Dynamics of Sovereign Investment, Debt, and Default*

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Abstract

In this paper, we study the interplay between the decision to default and capital accumulation. We find that conditional on a level of foreign indebtedness more capital reduces the likelihood of default and increases the price of debt. Large capital stocks have the additional benefit of taming the severity of the contraction following default. Our quantitative model delivers default episodes that mimic those observed in the data.

PRELIMINARY

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1 Introduction

Sovereign defaults are followed by periods of economic stress characterized by a persistent decline in output and consumption. Yet these are just two elements of the turmoil unraveling around default episodes. The collapse of investment is an example of the complexities surrounding crises in emerging economies. On average, the contraction in investment upon default is three times larger than that in output or consumption. To the extent that investment can be used to smooth out consumption, this disproportionately large decline suggests that investment plays a nontrivial role during default periods. A second pattern in the data is that the transition to default is surprisingly smooth. Output in sovereign countries peaks on average 4 quarters prior to default and then gradually marches to a severe contraction upon defaulting. In this paper, we document these features and propose a model that is capable of matching them.

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Understanding the dynamics of default and investment together is important because we find that countries with low investment-to-GDP ratios are more prone to default. For example, economies that have defaulted in the past like Argentina, Ecuador, Mexico, and Russia have ratios below 19%. In contrast, developed small open economies such as Australia and Norway have ratios in excess of 25%. At face value, these observations suggest that low capital accumulation makes a country more likely to default.

Furthermore, defaults are a pervasive feature of emerging economies. Indeed, the number of defaults and reschedulings in Latin America and Asia almost tripled in the period 1975-2006 relative to 1950-1974 (Reinhart and Rogoff, 2008). This means that business cycles in these economies need be studied under the presumption that countries sometimes renege their external obligations. The recent experience in some European countries reinforces this need for a model with fully-fledged business cycles and default. Our model (like Mendoza and Yue, 2011) is an attempt towards this ambitious goal.

For emerging economies, prosperous times are periods of high productivity and cheap financing from international markets. In response to these favorable conditions, developing economies tend to invest. For instance, investment in Argentina was on average 6.5% above trend in the three years preceding its 2001 default. This increased investment comes at the expense of raising the countries' level of indebtedness. As a consequence, more capital increases the likelihood of default. The temptation to stopping debt payments is further exacerbated by the observation that physical capital (an input to the production function) cannot be seized upon default.[1] At the same time, additional capital can delay default in the face of bad technology shocks because capital can be liquidated to meet outlays. What channel dominates is a central task of our quantitative investigation.

Yet shaping the theoretical relation between capital and default is nontrivial. To begin with, capital drastically changes the model because it enables smoothing that (absent adjustment costs) eliminates excess volatility in consumption. A second reason is that the consequences of default are much more complicated in the presence of capital. For example, default ends up being particularly costly during periods of high productivity since an inability to borrow from financial markets limits capital accumulation. Consequently, a defaulter must internalize all the costs, both direct and indirect, associated with default. When we simplify the theory by assuming that capital is exogenous, stochastic, and randomly drawn every period, we show analytically that countries with low capital stock are more likely to default.

For the general case where investment is a choice variable, we solve our model numerically. The simulations show that our calibrated model captures many quantitative features found in the

[1] In modern history, physical assets within the borders of a sovereign country have not been seized by creditors upon default.

data such as the probability of default, the volatility of output, the ratio of debt-to-output, and the dynamics of investment. More importantly, our quantitative exercise reveals that, conditional on a level of debt, additional capital reduces the likelihood of default and increases the price of debt. That is, the supply of credit is increasing in the stock of capital.

Our simulations also indicate that capital makes the recession following default less painful. We find that the collapse of consumption is substantially smaller in an economy with investment. Furthermore, the more flexible capital accumulation is, the smaller the contraction in consumption is. Indeed, when capital is very flexible the economy with default looks like a developed small open economy in which consumption is less volatile than output. As a consequence, we find that to capture simultaneously the dynamics of consumption and investment under default, capital accumulation has to be subject to adjustment costs.

Additionally, we find long-term debt is necessary to capture the smooth dynamics leading to default. The reason is that, contrary to short-term obligations, households in good standing only repay a fraction (about 5% in our calibration) of their total indebtedness each period. Hence, when the sovereign faces adverse shocks, debt imposes a small burden on its budget so the default decision can be (at least temporary) postponed. However, if difficult times linger, the economy is eventually forced to renege on its obligations. With short-term debt, the sovereign defaults immediately in response to an adverse shock preventing a smooth transition. Investment also smooths dynamics around default, albeit to a lesser extent. The reason for this is that households can only change slowly the stock of capital when facing negative productivity innovations.

Our paper joins recent contributions that study the interactions of default and business cycles. Some recent papers include Aguiar and Gopinath (2006; henceforth, AG), Arellano (2008), Arellano and Bai (2012), Chatterjee and Eyigungor (2011; henceforth, CE), Hatchondo, Martinez, and Sapriza (2010), and Mendoza and Yue (2011; henceforth, MY). In particular, we extend CE's long-term debt model to include labor and capital accumulation. The resulting model is well equipped to analyze not only default and investment but also to capture the transitional dynamics to default.

In the consumer finance literature, several authors have examined the effect of non-seizable wealth on default. Pavan (2008) studied an economy where households borrow and invest in durable goods. She explored several scenarios about what happens to the durable goods upon default. In her work, allowing lenders to only seize a fraction of durable goods following default increases the probability of default. Mitman (2011) found that bankruptcy rates are decreasing in the amount of home equity that can be retained after declaring bankruptcy. He further showed that for a given level of net worth, more home equity increases the likelihood of declaring bankruptcy.

The rest of the manuscript is organized as follows. In the next section, we report evidence on investment around default episodes in several emerging economies. The model is spelled out in

Section 3. The calibration, solution and main results are reported in Sections 4 and 5. Finally, Sections 6 and 7 provide some robustness checks and concluding remarks.

2 Investment and Default

Figure 1 displays a typical default episode in our sample of emerging economies 12 quarters before and after defaulting (the appendix provides description on the data and countries included). The values are reported as percentage deviations from trend calculated with the HP filter (the standard smoothing parameter value of 1600 is used). Dotted lines correspond to plus- and minus-one-standard deviations around the mean. Relative to previous studies, the novel element in this figure is the dynamics of investment.

One can see that investment follows a path that is qualitatively similar to the dynamics of output. For example, investment is above trend in the periods leading up to default, subsequently falling during default and remaining depressed for several quarters.[2] On closer look, investment is far more responsive than output and consumption in the onset of the crises. Whereas consumption declines by 5% in the quarter of default, the collapse of investment is 3 times larger (15%). At the country level, the demise can be as large as -44% like during the Argentinean crisis in 2001. From peak to bottom, investment contracts by a whopping 27% (compare this with the milder decline of output of 8%). Clearly, investment does move substantially during default episodes. Furthermore, this movement indicates that sovereign economies use investment to ameliorate the negative consequences of defaulting, i.e., to smooth out consumption.

[2] Mendoza (2010) documents the decline of investment during sudden stops in several countries. His data set is broader because sudden stop episodes include cases in which the country did not declare default like the Mexican experience during the Tequila crisis in 1994. As a consequence, the dynamics of investment in that paper does not necessarily reflect those during default.

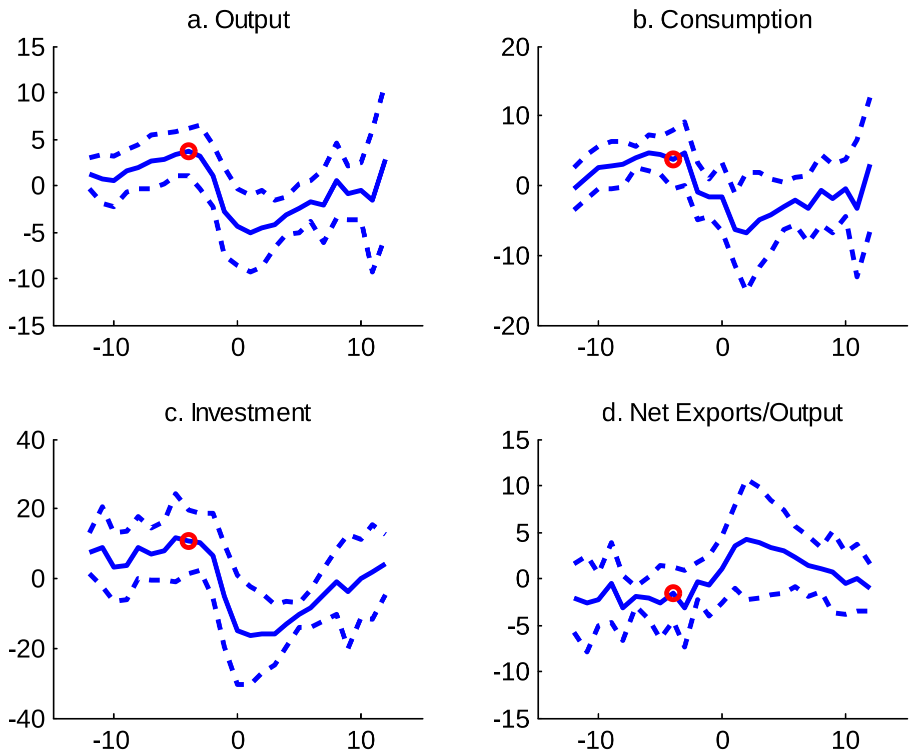

Figure 1: Average Default Episode in Emerging Countries.

A second interesting (and under-appreciated) feature of the data is that the transition to and from the default period is surprisingly smooth. Indeed, investment as well as output and consumption peak about 4 quarters (as indicated by the red circle) prior to reneging on debt. Further, the recovery is rather slow, taking an average of two years to return to trend.

At face value, the slow-moving transition to default indicates that a sequence of moderately bad shocks is what brings the sovereign to stop meeting its foreign obligations. In other words, the default dynamics suggests that to some degree default episodes are forecastable. That is, if one were able to filter out productivity in real time, one could in principle anticipate default. This observation is in stark contrast with default models in which the collapse is sudden and due to a large and unanticipated productivity shock.

3 Model

We modify CE's long-term debt model to include capital investment. Households consume, supply labor, and invest in capital to maximize utility:

$$\max_{c,k,l} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t\right),$$

subject to

$$\begin{aligned} c_t + i_t &= w_t l_t + r_t^k k_t + m_t + \tau_t - \frac{\Theta}{2}\left(k_{t+1} - k_t\right)^2, \\ i_t &= k_{t+1} - \left(1 - \delta\right) k_t. \end{aligned}$$

Here, l_t is labor supply in period t; τ_t is transfers from the government; and Θ controls the cost of adjusting capital. We assume period utility of the type in Greenwood, Hercowitz, and Huffman (1988): $u\left(c_t, l_t\right) = \frac{\left[c_t - \eta \frac{l_t^{\omega}}{\omega}\right]^{1-\sigma}}{1-\sigma}$. The FONC's are the standard ones except for the presence of the adjustment cost. As we explain below, m_t is an additional and small shock present only in good standing, which is needed for computational reasons.

A neoclassical firm rents capital k_t and labor l_t to produce output y_t using the Cobb-Douglas production function $y_t = A_t k_t^{\alpha} l_t^{1-\alpha}$. It is assumed that productivity follows $\log A_t = (1 - \rho_A) \log \mu_A + \rho_A \log A_{t-1} + \varepsilon_A$, where $\varepsilon_A \sim N(0, \sigma_A^2)$.

Following CE, the sovereign government has access to long-term debt contracts, in which outstanding debt matures with probability λ. If debt does not mature, it delivers a coupon payment z. As shown by the authors, this debt structure captures not only the long-term nature of debt in the data but also helps improve the quantitative properties of the model without making its computation too onerous. Following the convention in the literature, we treat debt as negative wealth. Debt is chosen from a finite set $B \subset R_{-}$ which contains zero and strictly negative elements.[3]

A sovereign with debt $b_t < 0$ may always default. When he does so, four things happen. First, his debt goes away. Second, he is excluded from credit markets. Third, he is readmitted to credit markets with probability ϕ. Lastly, for the duration of autarky, a fraction κ of output is lost. This last assumption captures what is endogenized in MY, namely, that default impairs a country's ability to produce by limiting its access to imports. Since capital refers to assets physically located within the borders of an economy, we further assume that capital cannot be expropriated in default and cannot be pledged as collateral.

[3] Default and debt maturity have been studied in recent superb contributions by Arellano and Ramanarayanan (2012) and Hatchondo and Martinez (2010).

Each period the sovereign decides whether to repay debt and, if so, how much new debt to issue subject to households' and the firm's optimal allocations. The default decision is based on

$$V(b_t, k_t, m_t, A_t) = \max\left\{V^{nd}(b_t, k_t, m_t, A_t), V^d(k_t, A_t)\right\}, \tag{1}$$

where V^{nd} is the value of repaying debt (i.e. not defaulting) and V^d is the value of defaulting (and also the value of being excluded from credit markets). Capital remains a state variable after default because we assume that capital cannot be expropriated. A major difference in our model relative to previous papers is that the default decision, and consequently the price of debt, depends on capital and productivity rather than an exogenous output level. In particular, the price of debt depends on the technology shock, A_t, the amount of debt, b_{t+1}, and next period's stock of capital, k_{t+1}: $q_t(b_{t+1}, k_{t+1}, A_t)$. Since q_t is the price of debt to be repaid in the future, the relevant stock of capital is the one that will be used to produce tomorow, i.e. k_{t+1}.

The value of repaying debt is

$$V^{nd}(b_t, k_t, m_t, A_t) = \max_{c_t, b_{t+1}, i_t, k_{t+1}, l_t} u(c_t, l_t) + \beta \mathbb{E}_t V(b_{t+1}, k_{t+1}, m_{t+1}, A_{t+1}) \tag{2}$$

subject to:

$$\begin{aligned} c_t + q_t(b_{t+1}, k_{t+1}, A_t) b_{t+1} + i_t \quad &\leq \quad A_t k_t^{\alpha} l_t^{1-\alpha} + m_t - \frac{\Theta}{2}(k_{t+1} - k_t)^2 \\ &\quad + [\lambda + (1-\lambda) z] b_t + q_t(b_{t+1}, k_{t+1}, A_t)(1-\lambda) b_t, \end{aligned}$$

$$i_t = k_{t+1} - (1-\delta) k_t.$$

The right-hand side term $[\lambda + (1-\lambda) z]$ captures payments from the fraction λ that matures and the coupon from the fraction $(1-\lambda)$ that remains outstanding. The second term $q_t(b_{t+1}, k_{t+1}, A_t)(1-\lambda)$ corresponds to the valuation of the outstanding debt at current bond prices. We incorporate the *iid* term m_t to facilitate the computation of the model. Following CE, this innovation is drawn from a bounded normal with standard deviation σ_m and support $[\underline{m}, \bar{m}]$.

As already argued, a key contribution of this paper is the inclusion of capital accumulation in a way that captures the dynamics of investment found in the data. To this end, we found it necessary to include a variable quadratic cost paid any time the capital stock deviates from its previous value. This is because without adjustment costs, adverse productivity shocks result in two effects that make investment fluctuate drastically. First, an adverse shock makes the sovereign want to smooth consumption by borrowing against higher expected productivity in the future. Second, such a shock also increases the sovereign's default probability and so causes interest rates on debt to rise. Without adjustment costs, the cheapest way for the sovereign to "borrow" is by sharply reducing investment rather than borrowing on the world market. Consequently, investment ends up being too volatile relative to the series in the data. Adjustment costs make borrowing using capital more costly, and so tame the fluctuations in investment.

The value of defaulting is

$$V^d(k_t, A_t) = \max_{c,i,k',l} u(c_t, l_t) + \beta(1-\phi)\mathbb{E}_t V^d(k_{t+1}, A_{t+1}) + \beta\phi\mathbb{E}_t V(0, k_{t+1}, m_{t+1}, A_{t+1}), \qquad (3)$$

subject to:

$$\begin{aligned} c_t + i_t &\leq (1-\kappa) A_t k_t^{\alpha} l_t^{1-\alpha} - \frac{\Theta}{2}(k_{t+1} - k_t)^2, \\ i_t &= k_{t+1} - (1-\delta) k_t. \end{aligned}$$

As in the related literature, the sovereign is excluded from financial markets during the defaulting period t. With probability ϕ, the economy regains access to credit markets at $t+1$. In this case, the economy starts with zero debt. With probability $1-\phi$, it remains in autarky. Furthermore, upon default a fraction κ of production is lost. We assume the loss of output during default depends on the state of the economy. In particular, the cost is

$$\kappa = \max(0, d_0 + d_1 A_t).$$

Ours is a convenient modification of the one considered in CE. Assuming that the cost depends on the state of technology (rather than output) greatly simplifies the computation of our model.[4]

For a debt level b_t and capital stock k_t, it is optimal to default for TFP values and iid shock values in

$$D(b_t, k_t) = \left\{A_t, m_t : V^{nd}(b_t, k_t, m_t, A_t) < V^d(k_t, A_t)\right\}. \qquad (4)$$

Without capital, the sovereign defaults only when $b_t < 0$. A similar results follows with capital. The probability of default is

$$p_t(b_{t+1}, k_{t+1}, A_t) = \int_{D(b_{t+1}, k_{t+1})} dF(A_{t+1}, m_{t+1} | A_t).$$

In the absence of capital, it is well understood that the default set shrinks with b_t, i.e. lower debt increases the likelihood of repayment (Arellano, 2008; CE; MY). The proof is straightforward because debt affects the default set (4) only through the good-standing value function (2). However, capital accumulation introduces several layers of difficulty to characterize $D(b_t, k_t)$. The first and

[4] If we were to assume a function that depended on output ($\kappa = \max(0, d_0 + d_1 y_t)$), solving for labor under default would involve the solution of a highly non-linear problem. This is because the sovereign would have to internalize the impact that labor would have on the default cost.

obvious obstacle is that the value functions V^{nd} and V^d may not be monotonic in capital due to the nonlinearity. Second, even with monotonicity for each value function, a change in the capital stock can have uneven effects on the two value functions and cause the spread $V^{nd} - V^d$ to vary nonlinearly. Hence, the inequality in (4) can easily reverse complicating the characterization of the default set.

Foreign lenders are risk neutral. A zero profit condition implies that investors charge q_t for debt:

$$q_t\left(b_{t+1}, k_{t+1}, A_t\right) = \mathbb{E}_t\left(1 - d_{t+1}\left(b_{t+1}, k_{t+1}, m_{t+1}, A_{t+1}\right)\right) \frac{\lambda + (1-\lambda)\left[z + q_{t+1}\left(b_{t+2}, k_{t+2}, A_{t+1}\right)\right]}{1 + r^*},$$

where r^* is the risk-free international rate on a one-period bond.

The presence of capital accumulation and long-term debt makes the computation of our model quite challenging. As a consequence, we follow Arellano (2008), CE and MY and abstract from the important issue of debt renegotiation. Bai and Zhang (2012) and Yue (2011) provide a vivid discussion of default and debt renegotiation.

3.1 Understanding Capital and Default

As argued in the previous paragraphs, capital accumulation introduces a nontrivial element to the default decision. To shed some light into the problem, let us consider the case of exogenously fixed capital with zero depreciation. Further, suppose that the cost of default κ is constant, the *iid* shock m is equal to zero, and that debt matures in one period $\lambda = 1$. Under these assumptions, the sovereign's value of repaying their debt is

$$V^{nd}\left(b_t, k, A_t\right) = \max_{c, b', i, k', l} \left\{u\left(c_t, l_t\right) + \beta \mathbb{E}_t V\left(b_{t+1}, k, A_{t+1}\right)\right\}$$

subject to

$$c_t + q_t\left(b_{t+1}, k, A_t\right) b_{t+1} \leq A_t k^\alpha l_t^{1-\alpha} + b_t.$$

The value of defaulting is

$$V^d\left(k, A_t\right) = \max_{c, i, k', l} \left\{u\left(c_t, l_t\right) + \beta(1-\phi)\mathbb{E}_t V^d\left(k, A_{t+1}\right) + \beta\phi\mathbb{E}_t V\left(0, k, A_{t+1}\right)\right\},$$

subject to

$$c_t \leq (1-\kappa) A_t k^\alpha l_t^{1-\alpha}.$$

One can easily show that conditional on a level of capital, the model preserves the Eaton-Gersovitz property. That is if the economy defaults for b, it will default for $b' < b \leq 0$, i.e.

$D(b,k) \subseteq D(b',k)$. This is because the value function of defaulting is independent of debt. A closer look at the value functions reveals that capital acts like an additional productivity component. This observation highlights the complication behind characterizing the default set in the baseline model. Arellano (2008) provides an analytical description of the repayment decision in a model with *iid* endowment shocks. This assumption makes the analysis possible because expectations of tomorrow's value functions independent of today's endowment (productivity) innovations. Allowing for persistent shocks, which has the same flavor as allowing for investment, breaks this independence rendering an analytical exercise unfeasible. However, if we are willing to impose more structure, it is possible to sharpen the impact that capital has on default.

Proposition 1 *Suppose that capital and productivity are iid and drawn from distributions with positive support. Further, there is no output loss but the country is permanently excluded from international markets if defaulted. If $0 < \alpha < 1$ and $\omega > 1$, then lower capital increases the temptation to default: For all $k < \hat{k}$, if $A \in D(b, \hat{k})$ then $A \in D(b, k)$.*

Proof. See Appendix. ■

Equation (5) shows that labor supply is increasing in capital. Conditional on a bad realization of technology, households work harder in economies with more capital. This means that more resources are available to repay debt in economies with capital, which reduces the temptation to default.

Corollary 2 *Continue to assume the same assumptions as in proposition 1. Suppose that the sovereign defaults when productivity and capital are A and k. Then the country defaults for any pair $(\hat{A}, \hat{k}) \in R_+ \times R_+$ such that $\hat{A}\hat{k}^\alpha < Ak^\alpha$. In other words, the likelihood of default increases when the economy's endowment (Ak^α) is lower.*

We fall short of making a claim on the monotonicity of the default set with respect to capital in the general case when capital is choice variable. In fact, numerically we have found that it is usually, but not always, the case that the default set is decreasing in capital. On one hand, additional output implies more resources to repay today's debt, b_t. This alone should reduce the likelihood of default. Yet the extra capital increases output in default too and hence the resources available for consumption. Since capital cannot be seized, the sovereign can accumulate extra capital if it defaults. The additional capital tomorrow alleviates the effects from being excluded from financial markets. This increases the chances of default. These two forces feedback to the model through their effect on the price of debt today and their impact on the value functions tomorrow. For the default set to be monotone in capital, we need one of these channels to dominate. That is the value of default has to rise by more than the value good standing following the increase in capital or vice versa. In sum, if we want to analyze the general case of capital accumulation, a numerical exercise is the only option. Next we turn to the calibration and dynamics implied by our benchmark specification.

4 Calibration

In our calibration exercise, a period is a quarter. We divide the parameter space into two groups. The first group of parameters are set to values that have been previously used in the literature (upper panel in Table 1). Some of these parameters are worth mentioning. The structure of long-term debt is taken from CE with a coupon payment of 3% ($z = .03$) and a probability of maturity of 5% ($\lambda = .05$) each quarter. The parameter controlling risk aversion, σ, is set to a standard value of 2. The support of the continuous shock m_t is the same as in CE. Following AG, the probability of redemption is 10%, which corresponds to an average stay in autarky of 2.5 years. Conditional on the other parameters, we choose mean productivity μ_A and the labor disutility parameter η so that, in the non-stochastic steady state without foreign lending, output and labor are both equal to one.[5]

The second set of parameters in Table 1 were chosen to match the empirical moments reported in Table 2. The discount factor is close to the value calibrated in CE. The persistence and volatility of productivity (ρ_A, σ_A) are in the range used by AG, $(.90, 3.4\%)$, and MY, $(.95, 1.7\%)$. Our calibration exercise requires a highly inelastic labor supply. Because ours is the first study bringing the capital, labor, and default decisions together, it is difficult to comment about the plausibility of this elasticity relative to the one proposed in, for example, MY. The depreciation rate is in line with the value in Neumeyer and Perri (2005). The model requires a seemingly large adjustment cost to deliver the dynamics of investment observed in our sample of sovereign countries. Finally, the parameters of the output loss function imply an asymmetric punishment of defaulting with a large loss if productivity is high but low or zero loss if productivity is low.[6] Our values are in absolute value slightly larger than those in CE.

[5] This aids in calibrating the model as the grids do not need to be adjusted when different parameters (such as β) change the average capital stock.

[6] Specifically, κ is 0 whenever $\log A_t$ is more than 1.9 unconditional standard deviations below its mean. The case of $\kappa \geq 1$ never occurs in our computation as $\log A_t$ would have to be more than 10 unconditional standard deviations above its mean.

Table 1: Parameter Values

Calibrated Independently

Description	Parameter	Value
Constant relative risk aversion parameter	σ	2
Capital share of income	α	.36
World risk-free interest rate	r^*	.01
Coupon payment	z	.03
Probability of maturity	λ	.05
Probability of regaining credit market access	ϕ	.10
Standard deviation of the iid shock	σ_m	.003
Support of the iid shock	$\underline{m}, \bar{m}$	$\pm$.006
Normalization constant for productivity	μ_A	.6905
Normalization constant for labor supply	η	.64

Calibrated Jointly

Description	Parameter	Value
Discount factor	β	.9304
Depreciation rate	δ	.0539
Default fixed cost parameter	d_0	-.4692
Default proportional cost parameter	d_1	.7995
TFP shock persistence	ρ_A	.943
Standard deviation of the TFP shock	σ_A	.0285
Inverse of Frisch Elasticity $(1/(\omega-1)=0.04)$	ω	26.995
Adjustment cost	Θ	5.707

5 Results

Table 2 presents the empirical moments from Argentina 1993.Q1 - 2011.Q3. Some of the empirical findings such as the excess volatility of consumption or the countercyclicality of the trade balance have been extensively discussed elsewhere. Yet the excess volatility of investment and its strong and positive correlation with output are two elements omitted in the default literature.

Table 2: Moments in Data and Model

	σ_y	$\frac{\sigma_c}{\sigma_y}$	$\frac{\sigma_i}{\sigma_y}$	$\sigma_{nx/y}$	$\rho_{y,c}$	$\rho_{y,i}$	$\rho_{y,nx/y}$	$\mathbb{E}b/y$	$\mathbb{E}I/y$	$\mathbb{E}d$
Empirical Moments	4.82	1.23	2.66	2.34	0.93	0.85	-0.68	-0.35†	0.19	3
Benchmark	4.58	1.09	2.07	2.36	0.90	0.82	-0.19	-0.39	0.15	2.96
Default Episode	4.27	1.14	2.12	2.27	0.90	0.81	-0.27	-0.42	0.15	2.45
Alternative Scenarios										
Fixed Capital*	4.39	1.41	-	2.03	0.93	-	-0.24	-0.39	-	2.80
Flexible Capital*	4.99	0.77	3.76	2.94	0.91	0.81	-0.11	-0.40	0.15	3.11
Low Depreciation**	4.12	1.08	2.40	1.63	0.94	0.68	-0.22	-0.25	0.12	3.35
High Depreciation**	5.97	1.24	1.65	4.27	0.84	0.89	-0.19	-0.82	0.22	2.19

†Value taken from Mendoza and Yue (2011).

*Fixed Capital has investment as fixed number.

**Flexible Capital has Θ equal to 1/5 the benchmark value.

***Low Depreciation has δ 1/3 lower than its benchmark value. High Depreciation has δ 1/2 higher.

Overall our calibrated model does a good job matching Argentina's business cycles (second row in Table 2). We capture simultaneously the large fluctuations of output, the volatile consumption path, and the volatility of net exports. Although the model correctly predicts the countercyclicality of the trade account, we fall short of delivering its magnitude (this is also the case in the work of Arellano, 2008; CE; and Mendoza and Yue, 2011). Interestingly, our model gives correct predictions for the average debt relative to output found in the data (39% versus 35%, respectively). The default probability is 2.96 in our model, which matches the frequency of default in Argentinean data reported in the literature (Arellano; 2008).

More relevant to our purpose, the model gets close to replicating the excess volatility of investment and its correlation with output. Consistent with the data, the model predict a ratio of investment to output of 15%. Matching the dynamics of investment in the data is crucial because, in the spirit of incomplete market models (Aiyagari; 1994), the sovereign country can use capital to hedge against bad outcomes. That is, having more capital ameliorate the cost of defaulting because capital can be freely transformed into consumption goods. At the same time, too much capital increases the sovereign's exposure to negative productivity shocks (recall that the return on capital is decreasing in capital). We find that the first channel dominates and hence the small open economy overaccumulates capital. Our model benchmark model predicts an average capital-to-output ratio of 2.87 (this is the mean of the ergodic distribution). In contrast, this ratio is 2.79 in the steady state of our model that excludes default and foreign borrowing (or 3% lower than in the baseline model with default).

For completeness, the third row in Table 2 shows the model's moments prior to a default episode. To this end, we do a long simulation of the model, locate 100 default events, and then compute the moments using the 74 observations leading up to (but not including) the default period (this is the same approach in Arellano; 2008). The table reports the average across the 100 events. Conditioning on this episode does not change the predictions of our model.

Based on these results from our model, we move on to study in more detail how capital accumulation interacts with the repayment decision and business cycles around a default episode. Figure 2 presents the dynamics of a typical default episode predicted by our benchmark model (solid lines). For clarity, we also plot one-standard deviation bands (dotted lines). Our benchmark model is capable of capturing the slow transition from good standing to default. Consistent with the data, our model predicts that the economy peaks exactly 4 quarters before stopping repayments. Further, the transition from good to bad standing is smooth and it is the result of worsening productivity prior to default. The investment series presents the same smooth pattern: a slow decline followed by a sluggish recovery. In sum, the typical default episode in our benchmark economy closely tracks the one present in the data.

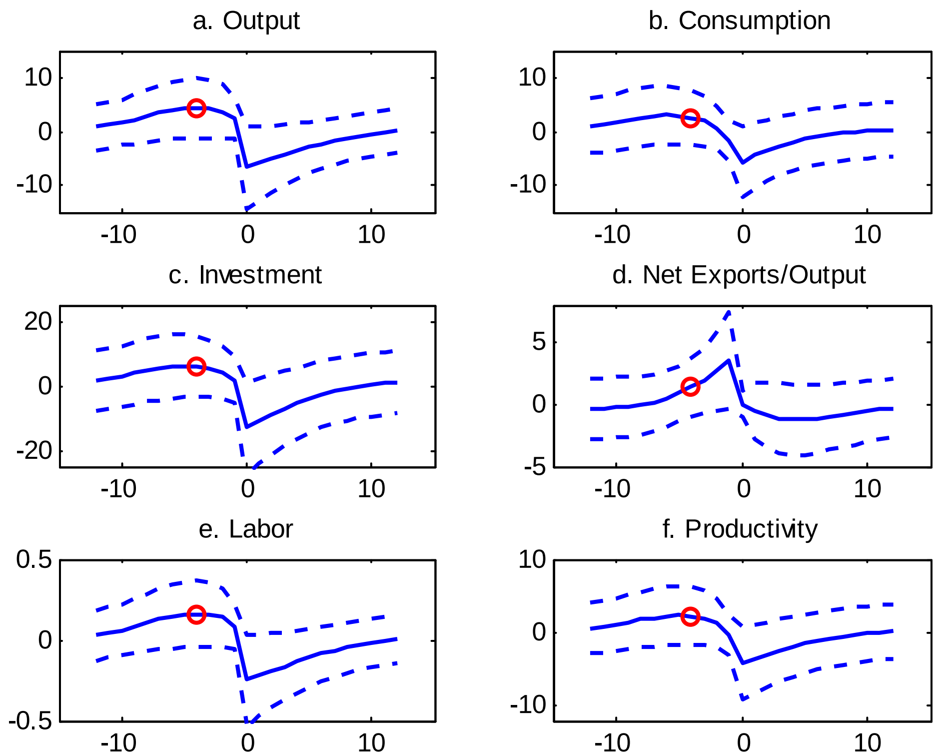

Figure 2. Default Episode in Model.

Interestingly, our model predicts that the economy is in trade balance (i.e. net exports are zero) when default is declared and then experiences a mild trade deficit. To understand this subsequent worsening of the trade account, first note that net exports in the economy can be written as $NX_t = q_t(b_{t+1}, k_{t+1}, A_t)(b_{t+1} - (1-\lambda)b_t) - (\lambda + (1-\lambda)z)b_t$. When a sovereign defaults, b_t is set to 0 and, for as long as the sovereign remains in autarky, both b_{t+1} and NX_t are 0. When the economy is readmitted to financial markets, b_{t+1} is chosen from $B \subset R_-$ and so NX_t must be less than 0.[7]

The model also predicts an improvement of the trade balance prior to declaring default. To understand this, first note that higher productivity and capital (due to additional investment) occur during the pre-crisis periods. Second, note that, as shown in Figure 4, the bond price schedule is increasing both in productivity and capital. Indeed, new debt is barely discounted for small enough levels of existing debt. Consequently, in the pre-crisis periods the sovereign can meet its financial obligations with only a small issuance of debt, resulting in a trade surplus. At

[7] This is a feature common to most sovereign default models. An exception is Mendoza and Yue (2011) where the trade balance improves following default. In their model, upon reentry to international markets the sovereign has access to debt as well as intermediate imported goods. This means that if imports of these goods drop sufficiently (as they do in their paper), the economy can experience a trade surplus following default.

first look, this prediction seems at odds with the average default episode in the data. However, it is consistent with the default episodes of Indonesia (1998.Q3), Peru (1983.Q2), and South Africa (1998.Q1). Furthermore, Arellano (2008) and Mendoza and Yue (2011) report similar dynamics prior to default.

We now study the implications of capital accumulation on the price of debt. Figure 3 shows the bond price schedule along the capital and bond dimensions conditional on a typical productivity shock. For a given capital value, we obtain the standard result that lower levels of debt are associated with higher bond prices (in our figures, debt is expressed as a fraction of output in steady state, which we normalize to 1). More important, the figure reveals one important finding of this paper: Additional capital helps to sustain higher levels of debt. To further appreciate this point, Figure 4 plots the price schedule for different levels of capital (the arrow indicates the direction in which capital increases). At the lowest capital level in our grid, debt (as a fraction of steady state output) starts being valued at around $b = -0.64$. In contrast, when capital is at its highest level (farthest northwest line), debt has a positive price at a debt value that is almost twice (in absolute terms) as large as its value when capital is low. This observation may help in part explain why more developed—that is, more heavily capitalized—countries are able to support higher debt-output ratios than less developed countries.

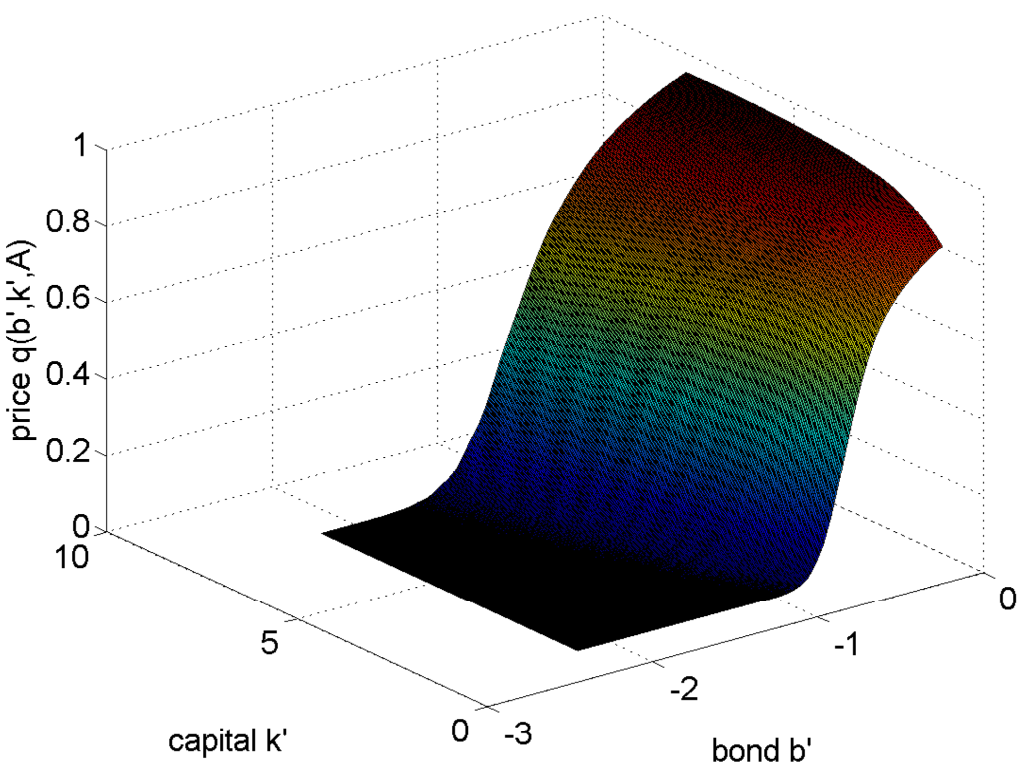

Figure 3. Bond Price with Typical Productivity.

Figure 4 also indicates that the stock of capital today not only helps to sustain higher debt but also raises the price of debt. Even when the sovereign issues a nil amount of debt, the price of debt increases with capital holdings. The effect is even more pronounced for higher values of debt. Let us take, for example, $b = -0.5$. The price of debt at the highest level of capital is about twice as large as than at the lowest level of capital.

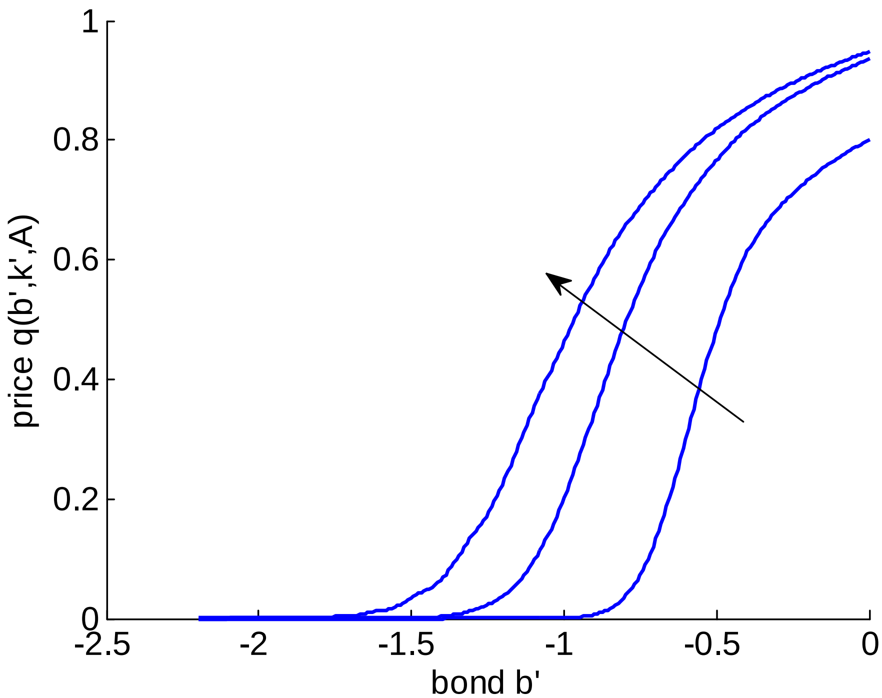

Figure 4: Bond Price at Different Capital Levels.

When we move to Figure 5, we observe that higher productivity amplifies the effect of capital holdings on debt sustainability and bond pricing (for this figure, productivity is set to 68% of the highest value allowed in our grid). As previously mentioned, when existing debt is sufficiently low, good productivity with high capital implies that new debt is bought at roughly face value, i.e. the discount is mild or inexistent. Even at very low capital stocks, the sovereign is capable of issuing debt at finite interest rates.[8]

[8] The curious reader may notice that the price of debt is above one for low values of debt and high levels of capital. This is a feature of the long-term debt contract which has a risk-free price $\bar{q}$ solving $\bar{q} = (\lambda + (1-\lambda)(z+\bar{q}))/(1+r^*))$. For our calibration $\bar{q}$ is 1.308.

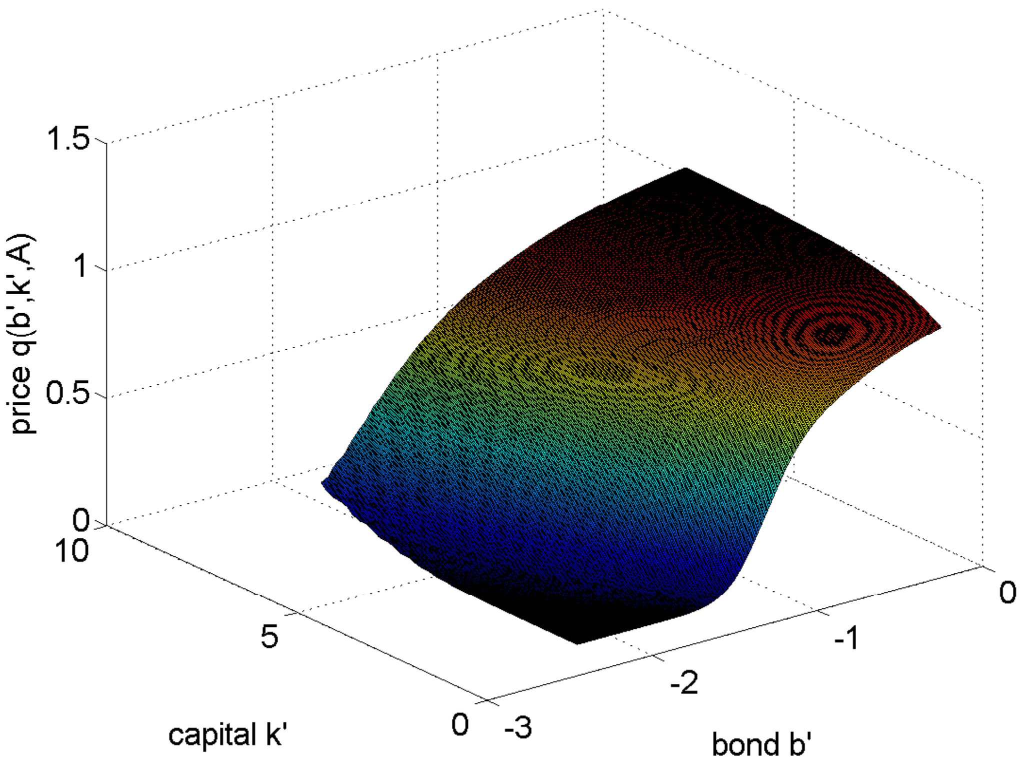

Figure 5: Bond Price with High Productivity

In contrast, Figure 6 shows that capital helps sustain debt even when productivity is low (32% of the highest value in our grid). The effect, however, is marginal compared to that when productivity is high. The reason is that low productivity suddenly makes capital a risky asset. This is because having too much capital amplifies the negative impact of low productivity. That is, for a given decline in productivity, the drop in output is increasing in capital. These factors amplify the likelihood of default (relative to case of high productivity). Hence, debt is sustainable at very low levels.

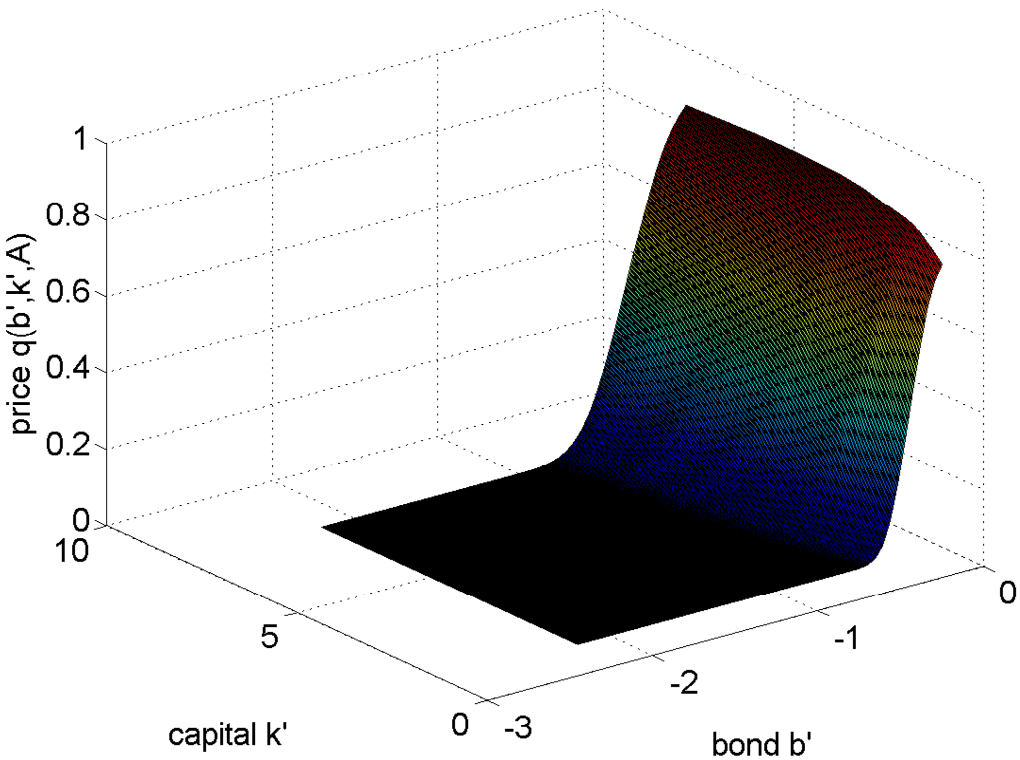

Figure 6: Bond Price with Low Productivity.

Figure 7 further clarifies the impact that the capital stock has on default. It plots the probability of default conditional on a typical productivity shock for different stocks of capital (as before, the arrow indicates the direction of increasing capital). More capital reduces the probability of default for a fixed level of debt. For example, the sovereign defaults with 50% probability at the lowest capital level (farthest east line) when debt is -0.44. This debt level rises in absolute value to -0.60 and -0.70 when the capital stock is at its medium and highest levels, respectively. From a different perspective, Figure 7 shows that the lesson of Proposition 1 seems to apply to our benchmark economy: for a given level of debt, a decline in capital increases the temptation to default.

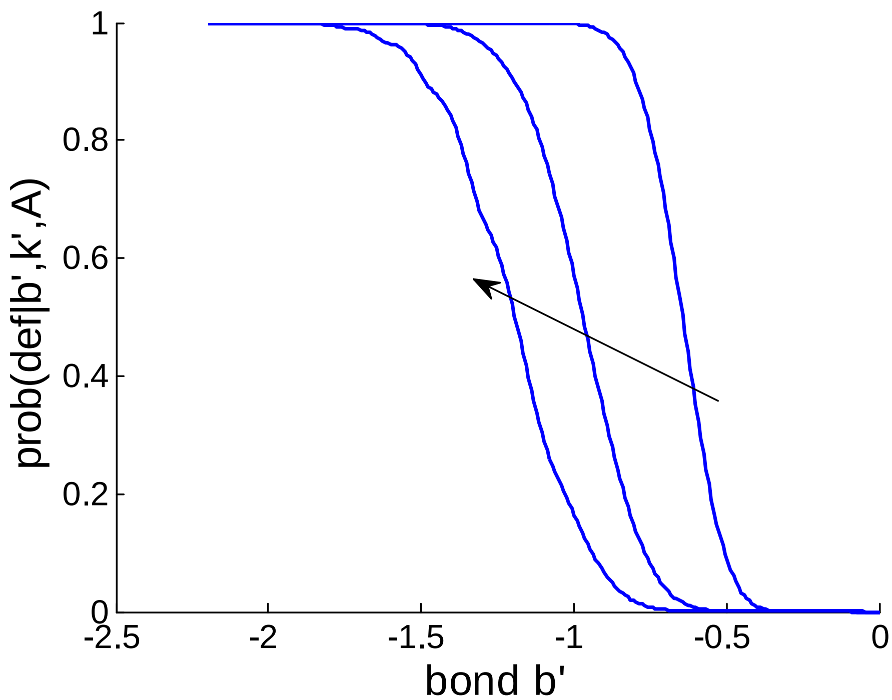

Figure 7: Probability of Default at Different Capital Levels.

6 Alternative Scenarios

This section explores results from alternative economies.

6.1 Cost of Adjusting Capital

Figure 8 displays default episodes for different specifications of the capital adjustment cost. The solid line corresponds to our baseline calibration; the squares in turn represent the model with fixed capital ($\Theta \rightarrow \infty$); and the red dots are the model with flexible capital accumulation where the adjustment cost Θ is set to $1/5$ its benchmark value. A low adjustment cost makes capital flexible, which results in more volatile investment. Not surprisingly, investment collapses upon the sovereign defaulting on its obligations. Moreover, the sovereign exploits this flexibility to reduce the fluctuations in consumption. In the default period ($t = 0$), consumption is around 3 percentage points higher with low adjustment cost than in the baseline scenario. Similarly, the contraction at $t = 0$ is smaller for output and labor although the difference is less noticeable. In the periods before stopping repayment, the expansion is stronger when the adjustment cost is low. For completeness, the second panel in Table 2 shows the moments from our model with flexible capital. Consistent with the dynamics in Figure 8, we observe that consumption is substantially smoother than in the benchmark case. This result illustrates how the planner uses capital as a hedging instrument to dampen the volatility of consumption. There is an increase in the debt-to-output ratio, which squares well with our previous claim that more capital (in this case due to flexible capital accumulation) helps to sustain more debt.

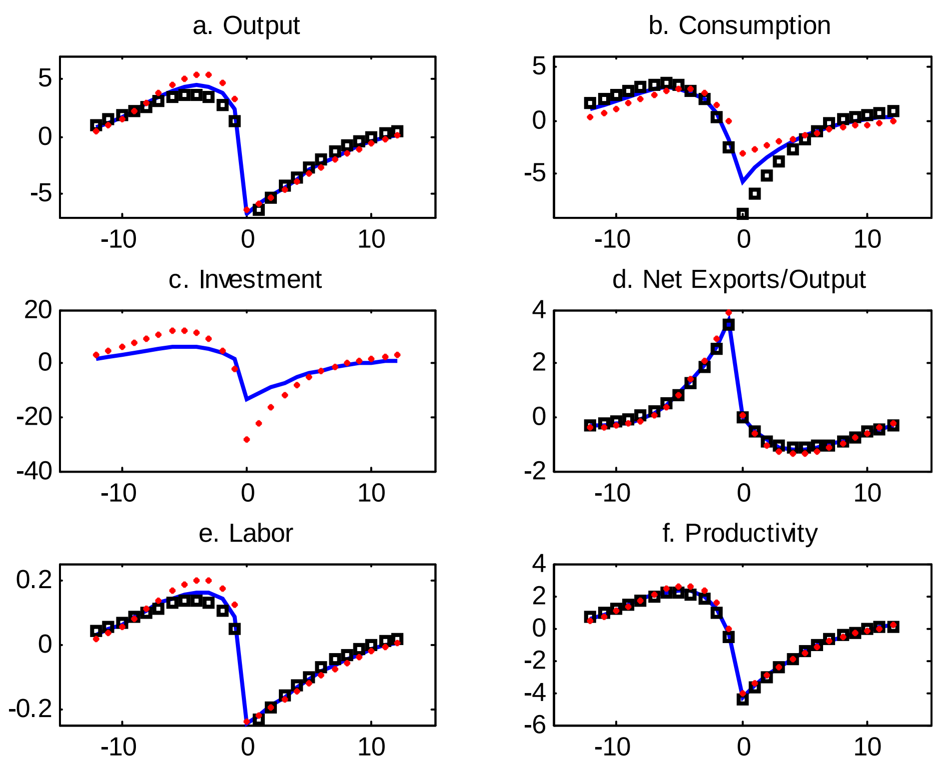

Figure 8: Default Episode under Alternative Adjustment Costs.

In contrast, the recession following default is sharper when capital is inflexible. The sovereign has less degrees of freedom to face adverse outcomes. Investment is now fixed and hence can no longer smooth out the impact that default has on consumption. Consequently, consumption has to absorb the bulk of the recession when the sovereign defaults. At $t = 0$, consumption under fixed capital is 3 percentage points lower than in the baseline scenario. Table 2 shows that the lack of flexibility in investment makes consumption more volatile not only during default but everywhere.

An interesting feature in Figure 8 is that capital accumulation by itself does not guarantee the smooth dynamics of output during default. Note how the transitions look qualitatively similar regardless of the degree of adjustment costs. This result suggests that having access to long-term debt is what smooths out the default episode. Indeed, in simulations not reported here we find that having a short-term (rather than long-term) debt delivers much less smooth paths. Access to long-term debt implies that the emerging economy only needs to repay a fraction $\lambda < 1$ of outstanding obligations each period. Hence, if the country faces a sequence of adverse shocks, it can initially afford repayment because the debt burden is smaller than if debt were short-term.

6.2 Depreciation

To understand the role of depreciation, let us recall that in the standard RBC model investment (and hence capital) in the ergodic mean is increasing in the depreciation rate. This means that if depreciation is low to begin with, then so is the stock of capital in the stochastic steady state. In other words, there is no need to keep a large capital stock because a larger fraction of it survives to next period. Based on Proposition 1, we then expect that the default rate will be higher than in the benchmark model. This is precisely what we observe in Table 2 (in the Low Depreciation row). Figure 9 (red circled line; low δ) shows that with more capital being preserved from period to period, there is less need for sudden movements in investment in the periods priors to default. Relative to the benchmark model, consumption remains unchanged and there is a more modest contraction in output.

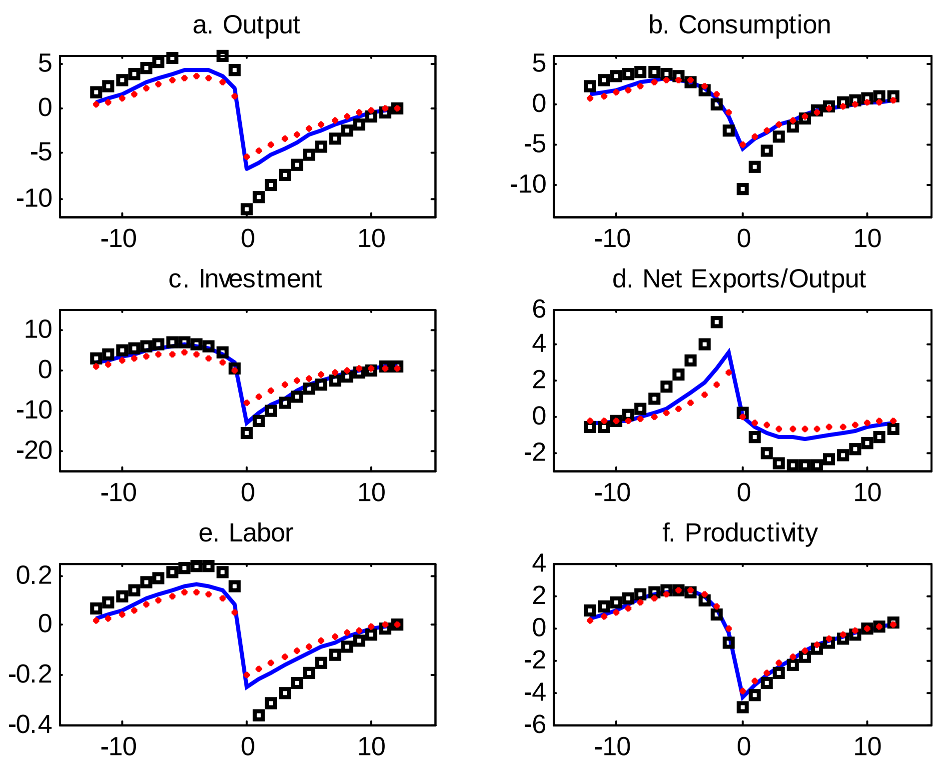

Figure 9: Default Episode under Different Depreciation Rates.

In contrast, high depreciation leads to a larger capital stock in the mean of the ergodic distribution. More capital is needed to make up for the loss due to depreciation. Consequently, the ratio of investment to GDP in the model is 7 percentage points higher than in the benchmark model. The additional capital in turn leads to a decline in the likelihood of default and a higher debt-to-GDP ratio (High Depreciation row in Table 2). The large stock of capital makes it a risky hedge during bad times because more resources have to be devoted to adjust it to the desired level. We then expect that the dynamics of investment be more volatile with high depreciation rates (black squared line in Figure 9). Further, the sluggish capital accumulation requires more labor to keep up with production. Ultimately, these factors make output more volatile during the default episode.

7 Conclusion

We study the joint determination of capital and default in emerging economies. We find that access to capital accumulation significantly changes the dynamics of debt.

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8 Appendix

8.1 Data Description

The data are collected from the International Financial Statistics and OECD's statistical database. Figure 1 is generated using the following countries: Argentina (1993.Q1-2011.Q3; 2002Q1), Ecuador (1991.Q1-2002.Q2; 1999Q3), Indonesia (1997.Q1-2011.Q3; 1998.Q3), Mexico (1981.Q1-2011.Q3; 1982.Q4), Peru (1979.Q1-2011.Q3; 1983.Q2), Philippines (1981.Q1-2011.Q2; 1983.Q4), Rusia (1995.Q1-2011.Q3; 1998.Q4), South Africa (1960.Q1-2011.Q3; 1985.Q4 and 1993.Q1), and Thailand (1993.Q1-2011.Q2; 1998.Q1). The first numbers for each country correspond to the sample length while the second observation indicates the year and quarter of default.

The variables were seasonally adjusted. Nominal variables were deflated by the GDP deflator. All variables except net exports were expressed in logs. Finally, the HP-filter with smoothing parameter 1600 was used to detrend all observations.

8.2 Proof Proposition 1

Proposition 1. *Suppose that capital and productivity are iid and drawn from distributions with positive support. Further, there is no output loss but the country is permanently excluded from international markets if defaulted. If $0 < \alpha < 1$ and $\omega > 1$, then lower capital increases the temptation to default: For all $k < \hat{k}$, if $A \in D(b, \hat{k})$ then $A \in D(b, k)$.*

Proof. The strategy consists on demonstrating that utility is concave and increasing in the stock of capital and then invoking some results in Arellano (2008). In the model, labor is a static choice regardless of the default decision. The optimal labor allocation is

$$\widetilde{l}_t = \left[\frac{1-\alpha}{\eta} A_t k_t^{\alpha}\right]^{\frac{1}{\alpha+\omega-1}}. \tag{5}$$

Note that labor is increasing in both productivity and capital because $\alpha + \omega > 1$ and $0 < \alpha < 1$. Then the indirect utility function is

$$\widetilde{u}(k, A) = \frac{\left[A_t k_t^{\alpha} \widetilde{l}_t^{1-\alpha} - \eta \frac{\widetilde{l}_t^{\omega}}{\omega}\right]^{1-\sigma}}{1-\sigma}.$$

For simplicity, the terms involving borrowing $(q_t b_{t+1} - b_t)$ have been omitted since they do not depend on capital and productivity (recall they are assumed to be iid). As with the labor decision, the indirect utility is increasing in capital:

$$\frac{\partial \widetilde{u}}{\partial k} = \left[A_t k_t^{\alpha} \widetilde{l}_t^{1-\alpha} - \eta \frac{\widetilde{l}_t^{\omega}}{\omega} \right]^{-\sigma} \alpha A_t \left(\frac{\widetilde{l}_t}{k_t} \right)^{1-\alpha} > 0,$$

Regarding concavity of the indirect utility with respect to capital we have:

$$\frac{\partial^2 \widetilde{u}}{\partial k^2} = \frac{\alpha A (1-\alpha)(1-\omega)}{\alpha + \omega - 1} \widetilde{u}_c k_t^{-(2-\alpha)} \widetilde{l}_t^{1-\alpha} + \left[\alpha A \widetilde{l}_t^{1-\alpha} k_t^{-(1-\alpha)} \right]^2 \widetilde{u}_{cc},$$

$$\widetilde{u}_c = \left[A_t k_t^{\alpha} \widetilde{l}_t^{1-\alpha} - \eta \frac{\widetilde{l}_t^{\omega}}{\omega} \right]^{-\sigma}, \ \widetilde{u}_{cc} = -\sigma \left[A_t k_t^{\alpha} \widetilde{l}_t^{1-\alpha} - \eta \frac{\widetilde{l}_t^{\omega}}{\omega} \right]^{-\sigma-1}.$$

The second derivative $\frac{\partial^2 \widetilde{u}}{\partial k^2}$ is negative as long as $\omega > 1$ and $0 < \alpha < 1$, which implies that the indirect utility is concave in capital (in our simulations below, these two conditions are always verified). Since capital is identically and independently distributed, the proof follows from the observation that under the stated assumptions the model reduces to the case of *iid* endowment shocks discussed in Arellano (2008). For us, the economy's endowment is now defined as Ak^{α}. ■

8.3 Computational Algorithm

As emphasized in CE, long-term debt models suffer from convergence problems which can be mitigated by including a continuous iid shock in the computation. In fact, the convergence problems are even worse when capital is introduced. Relying on monotonicity of the policy function, CE construct a computational algorithm to explicitly handle a continuous shock. Unfortunately, with two assets, bonds and capital, we do not have a proof of monotonicity. However, we now present a computational algorithm that does not rely on monotonicity and can be trivially extended to a general choice set. As the only non-trivial part of the computation is incorporating a continuous iid shock, we focus on this part of the computation.[9]

For compactness, we switch to recursive notation with a " $'$ " denoting next period's value and use a instead of A. The algorithm's objective is, for a given (b, k, a), to find policies $c^{nd}(b, k, m, a)$, $b'^{nd}(b, k, m, a)$, $k'^{nd}(b, k, m, a)$, and $V^{nd}(b, k, m, a)$ for all $m \in [\underline{m}, \bar{m}]$ such that the budget set is nonempty. To this end, we fix (b, k, a), suppress dependence on it, and begin with the following definitions:

1. Define $X = \{(b', k') | (b', k') \in B \times K\}$ with typical element $x = (b', k')$. X is the choice space.

[9] Hatchondo, Martinez and Sapriza (2010) discuss alternative methods and their accuracy to solve models of default like those of Arellano (2008) and Aguiar and Gopinath (2006).

2. Define $c : X \to R$ by

$$c(x) = -q(b', k', a)(b' - (1-\lambda)b) + (\lambda + z(1-\lambda))b - k' + (1-\delta)k \\ - \frac{\Theta}{2}(k'-k)^2 + ak^{\alpha}l(k,a)^{1-\alpha} - \eta \frac{l(k,a)^{\omega}}{\omega}.$$

$c(x)$ is the consumption—net of labor disutility and m—arising from a choice of x. Note that we have written $l(k, a)$ using the convenient Greenwood, Hercowitz, and Huffman (1988) property.

3. Define $W : X \to R$ by

$$W(x) = \beta \mathbb{E} V(b', k', m', a').$$

W is the continuation utility associated with the choice of x.

4. Define $\tilde{X}(m) \subseteq X$ by

$$\tilde{X}(m) = \{x \in X | c(x) + m \geq 0\}.$$

$\tilde{X}(m)$ contains all feasible choices of X given m. Note that $\tilde{X}(m_1) \subseteq \tilde{X}(m_2)$ whenever $m_1 < m_2$.

5. Define $V(x, m)$ for all m and $x \in \tilde{X}(m)$ by

$$V(x, m) = u(c(x) + m) + W(x).$$

Then V represents the indirect utility from choice x given the current state and some shock m.

By definition, we have

$$x^{nd}(b, k, m, a) \in \arg\max_{x \in \tilde{X}(m)} V(x, m)$$
$$c^{nd}(b, k, m, a) = c(x^{nd}(b, k, m, a)) + m + \eta \frac{l(k,a)^{\omega}}{\omega}, \text{ and}$$
$$V^{nd}(b, k, m, a) = V(x^{nd}(b, k, m, a), m),$$

which are well-defined if and only if $\tilde{X}(m)$ is nonempty.

We now simplify the discussion of the algorithm by providing some theoretical results:

Lemma 3 *$V(x_1, m) = V(x_2, m)$ for all feasible m if and only if $c(x_1) = c(x_2)$ and $W(x_1) = W(x_2)$.*

Proof. Since V is differentiable in m, we have

$$V(x_1, m) = V(x_2, m) \Leftrightarrow$$
$$\exists \hat{m} \ s.t.\ V(x_1, \hat{m}) = V(x_2, \hat{m}) \text{ and } \frac{\partial V(x_1, m)}{\partial m} = \frac{\partial V(x_2, m)}{\partial m} \forall m$$

Because $\partial V(x, m)/\partial m = u'(c(x) + m)$, the second part is true if and only if $c(x_1) = c(x_2)$. The first part is true for $c(x_1) = c(x_2)$ if and only if $W(x_1) = W(x_2)$. ■

Lemma 4 *Given two choices x_1 and x_2, $\{m | V(x_1, m) = V(x_2, m)$ and $x_1, x_2 \in \tilde{X}(m)\}$ is either empty, a singleton, or equal to $[-c(x_i), \infty)$ for both $i = 1$ and 2. If $c(x_1) \neq c(x_2)$, then it is either empty or a singleton and so $V(x_1, \cdot)$ and $V(x_2, \cdot)$ cross at most once.*

Proof. For all m such that $x_1, x_2 \in \tilde{X}(m)$, define $H(m) = V(x_2, m) - V(x_1, m)$. Then $H'(m) = u'(c(x_2)+m) - u'(c(x_1)+m)$. It is then obvious that $H'(m)$ never changes sign, and so $H(m)$ has at most one zero (or is always zero if $c(x_1) = c(x_2)$ by Lemma 3). ■

Proposition 5 *Let (x^*, m^*) be such that $m^* \in [\underline{m}, \bar{m}]$, $V(x^*, m^*) \geq V(x, m^*)$ for all $x \in \tilde{X}(m^*)$, and, if $V(x^*, m^*) = V(x, m^*)$, then $c(x^*) \geq c(x)$. Then x^* is optimal for all m in $[m^{**}, m^*]$ where*

$$m^{**} := \begin{cases} \max\{-c(x^*), \underline{m}\} & \textit{if } \{x | c(x) > c(x^*)\} \textit{ is empty} \\ \max\left\{-c(x^*), \underline{m}, \max_{\{x|c(x)>c(x^*)\}} \tilde{m}(x)\right\} & \textit{otherwise} \end{cases}$$

and $\tilde{m}(x)$ is defined implicitly by

$$V(x^*, \tilde{m}(x)) = V(x, \tilde{m}(x)).$$

Further, $\tilde{m}(x)$ is well-defined (unique and exists) for each x having $c(x) > c(x^*)$ and lies in the interval $(-c(x^*), m^*)$.

Proof. Define $\hat{m} := \max\{-c(x^*), \underline{m}\}$. Then x^* is feasible for all $m \geq \hat{m}$.

First we prove that, for $m \in [\hat{m}, m^*]$, any $x \in \tilde{X}(m^*)$ with $c(x) \leq c(x^*)$ delivers weakly lower utility than x^* or is not feasible. Note that where feasible, $\partial V(x, m)/\partial m \geq \partial V(x^*, m)/\partial m$. Consequently, for $m \leq m^*$ where x is feasible, we have $V(x, m) \leq V(x^*, m)$. Moreover, where x is feasible, x^* is feasible.

Now consider an x with $c(x) > c(x^*)$. Then where x^* is feasible, x is feasible. So we have $\partial V(x, m)/\partial m < \partial V(x^*, m)/\partial m$ for all $m \in (\hat{m}, m^*]$. Further, there exists an $\tilde{m} \in (-c(x^*), m^*)$ s.t. $V(x, \tilde{m}) = V(x^*, m)$ because as $m \downarrow -c(x^*)$, $V(x^*, m) \downarrow -\infty$ for the preferences we have chosen while $V(x, m)$ is not arbitrarily negative. Then by virtue of the single crossing property of $V(x, m) - V(x^*, m)$ (see Lemma 4), we have $V(x^*, m) \geq V(x, m)$ for all $m \in [\max\{\tilde{m}, \hat{m}\}, m^*]$.

Let the $\tilde{m}$ for x with $c(x) > c(x^*)$ be denoted $\tilde{m}(x)$. Define, when there is at least one x s.t. $c(x) > c(x^*)$, $m^{**} = \max\{\hat{m}, \max_{\{x|c(x)>c(x^*)\}} \tilde{m}(x)\}$. Otherwise, define $m^{**} = \hat{m}$. Then from the preceding arguments we have $V(x^*, m) \geq V(x, m) \forall m \in [m^{**}, m^*]$. ■

The preceding proposition suggests a natural algorithm for computing the optimal policy. Beginning with the optimal choice at $\bar{m}$, iterate down until one has reached $\underline{m}$ (or exhausted feasible choices). We state it formally now:

Algorithm

<u>Objective</u>

Compute the optimal policy and value function.

<u>Initialization</u>

If $\tilde{X}(\bar{m}) = \emptyset$, then no feasible policies exist for any $m \in [\underline{m}, \bar{m}]$ and we are done, so STOP. Otherwise, define $m_1^* = \bar{m}$. Set $i = 1$ and go to Step 1.

<u>Step 1</u>

For all $x \in \tilde{X}(m_1^*)$, compute $V(x, m_1^*)$. Let $x_1^* \in \arg\max V(x, m_1^*)$ such that if the arg max is not a singleton, then x_1^* has the smallest value of $c(x)$. If even this is not unique, employ a tie breaking rule as Lemma 3 implies the choices offer the same utility everywhere.

One may proceed to Step 2 and the algorithm works. However, it is advantageous computationally to discard any $x \in \tilde{X}(\underline{m})$ that have $V(x, \underline{m}) \leq V(x_1^*, \underline{m})$ as, by Lemma 4, these x cannot be optimal. Because $\bar{m} - \underline{m}$ is small, this typically involves discarding most of X.

Additionally, note that because $x_1^* = x^{nd}(b, k, m_1^*, a)$ is an optimal policy for m_1^*, if there is special knowledge about the problem (such as monotonicity w.r.t. m), then wlog one may discard elements of $\tilde{X}(m)$ that cannot possibly be optimal from the rest of the procedure.

<u>Step 2</u>

Now use Proposition 5 to find the interval $[m_{i+1}^*, m_i^*]$ over which x_i^* is optimal. To do this, compute for each x having $c(x) > c(x_i^*)$ the unique value $\tilde{m}(x)$ such that $V(x_i^*, \tilde{m}(x)) = V(x, \tilde{m}(x))$. Take the largest such $\tilde{m}$ and the corresponding value of x, and let them be denoted $(\hat{m}, \hat{x})$. If x is not unique, choose the one with the largest value $c(x)$ (if even this isn't unique, use a tie-breaking rule). Then define $m_{i+1}^* = \max\{-c(x_i^*), \underline{m}, \hat{m}\}$. If $\hat{m}$ is largest, let $x_{i+1}^* = \hat{x}$. Otherwise, let $x_{i+1}^* = x_i^*$.

If $m_{i+1}^* = \max\{-c(x_i^*), \underline{m}\}$, go to Step 3. Otherwise, increment i and repeat Step 2.

<u>Step 3</u>

We have found a sequence of n pairs $\{(x_i^*, m_i^*)\}_{i=1}^n$ such that x_i^* is optimal in $[m_{i+1}^*, m_i^*]$ with $m_1^* = \bar{m}$.

It is more natural to work with a reordered sequence that has increasing m, so redefine $m_{n-i+1}^* = m_i^*$ and $x_{n-i}^* = x_i^*$ for all $i = 1, \ldots, n$ (note the reordering is slightly different for m and x). Now we have $\{x_i^*\}_{i=0}^{n-1}$ and $\{m_i^*\}_{i=1}^n$ such that x_i^* is optimal in $[m_i^*, m_{i+1}^*]$ for all $i = 1, \ldots, n-1$ with

$m_n^* = \bar{m}$. Discard x_0^*.

The optimal policy is defined everywhere (that a feasible policy exists) by

$$x^{nd}(b,k,m,a) = \mathbf{1}_{m \in [m_{n-1}^*, m_n^*]} x_{n-1}^* + \sum_{i=1}^{n-2} \mathbf{1}_{m \in [m_i^*, m_{i+1}^*)} x_i^*$$

giving b'^{nd} and k'^{nd}.

With the optimal asset policies, the optimal consumption choice $c^{nd}(b,k,m,a)$ and the value function $V^{nd}(b,k,m,a)$ can be calculated in a straightforward fashion. STOP.

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One loose end from the preceding algorithm is how to compute $\tilde{m}(x)$ in Step 2. However, this can be efficiently done by solving the non-linear equation

$$f(m) = 0 \text{ where } f(m) := u(c(x_i^*) + m) + W(x_i^*) - u(c(x) + m) - W(x).$$

Evidently, this is a smooth problem in m and we can use Newton's method to quickly compute the solution. For reasonable starting guesses, this always converged.

The last loose end is how to compute $\int \max\{V^{nd}(b,k,m,a), V^d(k,a)\} dF(m)$ and $q(b',k',A)$ which both involve integration. We use a 10-point Gauss-Legendre quadrature rule, but there are many ways to do this.[10]

[10] In a previous version we assumed the m shock was uniformly distributed and calculated the integral exactly. The new approach is nearly as accurate but works with any m distribution.