real interest rate or the real stock of debt, and there is no need for it to cooperate with the Fed in any way except to borrow just enough, at each date, to cover the portion of the budget deficit that is not covered by revenue from currency seigniorage.

We conclude this section by reconciling our description of the mechanics of monetary policy changes with descriptions presented elsewhere in the literature. Let β represent the ratio of the real present value of the stock of government bonds to real balances of fiat currency, so that $\beta \equiv B/M$. As we noted in our introduction, Wallace (1984) and Espinosa and Russell (1998a), Bhattacharya, Joydeep, Mark G. Guzman and Bruce D. Smith (1998) and Bhattacharya and Kudoh (2001), among others, identify the bonds-currency ratio as the central bank's active policy variable, with z and R^m changing passively to accommodate changes in the ratio.[10] They describe an increase in bonds-currency ratio as a monetary tightening, and vice-versa. In our model, conditions (15) and (16) imply that across steady states,

$$\beta = \frac{B(R)}{\theta\, S(R)} = \frac{1-\theta}{\theta} - \frac{K(R)}{\theta\, S(R)}. \tag{28}$$

It is readily seen that for a given value of θ, the relationship between β and R is monotone increasing. Stated formally,

Proposition 2 *When θ is fixed, the relationship between β and R across binding steady states is monotone increasing.*

Proof. Equation (27) implies

$$\beta'(R) = \frac{[(1-\theta)\, S'(R) - K'(R)]\, \theta\, S(R) - \theta\, S'(R)\, [(1-\theta)\, S(R) - K(R)]}{[\theta\, S(R)]^2}.$$

Since $S'(R) > 0$, and $(1-\theta)\, S(R) - K(R) \geq 0$ on $[\underline{R}_\theta, \Psi + c]$, Lemma 1 implies that $\beta'(R) > 0$ on $[\underline{R}_\theta, \Psi + c]$.

Thus, for initial equilibria on the left "PMA" side of our seigniorage revenue curve, the two natural definitions of monetary tightening are equivalent. An active decrease in the

[10] Other papers with this feature include Miller and Todd (1995), Bhattacharya *et. al* (1997,1998) and Schreft and Smith (1998). In this paper we define $\beta_t \equiv B_t/M_t$. A few of the papers just cited use this definition, but most follow Wallace (1984) by defining $\beta_t \equiv \mathcal{B}_t/H_t = \mathrm{B}_t/M_t$. If we call the Wallace version of the ratio β_t, then our $\beta = \beta_0\,(R/R^m)$. We adopt our definition because it simplifies the analysis.

money growth rate z, which is the definition of monetary tightening we use here [following Sargent and Wallace (1981) and Miller and Sargent (1984), among many others] produces both an increase in the real interest rate and a passive increase in the bonds-money ratio β. It follows that an active increase in the bonds-money ratio would produce both an increase in the real interest rate and a passive decrease in the money growth rate. From a stability perspective, however, whether the instrument is θ or β makes a big difference as can be corroborated by contrasting our results in this paper against those in Bhattacharya and Kudoh (2001) .

3 Concluding remarks

In this paper, we have integrated the model of monetary policy devised by Wallace (1984) with Diamond's (1965) neoclassical model of production and capital. The result is a general equilibrium model in which monetary policy can have long-run real effects consistent with the Keynesian conventional wisdom: a "tightening" of policy, engineered by a decrease in the money supply growth rate or by an in the required reserve ratio, reduces the rate of inflation and increases the real interest rate. This "conventional wisdom" is also a version of the so-called PMA described above. The increase in the real interest rate produces a permanent decrease in the level of output, and thus persistent but ultimately temporary declines in the growth rate of output. Espinosa and Russell (1998b) show that, under certain conditions, the increase in the real interest rate produced by a move to tighten policy can be large enough to allow the nominal interest rate to rise. Under these conditions the ratio of the increase in the real interest rate to the associated decrease in the inflation rate can become quite large, which means that monetary policy can have substantial real effects. We show that if the government's monetary policy rule involves a fixed money supply growth rate, then a steady state is dynamically stable if and only if it is on the left (upward-sloping) side of the seigniorage revenue curve. If the criterion to choose relevant steady states is whether the equilibria are dynamically stable or unstable, one can conclude that PMA of the type studied in Espinosa and Russell (1998b) is the most likely equilibrium to be observed.

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Appendix

4

Proof of Proposition 1: [1] We need to show that

$$\frac{\partial}{\partial R}\left(\frac{\partial}{\partial R}\,\Gamma(R,R^m;\theta)\right)<0\,.$$

Differentiating equation (24) produces the derivative

$$(\Psi-R^m)\,\theta\,S''(R)+[\Psi-(R-c)]\,[(1-\theta)S''(R)-K''(R)]-2\,[(1-\theta)S'(R)-K'(R)]\;.$$

Sufficient would be

$$\Psi\,\theta\,S''(R)-2\,[(1-\theta)S'(R)-K'(R)]\;<0,$$

since $\Psi\,\theta\,S''(R)\geq(\Psi-R^m)\,\theta\,S''(R)$ for $R^m\in[0,\Psi)$, and $(\Psi-R-c)\,[(1-\theta)S''(R)-K''(R)]<0$ for $R\in(\underline{R}_\theta,\Psi+c)$. The latter inequality follows from the fact that $K''(R)>S''(R)$ on $(\underline{R}_\theta,\Psi+c)$, which was established in Lemma 1.

Now

$$\Psi\,\theta\,S''(R)-2\,[(1-\theta)S'(R)-K'(R)]\;<0\Leftrightarrow\Psi\,\theta<2\left[(1-\theta)\frac{S'(R)}{S''(R)}-\frac{K'(R)}{S''(R)}\right].$$

Equations (10) and (11) imply

$$\frac{S'(R)}{S''(R)}=-(1-\alpha)R\quad\text{and}\quad\frac{K'(R)}{S''(R)}=-\frac{\Psi}{s},$$

which gives us

$$\Psi\,\theta<2\left[\frac{\Psi}{s}-(1-\theta)\,(1-\alpha)R\right]\Leftrightarrow\frac{R}{\Psi}<\frac{\frac{1}{s}-\frac{\theta}{2}}{(1-\theta)\,(1-\alpha)}\,.$$

We have $R<\Psi+c\Leftrightarrow R/\Psi<1+c/\Psi$, so sufficient would be

$$1+\frac{c}{\Psi}<\frac{\frac{1}{s}-\frac{\theta}{2}}{(1-\theta)\,(1-\alpha)}\,.$$

Assumption I gives us

$$1+\frac{c}{\Psi}<\frac{1}{s\,(1-\alpha)},$$

so sufficient would be

$$\frac{1}{s\,(1-\alpha)} < \frac{\frac{1}{s}-\frac{\theta}{2}}{(1-\theta)\,(1-\alpha)} \Leftrightarrow \frac{1-\theta}{s} < \frac{1}{s}-\frac{\theta}{2} \Leftrightarrow s<2\,. \quad \square$$

[2] Equation (24) implies that at $R=\Psi+c$ the slope of the seigniorage revenue curve is

$$(\Psi-R^m)\,\theta\,S'(\Psi+c)-\,[(1-\theta)S(\Psi+c)-K(\Psi+c)]\,.$$

Lemma 2 gives us $(1-\theta)\,S(\Psi+c)-K(\Psi+c)>0$. Since $S'<0$, we have our result.

[3] Equation (24) implies that at $R=\underline{R}_\theta$ the slope of the seigniorage revenue curve is

$$(\Psi-R^m)\,\theta\,S'(\underline{R}_\theta)+[\Psi-(\underline{R}_\theta-c)]\;[(1-\theta)S'(\underline{R}_\theta)-K'(\underline{R}_\theta)]\;.$$

We need to show that this is positive. Since $S'(R)<0$ and $R^m\in[0,\Psi)$, sufficient would be

$$\Psi\,\theta\,S'(\underline{R}_\theta)+[\Psi-(\underline{R}_\theta-c)]\;[(1-\theta)S'(\underline{R}_\theta)-K'(\underline{R}_\theta)]>0,$$

which is equivalent to

$$\Psi\,\theta<[\Psi-(\underline{R}_\theta-c)]\;[\frac{K'(\underline{R}_\theta)}{S'(\underline{R}_\theta)}-(1-\theta)]\,. \tag{29}$$

Equations (A.1) and (20) imply

$$\frac{K'(\underline{R}_\theta)}{S'(\underline{R}_\theta)}=\frac{1-\theta}{\alpha} \quad \text{and} \quad \frac{K'(\underline{R}_\theta)}{S'(\underline{R}_\theta)}-(1-\theta)=(1-\theta)\,\frac{1-\alpha}{\alpha}\,.$$

This leaves us with

$$\Psi\,\theta<[\Psi-(\underline{R}_\theta-c)]\;(1-\theta)\,\frac{1-\alpha}{\alpha}\,. \tag{30}$$

Equation (20) also gives us

$$\Psi-(\underline{R}_\theta-c)=\Psi\,\left[1-\frac{\alpha}{1-\alpha}\,\frac{1}{s\,(1-\theta)}\right]+c,$$

producing the condition

$$\begin{aligned}\theta \;&<\; \left\{\left[1-\frac{\alpha}{1-\alpha}\,\frac{1}{s\,(1-\theta)}\right]+\frac{c}{\Psi}\right\}(1-\theta)\,\frac{1-\alpha}{\alpha}=\left[(1-\theta)\frac{1-\alpha}{\alpha}-\frac{1}{s}\right]+\frac{c}{\Psi}\left[(1-\theta)\,\frac{1-\alpha}{\alpha}\right]\\ \Leftrightarrow\;\; \theta &< \frac{(1-\alpha)-\frac{\alpha}{s}+\frac{c(1-\alpha)}{\Psi}}{1+\frac{c(1-\alpha)}{\Psi}}=\frac{\left(1+\frac{c}{\Psi}\right)-\frac{\alpha}{s(1-\alpha)}}{\frac{1}{1-\alpha}+\frac{c}{\Psi}}=\frac{\left(1+\frac{c}{\Psi}\right)-\frac{\alpha}{s(1-\alpha)}}{(1+\frac{c}{\Psi})+\frac{\alpha}{1-\alpha}}\,.\end{aligned}$$

Now define

$$\widetilde{\theta} \equiv \frac{\left(1+\frac{c}{\Psi}\right) - \frac{\alpha}{s(1-\alpha)}}{\left(1+\frac{c}{\Psi}\right) + \frac{\alpha}{1-\alpha}}.$$

We have

$$\widetilde{\theta} = \frac{\left(1+\frac{c}{\Psi}\right)}{\left(1+\frac{c}{\Psi}\right) + \frac{\alpha}{1-\alpha}}\,\theta_{\max}. \quad \square$$

Supplementary note: If $R^m > 0$ then the threshold value of θ will be higher than $\widetilde{\theta}$. In particular, if $R^m > 0$ then the appropriately revised version of inequality (A.3) is

$$(\Psi - R^m)\,\theta < (\Psi - \underline{R}_\theta - c)\,(1-\theta)\,\frac{\alpha}{1-\alpha}]$$

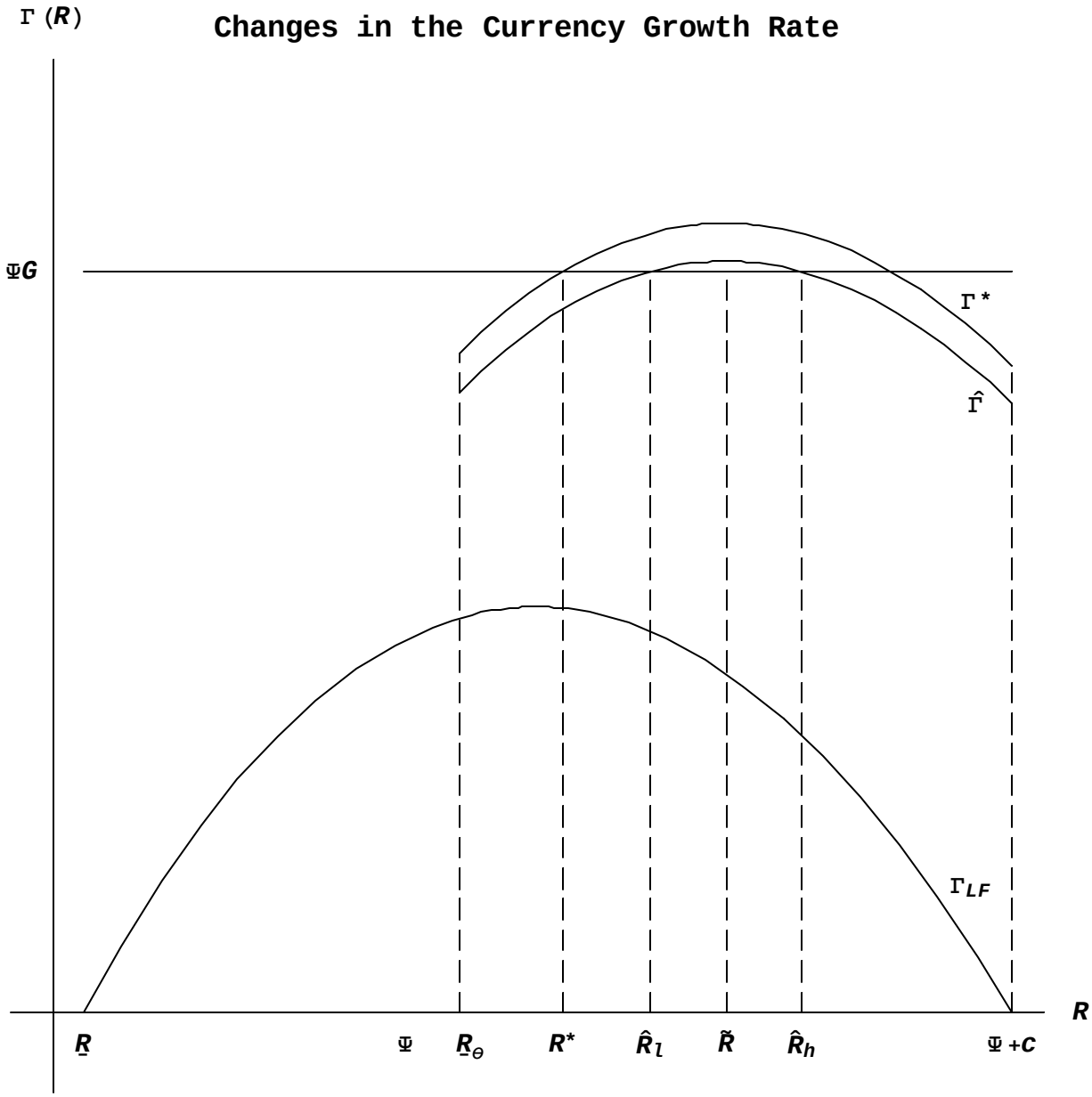
or equivalently

$$\frac{\theta}{1-\theta} < \frac{(\Psi - \underline{R}_\theta - c)}{(\Psi - R^m)}\frac{1-\alpha}{\alpha}\,.$$

If $R^m \geq \underline{R}_\theta$, then sufficient would be

$$\frac{\theta}{1-\theta} < \frac{1-\alpha}{\alpha} \Leftrightarrow \theta < 1-\alpha\ ,$$

and it is readily seen that this condition holds for any $\theta \in (0, \theta_{\max})$. Thus, in this case there does not need to be any additional restriction on θ.


Γ(R)
Changes in the Currency Growth Rate
ΨG
Γ*
Γ̂
Γ_LF
R
R̲
Ψ
R̲_θ
R*
R̂_l
R̃
R̂_h
Ψ+c