

# Optimal Collateralization with Bilateral Default Risk\*

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## Abstract

We consider over-the-counter (OTC) transactions with bilateral default risk, and study the optimal design of the Credit Support Annex (CSA). In a setting where agents have access to a trading technology, default penalties and collateral costs arise endogenously as a result of foregone investment opportunities. We show how the optimal CSA trades off the costs of collateralization against the reduction in exposure to counterparty risk. The results are used to provide insights on the drivers of different collateral rules, including hedging motives, re-hypothecation of collateral, and close-out conventions. We show that standardized collateral rules can have a detrimental impact on risk sharing, which should be taken into account when assessing the merits of standardized vs. customized CSAs in bilateral OTC transactions.

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# 1 Introduction

The importance of counterparty risk for the structuring and pricing of financial transactions has become apparent following the recent financial crisis. The master agreement of a derivative transaction, which traditionally would have been of interest to the legal department rather than the structuring desk of a bank, is now an integral part of the design and pricing of a deal. At a higher level, the mark-to-market gains and losses arising from counterparty risk have a significant impact on the earning results of financial institutions, and are systematically monitored and hedged.<sup>1</sup> Similarly, a variety of discount curves is now used for the valuation of assets and liabilities, to reflect the liquidity and counterparty risk profiles of different overnight indices and tenors.<sup>2</sup>

The new Dodd-Frank regulation in the US and EMIR provisions in Europe all emphasize the role of credit risk mitigation tools such as clearing and collateralization to improve the stability of OTC markets. Policymakers and regulators have initially focused on standardized OTC derivatives, such as the most liquid interest-rate swaps and swaptions, and credit-default swaps (CDS) written on large names, while promoting trading on exchanges or electronic platforms, clearing through central counterparties (CCPs), and reporting of transactions to trade repositories. In 2011, the reform programme was extended to derivatives that are not standardized and for which central clearing will not be feasible.<sup>3</sup> Proposals include the standardization of collateral pro-

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<sup>1</sup>A survey by Ernst & Young (2012) of the largest dealing houses found that all respondents record a Credit Valuation Adjustment (CVA) on their derivatives assets (i.e., an allowance for amounts owed to them that might not be paid due to counterparty default), while 70% of those also record a Debit Valuation Adjustment (DVA) (i.e., an adjustment to the measurement of their derivatives liabilities reflecting their own default risk). See Brigo et al. (2013) and Gregory (2012) for an extensive treatment of CVA/DVA and extensions.

<sup>2</sup>The interbank market now quotes very diverse credit and liquidity premia, even for plain vanilla products; see Brigo et al. (2013).

<sup>3</sup>In 2010 the International Monetary Fund (IMF) reported that one quarter of interest rate swaps, one third of credit default swaps, and two thirds of other OTC derivatives are not standardized enough

cedures via the posting of an initial margin and full collateralization during the life of a trade.<sup>4</sup> The bilateral market is sizeable: according to the Bank of International Settlements (BIS, 2013), after central clearing reforms take effect non-centrally cleared transactions will account for over half of OTC market activity as measured by gross notional amounts.<sup>5</sup> Typical counterparties of bilateral trades are large banks acting as dealers, and corporations, asset managers, institutional investors and hedge funds seeking a considerable degree of product customization to hedge risk<sup>6</sup> or implement trading strategies. Examples of customized instruments include long dated forward and option contract on a variety of asset classes, cross-currency and commodity swaps, as well as CDS on smaller names, and most hybrid derivatives written on multiple underlying assets or asset classes. Despite the importance and size of OTC markets that will not be centrally cleared, little theoretical guidance is available to quantify the costs and benefits associated with different collateral rules, and to understand the design of the Credit Support Annex (CSA), the legal document that supports the collateral of derivatives transactions (see ISDA, 1996, 2010b). This paper aims at shedding some light on these issues.

We consider two credit-risky counterparties that are homogeneous with respect to risk preferences, credit quality, and investment opportunities, but have opposite exposure to a random payoff that is nontradable, and is unspanned by the tradeable assets.

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to be cleared; see IMF (2010, ch. 3).

<sup>4</sup>In a similar vein, the International Swaps and Derivatives Association (ISDA) recently issued a (non-mandatory) standard aimed at streamlining collateralization procedures, limiting for example the optionality in eligible collateral type, thresholds, and currency (see ISDA, 2013).

<sup>5</sup>According to BIS (2013, pp. 29-30), the activity in non-centrally cleared derivatives is estimated to drop by 46% in gross notional amounts after central clearing mandates take effect.

<sup>6</sup>Although relatively common risk exposures (e.g., long dated interest/inflation rates for corporate pension plans, jet fuel for airline companies, etc.) may be partially hedged with a combination of more standardized products, operational costs and hedge accounting (e.g., IFRS, 2014) provide strong incentives for basis risk reduction via customization.

This is a prototypical situation where agents can improve their position by sharing risk. Because of default risk, however, any risk sharing arrangement will introduce counterparty risk. The latter can be mitigated by using a CSA, but the collateralization procedure will typically divert resources away from profitable trading opportunities, and give rise to an opportunity cost of posting collateral. The CSA should therefore trade off the costs of collateral against the reduction in default penalties that collateralization is able to deliver. We develop a model that allows us to formalize some of these issues, and to answer several interesting questions.

First, we provide a rationale for collateral rules commonly observed in bilateral trades. In line with CSAs specifying collateral thresholds and triggers aimed at economizing on collateral (e.g., ISDA, 2010a,b), we find that collateral costs result in ‘less-than-full’ collateralization being mutually optimal for the counterparties. The result applies even in the baseline case where agents face no borrowing costs, and hence there is no direct cost of posting collateral, as interest on cash collateral is rebated. The reason is that agents value collateral differently in different states of the world, where the states reflect the moneyness of the instrument and whether default has occurred or not, and the asymmetry is resolved in favor of less stringent collateral rules. When a counterparty defaults, the surviving party values collateral more in case she is in-the-money, as collateral reduces default losses and can be invested in the financial market. Exclusion of the defaulting party from the financial market, on the other hand, means that she will find collateral less valuable; and if she is out-of-the-money, the larger the collateral posted, the higher her default penalties. We discuss some extensions of the model (including collateral segregation and borrowing costs), which make our results stronger by reducing further the demand for collateral. When collateral amounts can be contingent on the value of the underlying exposure, or the path of the collateral account

itself, we find that the optimal CSA requires collateral to be posted gradually as the position moves more out-of-the-money and/or the balance of the collateral account is lower. This is consistent with CSAs specifying incremental collateral flows triggered by the performance of the underlying exposure or the collateral account itself, as often found in bespoke CSAs for customized transactions.<sup>7</sup>

Second, we use our framework to study the potential impact of new regulation on bilateral OTC trades. Dodd-Frank and EMIR provisions contemplate the introduction of standardized collateral rules in the form of a two-way initial margin and a variation margin. The former is a VaR-based amount posted by both counterparties with a custodian; the latter corresponds to a CSA prescribing full collateralization (e.g., BIS, 2013). Regulators have acknowledged potential impediments to trade arising from standardized collateral rules, due to the scarcity of good quality collateral and the opportunity cost of posting it. Under current EMIR proposals, for example, initial margins are compulsory only above a certain threshold.<sup>8</sup> Regulators are also looking for ways to facilitate the posting of non-cash collateral with CCPs by pension plans and other hedgers having liquidity constraints.<sup>9</sup> The discussion of these issues is typically limited to initial margins, on the grounds that variation margins represent transfers of resources from one party to the other, and hence entail no net liquidity cost (see Heller and Vause, 2012; Sidanius and Zikes, 2012; BIS, 2013). Our analysis demonstrates instead the importance of variation margins in shaping hedging demand, and allows us to quantify the impact on trading volume of standardized collateral rules. We show that the latter have

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<sup>7</sup>See Brigo et al. (2013), for example.

<sup>8</sup>“All covered entities must exchange, on a bilateral basis, initial margin with a threshold of EUR 50m. The threshold is applied at the level of the consolidated group to which the threshold is being extended and is based on all non-centrally cleared derivatives between the two consolidated groups” (BIS, 2013, p. 8).

<sup>9</sup>See European Commission (2013), for example.

detrimental effects on risk sharing that should be taken into account when assessing the merits of standardized vs. bespoke CSAs in non-centrally cleared OTC contracts. For example, it may be desirable for initial margins to co-exist with bilateral CSAs, as the collateralization of non standard instruments can only be ‘full’ to the extent of the replacement cost defined in the CSA: indeed the role of customized CSAs is to allow counterparties to specify models/proxies to determine collateral and close-out amounts, which can better align the views and needs of end users, thus enhancing risk sharing.

Third, we study the hedging demand against a non tradeable source of risk in the presence of bilateral counterparty risk and a CSA. As a simple example, we consider forward contracts and show how suboptimal collateral rules (such as full collateralization) reduce hedging. When both the CSA and the position in the hedging instrument are chosen optimally, agents collateralize less in favor of hedging more. The reason is that a larger position in the hedging instrument allows the defaulting party to increase resources in her in-the-money states, without increasing collateral losses in her out-of-the-money states. At an optimum, these benefits are in expectation just enough to compensate the counterparties for the cost of an imperfect hedge, the imperfection arising from overhedging the risk exposure while underhedging counterparty risk. The results emphasize the importance of understanding the CSA in its entirety, including the close-out convention,<sup>10</sup> and show that the design of the CSA can have important implications for hedging demand and trading volume.

The paper is structured as follows. In the next section, we discuss how the paper contributes to the existing literature. In section 2, we outline a simple continuous-time model with two agents and two sources of risk, one tradeable and one illiquid. We

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<sup>10</sup>The close-out clause in a CSA defines the transfer at default between the surviving and the defaulting counterparty, depending on the moneyness of the counterparties and the reference market or valuation model used to compute the replacement cost of the OTC instrument.

then provide some results for the agents' private valuation of the illiquid position under CARA preferences. In section 3, we allow the agents to hedge the illiquid position with a forward contract, which introduces counterparty risk. We start with the benchmark case of no collateral, followed by a detailed study of the optimal CSA. We then allow the agents to choose jointly the CSA and the position size, demonstrating how collateral arrangements and trading volume are interrelated. We illustrate our results with several numerical examples. In section 4, we provide a comparison between the optimal CSA and standardized collateral rules. We distinguish the role of initial and variation margins, and quantify the detrimental impact on risk sharing of mandated suboptimal collateral rules. Regulatory implications are discussed by making reference to the stylized situation where counterparties are too-big-to-fail. We show that in case of joint default expected bailouts costs may be larger than in the case of optimal CSAs. Section 5 discusses more in detail some aspects related to collateral rehypothecation, close-out conventions, and mark-to-market/model approaches to computing collateral amounts. We discuss some realistic extensions of the setup, which all strengthen the predictions of the baseline model. Finally, section 6 concludes, while an appendix collects proofs and additional technical results.

## 1.1 Related literature

Our contribution is related to at least three strands of literature. First, we contribute to the literature on derivatives pricing with bilateral default risk. Starting from the insights of Duffie and Huang (1996), several studies have more recently looked at the valuation of contracts with bilateral default risk (e.g., Brigo et al., 2014; Crépey, 2015; Hull and White, 2012) and with CSAs (see Biffis et al., 2015; Brigo et al., 2012), but

they either abstract from collateral flows, or take collateral rules as exogenously given. Here, we explicitly consider the CSA, study its optimal design, and shed light on the interplay between collateral flows, close-out clauses, and trading volume. In particular, we obtain normative results that are in line with CSAs commonly used in customized OTC transactions. A related strand of literature explicitly recognizes the importance of collateral costs in more standardized OTC markets (see Collin-Dufresne and Solnik, 2001; Johannes and Sundareshan, 2007, for the case of interest-rate swap markets), and in corporate risk management (e.g., Rampini and Viswanathan, 2010; Rampini et al., 2014). In line with these studies, we show that collateral is an important dimension of OTC transactions. In our model, however, collateral rules are determined endogenously in response to expected default losses and the opportunity cost of collateralization.

Second, we contribute to the discussion on the regulation of OTC derivative markets. The main focus of recent contributions on counterparty risk is the market for standardized OTC instruments, such as credit default swaps on large names, which will be cleared via CCPs under the Dodd-Frank and EMIR reforms. Several interesting studies have addressed the question of whether CCPs may increase counterparty risk due to asymmetric information (Pirrong, 2011; Stephens and Thompson, 2014), may reduce counterparty risk by mutualizing it (Biais et al., 2012), or controlling the risk exposure of protection sellers (Acharya and Bisin, 2014), and of whether multiple CCPs may lead to efficiency gains or losses from netting (Duffie and Zhu, 2011; Cont and Kokholm, 2014). Here, we look at the sizeable market for customized OTC instruments, which will not be centrally cleared, and develop a continuous-time model that allows us to understand the main features of collateral rules observed in bespoke CSAs. Moreover, we contribute to the discussion of the potential impact of standardized collateral rules on the liquidity of the bilateral OTC market. As opposed to studies that quantify aggregate col-



lateral demand by considering exclusively initial margins (e.g., Singh and Aitken, 2009; Heller and Vause, 2012; Sidanius and Zikes, 2012), we explicitly allow for variation margins and for the impact of collateral rules on trading volume. Our results support the view that standardized collateralization can reduce the supply/demand for customized risk management solutions, leaving a variety of end users exposed to the costs of being unhedged (see Singh and Aitken, 2009; Duffie et al., 2010; French et al., 2010).

Finally, our work contributes to the literature on optimal portfolio choice when hedging gives rise to counterparty risk. The problem can be related to the literature on optimal insurance and portfolio choice with background risk (e.g., Doherty and Schlesinger, 1983; Eeckhoudt et al., 1996), and on dynamic hedging in incomplete markets (in particular Duffie and Jackson, 1990; Svensson and Werner, 1993; Teplá, 2000), but we explicitly consider bilateral default risk, as well as endogenous counterparty risk mitigation tools such as the CSA.

## 2 A model

Consider two risk-averse agents, denoted by  $A$  and  $B$ , with a trading horizon  $[0, T]$ , who have opposite exposure to a source of risk  $Z_T$  at the terminal date  $T$ . Assume, for example, that party  $A$  is exposed to the payment of a random amount  $Z_T$  (a liability equal to  $-Z_T$ ), whereas party  $B$  expects a random inflow equal to  $Z_T$ . For simplicity, the dynamics of  $(Z_t)_{t \geq 0}$  is described by a Wiener process

$$dZ_t = \sigma_Z dB_t^Z, \quad Z_0 = 0,$$

where  $B^Z$  is a standard Brownian motion and  $\sigma_Z$  a positive volatility coefficient. Agents have access to a financial market, where available are a money market account yielding the riskless rate  $r > 0$ , and a risky security with price  $S$  evolving according to

$$dS_t = \mu S_t dt + \sigma_S S_t dB_t^S, \quad S_0 = 1,$$

with  $\mu > r$  and  $\sigma_S > 0$ . We assume that the Brownian motions  $B^S, B^Z$  are uncorrelated, meaning that: i) tradable assets offer agents no way to span  $Z_T$ , and ii) the agents' exposure to  $Z_T$  is illiquid until time  $T$ , as there are no intermediate dividends. The assumption of an orthogonal risk exposure is only for tractability.<sup>11</sup> The presence of an illiquid source of risk makes the market incomplete, and we need to take a stance on the agents' preferences to identify a valuation functional consistent with the absence of arbitrage opportunities. We assume agents to both have CARA utility  $u(x) = -\frac{1}{\gamma}e^{-\gamma x}$ , with  $\gamma > 0$ , and to optimize their utility from terminal wealth.

Agents are credit risky. As we focus on marginal, non-standardized OTC transactions, we assume default to be exogenous. We take the agents' default times  $\tau^A, \tau^B$  to coincide with the first jumps of two independent Poisson processes with the same parameter  $\lambda > 0$ . Abstracting away from agent heterogeneity (safe for the opposite exposure to the illiquid source of risk) is essential to provide explicit solutions of the baseline model. Defining the default indicators by  $N_t^i = 1_{\tau^i \leq t}$  (for  $i \in \{A, B\}$ ), we write the dynamics of each agent's wealth,  $W_t^i$ , as

$$dW_t^i = (1 - N_{t-}^i) [(rW_t^i + \pi_t^i(\mu - r)) dt + \pi_t^i \sigma_S dB_t^S] + N_t^i W_t^i r dt, \quad W_0^i = w^i, \quad (2.1)$$

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<sup>11</sup>The case where the risk exposure is partially spanned by the traded assets can be covered along the lines of, for example, (Svensson and Werner, 1993; Teplà, 2000).

where  $\pi_t^i$  is the wealth amount allocated by agent  $i$  to the risky asset at each time  $t \in [0, T \wedge \tau^i)$ . We interpret the agents' wealth as the resources dedicated to a specific transaction, or the balance of a trading account, which can turn negative and hence attract interest rate charges. In the baseline model, we assume the latter to accrue at rate  $r$ .<sup>12</sup> Note that in the above we have assumed that the residual wealth of a defaulted party is invested in the money market account until maturity: exclusion from the financial market is a default penalty (Alvarez and Jermann, 2000). This allows us to mimic the common practice of subjecting structuring desks to risk limits and counterparty risk charges.<sup>13</sup>

In the absence of credit risk mitigation tools, which will be introduced in the next section, the agents solve the problem

$$\max_{\pi^i} E [u(W_T^i + Z_T^i)], \quad (2.2)$$

subject to the budget constraint (2.1), where we have defined agent  $i$ 's exposure to  $Z_T$  by  $Z_T^i = (1_{i=B} - 1_{i=A})Z_T$ , for  $i \in \{A, B\}$ . As  $Z$  has symmetric distribution, the agents achieve the same expected utility ex-ante. Their indirect utilities and optimal investment strategies can be computed explicitly, and are given in the next proposition.

**Proposition 2.1.** *On  $\{\tau^i > t\}$ , each agent's optimal value function is given by*

$$v(t, W_t^i, Z_t^i) = -\frac{1}{\gamma} \exp \left( \underbrace{-\gamma e^{r(T-t)} W_t^i - \frac{1}{2} s^2 (T-t) - \gamma Z_t^i + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-t)}_{=v^n(t, W_t^i, Z_t^i)} \right) F(t), \quad (2.3)$$

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<sup>12</sup>Section 5 discusses the extension to different lending and borrowing costs.

<sup>13</sup>An alternative approach is to assume that agents maximize their wealth at the terminal time  $\tau \wedge T$  and simply ignore the residual liability. This setting makes collateral posting more expensive and yields results stronger than in the current setup.

with

$$F(t) = \left\{ (1 - \alpha) \exp(-\lambda(T - t)) + \alpha \exp\left(\frac{1}{2}s^2(T - t)\right) \right\}, \quad (2.4)$$

where  $\alpha = \frac{\lambda}{\lambda + \frac{1}{2}s^2} \in (0, 1)$ ,  $s = \frac{\mu - r}{\sigma_S}$  denotes the Sharpe ratio, and  $v^n$  is the optimal value function in case of default-free counterparties. The optimal allocation to the risky asset is given by

$$\pi_t^* = \pi_t^{i,*} = \exp(-r(T - t)) \frac{\mu - r}{\gamma \sigma_S^2}. \quad (2.5)$$

The above shows that the optimal allocation to the risky asset is unaffected by the exposure to the illiquid payoff  $Z_T$  and default risk, as they are both unspanned by the tradeable assets. The indirect utility  $v$ , however, takes into account both sources of risk. It considers the indirect utility  $v^n$  of the case without default (Svensson and Werner, 1993; Teplá, 2000), and scales it by a factor  $F(t)$  reflecting the risk of default faced by the agent over the residual trading horizon. Specifically, one can easily show that  $F_{\lambda=0}(t) = 1$  and  $F(t) \geq 1$ . The scaling factor accounts for the cost of being excluded from the market, should default occur in the next small time interval. The interpretation of the default penalty is intuitive. For example, the better the forgone trading opportunities (as summarized by a larger Sharpe ratio) associated with default, the larger the scaling factor and, thus, the more pronounced the negative impact on the agent's indirect utility. Similarly, the higher the default intensity  $\lambda$ , the higher the penalty applied by the scaling factor, thus also decreasing the agent's utility.

Because of risk aversion and the opposite exposure to  $Z_T$ , agents have natural risk sharing opportunities. As a simple example, we consider the introduction of a forward contract, i.e., an agreement whereby at time  $T$  agent  $A$  will transfer  $k \geq 0$  units of the realized exposure  $Z_T^A$  to agent  $B$  upon payment of a price  $p_k \geq 0$  agreed at time zero. Since the parties are fully symmetric, we have  $p_k = -p_k$  so that  $p_k = 0$  and we work

with a zero forward price in the following.

We first consider in detail how the agents will optimally design a CSA for a unitary position in the forward contract ( $k = 1$ ). We will then allow the agents to choose optimally both the size of the position in the forward contract and the CSA.

### 3 Hedging with counterparty risk

By entering a forward contract, the agents expose themselves to counterparty risk. In the presence of a CSA, the latter affects the budget constraints through collateral flows<sup>14</sup> and the close-out transfer at default. We denote by  $C^i$  the collateral process as seen from party  $i \in \{A, B\}$ , with the understanding that  $C^A = -C^B$ . We assume collateral to be posted continuously and in cash.<sup>15</sup> We let  $C_0$  be zero, and require  $C_t$  to be measurable with respect to the information available to agents an instant before each time  $t > 0$ , in particular before a default may occur (that is,  $C_t$  is predictable with respect to the information generated by the state variables). Collateral amounts enter and exit the agents' trading account, depending on the moneyness of each agent's position and on the collateral rules indicated in the CSA. These are fully fungible, in the sense that they can be optimally invested in the financial market. This assumption is without loss of generality here, as we assume that interest on collateral is rebated and there are no borrowing costs (see section 5 for an extension to collateral segregation and borrowing

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<sup>14</sup>More generally, one may allow for other counterparty risk mitigation tools, such as credit default swaps. Our results are robust with respect to this dimension, as discussed in Section 5.

<sup>15</sup>We ignore the choice of collateral type/quality in our setting (see, for example, Gorton and Metrick, 2012, for an analysis of this important dimension in the repo market).

costs). The budget constraint (2.1) now takes the following form:<sup>16</sup>

$$\begin{aligned} dW_t^i = & (1 - N_{t-}^i) [(rW_t^i + \pi_t^i(\mu - r)) dt + \pi_t^i \sigma_S dB_t^S] + N_t^i W_t^i r dt \\ & + (1 - N_{t-}^i) \left[ dC_t^i - r C_t^i dt + \left( (R_{t-}^i)^+ - (C_{t-}^i)^+ \right) dN_t^i - \left( (R_{t-}^i)^- - (C_{t-}^i)^- \right) dN_t^j \right], \end{aligned} \quad (3.1)$$

with  $W_0^i = w^i$ ,  $i, j \in \{A, B\}$ , and where we have defined by  $\tau = \tau^A \wedge \tau^B$  the first default time, and by  $N_t = 1_{\tau \leq t}$  the corresponding default indicator.

In the above,  $R^i$  denotes the replacement cost of the forward contract for agent  $i$ . The idea is that conditional on its default, agent  $i$  would receive the full replacement cost of the forward contract from the surviving counterparty when in-the-money, and would otherwise lose the collateral posted with party  $j$ . Note that in the case of over-collateralization, i.e.,  $C^\pm > R^\pm$ , the surviving party that is in-the-money incurs the cost of giving back any collateral posted in excess of the replacement cost. See table 1 for a summary of the payments upon default.

We consider a common default-risk-free, and risk-neutral, close-out convention<sup>17</sup> (see Brigo et al., 2013), meaning that we set<sup>18</sup>

$$R_t^i = E_t \left[ e^{-r(T-t)} (-Z_T^i) \right] = -e^{-r(T-t)} Z_t^i. \quad (3.2)$$

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<sup>16</sup>We use the notation  $a^+ = \max\{0, a\}$  and  $a^- = \max\{-a, 0\}$  for  $a \in \mathbb{R}$ .

<sup>17</sup>See section 5 for a discussion of alternative close-out conventions, as well as the possibility to endogenize the replacement cost.

<sup>18</sup>The term  $R^i$  reflects the position in the forward contract of party  $i$ ,  $-Z_T^i$ . In the case of party  $A$ , for example, the agent faces a liability  $-Z_T = Z_T^A$ , and hence is long a forward contract, which will pay  $Z_T = -Z_T^A$  at maturity in case of no default.

Agents solve the problem

$$\max_{(\pi^i, C^i)} E \left[ u \left( W_T^i + (1 - 1_{\tau > T}) Z_T^i - 1_{\tau > T} C_T^i \right) \right], \quad (3.3)$$

subject to the budget constraint (3.1), with the objective function taking into account the netting of the payoff from the forward contract and the terminal collateral amount.

We solve the problem for three different classes of CSA: no collateral (zero CSA), deterministic proportional collateral, and contingent collateral rules. Note that the CSA is relevant even in the case of no collateral, as it spells out the close-out rules.

### 3.1 No collateral

We need to consider three relevant cases at each time before maturity:

- Agent  $i$  has already defaulted,  $\{\tau^i \leq t < \tau^j\}$ .
- The counterparty has already defaulted,  $\{\tau^j \leq t < \tau^i\}$ .
- No default has yet occurred,  $\{\tau > t\}$ .

The first case is uninteresting, as agent  $i$ 's wealth is simply invested in the money market account and any resources available at the terminal date  $T$  will be used to meet the random exposure  $Z_T$ . In the second case, the agent is faced with the original exposure, without the help of a hedging instrument, and we are back to the case covered by Proposition 2.1. The third case is more interesting, as each agent's indirect utility will reflect the trade-off between the risk sharing benefits of the forward contract and the counterparty risk it gives rise to. The result is summarized in the following proposition.

**Proposition 3.1.** *Consider problem (3.3), with the collateral strategy restricted to the null process. On the event  $\{\tau > t\}$ , the optimal value function of agent  $i \in \{A, B\}$  is given by*

$$v^0(t, W_t^i, Z_t^i) = \underbrace{-\frac{1}{\gamma} \exp\left(-\gamma e^{r(T-t)} - \frac{1}{2}s^2(T-t)\right)}_{=\hat{v}^n(t, W_t^i, Z_t^i)} F^d(t, Z_t^i), \quad (3.4)$$

with

$$\begin{aligned} F^d(t, Z_t^i) &= \underbrace{\int_t^T e^{-\lambda(u-t)} \lambda \exp\left(\frac{1}{2}\gamma^2\sigma_Z^2(T-u)\right) \left[ e^{\frac{1}{2}s^2(T-u)} \left(1 - \Phi\left(\frac{Z_t^i}{\sigma_Z\sqrt{u-t}}\right)\right) + F(u) \Phi\left(\frac{Z_t^i}{\sigma_Z\sqrt{u-t}}\right) \right]}_{(I)} du \\ &+ \underbrace{\int_t^T e^{-\lambda(u-t)} \lambda \exp\left(-\gamma Z_t^i + \frac{1}{2}\gamma^2\sigma_Z^2(T-u)\right) \left[ F(u) \left(1 - \Phi\left(\frac{Z_t^i - \gamma\sigma_Z^2(u-t)}{\sigma_Z\sqrt{u-t}}\right)\right) + e^{\frac{1}{2}s^2(T-u)} \Phi\left(\frac{Z_t^i - \gamma\sigma_Z^2(u-t)}{\sigma_Z\sqrt{u-t}}\right) \right]}_{(II)} du \\ &+ e^{-2\lambda(T-t)} \underbrace{1}_{(III)}, \end{aligned} \quad (3.5)$$

where as usual  $s = \frac{\mu-r}{\sigma_S}$ ,  $\Phi$  denotes the cumulative distribution function of the standard Normal distribution, and  $F(\cdot) \geq 1$  is given by (2.4). The optimal allocation to the risky asset is again given by (2.5).

In line with expression (2.3), but taking into account risk sharing now, we see that the agents' optimal value function with zero collateral,  $v^0$ , is shaped by the indirect utility without default,  $\hat{v}^n$ , and a scaling factor  $F^d(t, Z_t^i)$ . The latter is a weighted sum of different adjustments capturing different situations: a factor of one in case of no default (*III*); a factor greater than one accounting for default penalties and foregone risk sharing opportunities in case there is default and the defaulting party is in-the-money—so that at least the portion of the risk realized at time  $t$  is hedged (*I*); and



a factor that directly depends on  $Z_t^i$ , reflecting default situations where the defaulting party is out-of-the-money (II). The latter situations accounts for the counterparty's as well as own default, and a CSA makes the transfer of resources across the two states possible. Since the agents are risk-averse, such a transfer turns out to be optimal, as will be shown in more detail in the following sections.

### 3.2 Proportional collateralization

Consider now the case of collateral amounts expressed as a deterministic fraction of the risk-free close-out amount (3.2). We set  $C_t^i = c(t)R_t^i$ , for  $t < \tau \wedge T$ , and restrict ourselves to differentiable functions  $c : t \in [0, T] \rightarrow \mathbb{R}_+$ , with  $c(0) = 0$ . Problem (3.3) becomes a deterministic optimal control problem, which yields the following results.

**Proposition 3.2.** *The optimal deterministic CSA entails partial collateralization,  $c^*(t) < 1$ , for  $t < T$ . Full collateralization becomes optimal as the time to maturity shrinks to zero. The optimal collateral fraction is given by*

$$c^*(t) = 1 - \frac{1}{\gamma \sigma_Z \sqrt{t}} g^{-1} \left( 1 + \frac{\lambda + \frac{1}{2} s^2 \exp\left(-\left(\frac{1}{2} s^2 + \lambda\right)(T-t)\right)}{\frac{1}{2} s^2 \left(1 - \exp\left(-\left(\frac{1}{2} s^2 + \lambda\right)(T-t)\right)\right)} \right), \quad (3.6)$$

where  $s = \frac{\mu-r}{\sigma_S}$  denotes the Sharpe ratio and  $g(x) = \Phi(x) + \frac{\phi(x)}{x}$ , with  $\Phi$  and  $\phi$  the cumulative distribution and the probability density function of the standard Normal distribution, respectively.

In the proportional CSA space, the optimal solution results in partial collateralization,<sup>19</sup> because agents evaluate scenarios where collateral is paid differently from scenar-

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<sup>19</sup>Note that  $c^*(t)$  approaches negative infinity as  $t$  goes to zero, but of course  $Z_0 = 0$ , so the optimal collateral amount is zero. In particular, the optimal collateral amount is finite almost surely for all  $t > 0$ .

ios where collateral is received. When a counterparty defaults, the surviving party still has the opportunity to invest in the financial market. On the other hand, the defaulting party is excluded from the financial market, reflecting a default penalty. Thus, marginal utilities are greater in scenarios where collateral is paid relative to scenarios where collateral is received, making full collateralization suboptimal even in the symmetric setting considered here.

< TABLE 1 ABOUT HERE >

From expression (3.6), we can characterize the sensitivity of optimal deterministic collateral with respect to the key parameters of the model as follows:

**Proposition 3.3.** *For fixed  $t \in [0, \tau \wedge T]$ , the optimal deterministic collateral fraction  $c^*$  is decreasing in the Sharpe ratio  $s$ , and increasing in the volatility coefficient  $\sigma_Z$ , the risk aversion coefficient  $\gamma$ , and the default rate  $\lambda$ .*

The results above are intuitive. The more appealing the trading opportunities available (as summarized by a higher Sharpe ratio), the lower the demand for collateral, as it is more convenient to self-insure against counterparty risk. When risk aversion, default risk, or the volatility of the illiquid exposure increase, the demand for collateral is higher, as hedging demand dominates collateral costs. To illustrate our results, we present in Figure 1 the optimal collateral schedule  $c^*$  for a set of baseline parameters. In line with (3.6), partial collateralization is always optimal before maturity; the longer the time to maturity, the more the CSA economizes on collateral.

< FIGURE 1 ABOUT HERE >

### 3.3 Contingent collateralization

We now generalize the previous example, by allowing collateral rules to be contingent on the path of the state variable vector  $(W^i, Z^i, C^i)$ . An admissible collateral rule is a process with continuous paths, which is predictable with respect to the information generated by the state vector, and satisfies the symmetry conditions indicated below. For each time  $t \in [0, T \wedge \tau)$ , we define the collateral amount posted by party  $i$  as

$$C_t^i = e^{-r(T-t)} \int_0^t c(s, W_s^i, Z_s^i, C_s^i) dZ_s^i,$$

where  $c$  satisfies  $c(t, W_t^i, Z_t^i, C_t^i) = c(t, W_t^j, Z_t^j, C_t^j) = c(t, W_t^j, -Z_t^i, -C_t^i)$ , since  $C^i = -C^j$ . In the present setting, the optimal CSA has the following features.

**Proposition 3.4.** *Consider problem (3.3), with the space of admissible collateral rules defined above. Then, optimal collateral levels are independent of wealth,  $c^*(t, W_t^i, Z_t^i, C_t^i) = \widehat{c}(t, Z_t^i, C_t^i)$ , and are given by*

$$\widehat{c}(t, z, y) = -\frac{\tilde{v}_{zy}(t, z, y) + \tilde{v}_{zy}(t, -z, -y)}{\tilde{v}_{yy}(t, z, y) + \tilde{v}_{yy}(t, -z, -y)},$$

where  $\tilde{v}$  solves the partial differential equations with deviating arguments (B.11).

To obtain more insights into the form of optimal contingent collateral rules, we solve (B.11) numerically.<sup>20</sup> Figure 2 shows the optimal collateral fraction  $\widehat{c}$  for different times to maturity, and for the average value of  $(Z^i, C^i)$ . Figure 3 depicts  $\widehat{c}$  for different configurations of the pair  $(Z^i, C^i)$ . The results show that ‘less-than-full’ collateralization is optimal in general, and that more collateral is posted as the balance of the collateral

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<sup>20</sup>The symmetry conditions we impose on  $c(\cdot)$  make the HJB equation a PDE with distributed arguments, which can be solved by using a suitable extension of standard finite difference methods for boundary value problems.

account decreases and  $Z^i$  increases (meaning that party  $i$  is more out-of-the-money). The more general contract space considered here allows the agents to mitigate their exposure to counterparty risk by modulating collateral flows in response to changes in market conditions. In line with bespoke CSAs found in non standardized transactions, our results provide a rationale for CSAs specifying incremental collateral flows triggered by movements in the underlying exposure or the performance of the collateral account (see ISDA, 2010b).

< FIGURE 2 ABOUT HERE >

< FIGURE 3 ABOUT HERE >

### 3.4 Trading volume

We now consider the case when agents can optimally choose the position in the forward contract, as well as the CSA. In the absence of counterparty risk, the independence of  $Z_T^i$  and  $W_T^i$  implies that full insurance is optimal ( $k^* = 1$ ), exactly as in Mossin (1968). The same may not hold true when counterparty risk is explicitly taken into account, for the following reasons. First, with counterparty risk each agent's objective function includes the random amount  $(1 - k1_{\tau > T})Z_T^i$ . Second, the first default time  $\tau$  introduces dependence between  $Z_T^i$  and  $W_T^i$ , as payments at default depend on  $Z_\tau$ . Finally, the CSA introduces an additional source of dependence via the collateral flows entering the dynamics of wealth. In line with the literature on background risk and optimal portfolio choice (Doherty and Schlesinger, 1983; Eeckhoudt et al., 1996), the choice of the optimal trading volume and the CSA will be closely linked. Although it is difficult to predict how the dependencies listed above will play out in the optimal solutions, the

following results suggest that trading volume will generally increase at the expense of collateral levels.

**Proposition 3.5.** *Within the class of proportional collateral rules, consider the case of a fixed collateral fraction  $c \geq 0$ , i.e.,  $C_t^i(k) = cR_t^i(k)$ , with  $R_t^i(k) = kR_t^i$ . Then, the optimal pair  $(c^*, k^*)$  satisfies  $c^* < 1$  and  $k^* > 1$ .*

Agents overhedge at the expense of collateralization, because that allows them to improve their situation in default states. The larger the position in the hedging instrument, the larger the payout the defaulting party will be able to receive in case she is in-the-money at default, without increasing her default losses in case she is out-of-the-money (see table 1). At trading level  $k^*$ , these benefits will be large enough to offset the cost of an imperfect hedge. We can appreciate these trade-offs more clearly by extending the results of Proposition 3.2 to the current setting.

**Proposition 3.6.** *The optimal deterministic CSA entails partial collateralization,  $c_k^*(t) < 1$ , for all  $t \leq T \wedge \tau$ . The optimal collateral fraction is given by  $c_{k^*}^*(t) = \frac{1}{k^*}c^*(t)$ , where  $c^*(t)$  is defined in (3.6) and  $k^* > 1$ .*

Hence, the optimal collateral fraction is less than 100% even at maturity, as overhedging ensures that recovery would be full, should default occur at expiration. The longer the time to maturity, the lower the collateralization level, which is uniformly lower than the one we obtained in Proposition 3.2 for the case of fixed  $k = 1$ . We note that the optimal trading volume and collateral fraction leave the overall collateral amount unchanged, as  $C_t^{i,*}(k^*) = c_{k^*}^*(t)R_t^i(k^*) = c^*(t)R_t^i = C_t^{i,*}$ . So, it is the process  $C^{i,*}$  that really captures the agents' insurance demand against counterparty risk. But why is it more convenient to post  $C^{i,*}$  via a pair  $(c_{k^*}^*, k^*)$  instead of  $(c^*, 1)$ ? Again, from table 1, we see that a larger position in the hedging instrument will allow the defaulting party

to receive a higher transfer when she is in-the-money, without increasing her collateral losses when she is out-of-the-money. At an optimum, these benefits are ex-ante large enough to offset the costs of an imperfect hedge at maturity, as well as the higher losses that the surviving party will have to incur at default in her out-of-the-money states.

We then consider the sensitivity of trading volume to key model parameters. Figure 4 shows that, as the default rate increases, trading volume increases and collateral levels decrease. This is because default states get a larger weight and hence it is more beneficial for the defaulting party to reduce collateral losses in her out-of-the-money states. Figure 5 shows that if we reduce the volatility of the illiquid exposure, or increase the Sharpe ratio, trading volume increases relative to the baseline case. This is due to the higher hedging demand against the illiquid exposure (the forward contract is more valuable) and the higher opportunity cost of posting collateral for the agents. The opposite effect is induced by the risk aversion parameter, suggesting that an increase in  $\gamma$  makes counterparty risk aversion dominate across states. Combining these results with the predictions of Proposition 3.3, we see that the interlinkage between optimal trading volume and CSA is far from trivial.

< FIGURE 4 ABOUT HERE >

< FIGURE 5 ABOUT HERE >

## 4 Standardized margins vs. CSAs

We now provide a comparison between optimal CSAs and standardized collateral rules, taking the current Dodd-Frank and EMIR proposals for non-centrally cleared OTC derivatives as a benchmark. The latter include a two-way initial margin posted with a

custodian, and a variation margin corresponding to full collateralization. For the initial margining, we adopt the convention of a 99% Value-at-Risk over a 10-day horizon (see BIS, 2013). To carry out the comparison within our setting, we express the collateral process as

$$C_t^i = \tilde{c} + c(t)R_t^i, \quad (4.1)$$

where  $\tilde{c}$  denotes the initial margin. For simplicity, we assume the initial margin to be fixed at inception and to be symmetric. Note that the setting of section 2 applies to this new case, even when the initial margin is segregated, as long as interest on collateral is fully rebated; see the next section for a discussion. We also adopt the proportional collateral rule of section 3.2, which is easier to analyze and includes the case of prescribed variation margins as a special case ( $c(t) = 100\%$ ). In practice, these assumptions could all be relaxed.<sup>21</sup>

Table 2 summarizes the three main cases that we would like to compare: no collateral, optimal proportional collateral, and Dodd-Frank/EMIR rules with initial and variation margins. As indicated in the last line of the table, mandating full collateralization will discourage hedging. One can indeed show that under full collateralization the risk sharing level is  $k^* = 1$ , irrespective of the initial margin. If collateral segregation is costly (see the next section), then the optimal hedge position size is smaller than one.

< TABLE 2 ABOUT HERE >

In Figure 6, we depict the expected utilities of the counterparties for the three base cases. To gauge the utility loss from counterparty risk, we also report the expected utilities for the case of no default risk ( $\lambda = 0$ ). As expected, the figure shows that the

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<sup>21</sup>Also in CCPs, initial margins can be asymmetric and updated on a regular basis (e.g., Brigo et al., 2013; Sidanius and Zikes, 2012).

use of collateralization mitigates the utility losses from counterparty risk. Although the standardized margin rules seem to deliver utility gains close to the case of an optimal proportional CSA, two considerations are in order. First, proportional collateral rules represent a very narrow CSA space: we depict the utility gains associated with the contingent collateral rules considered in section 3.3 to show how the agents could gain considerably from the adoption of a more general CSA. Second, when moving from optimal proportional collateral to Dodd-Frank/EMIR margin rules, agents can keep utility gains almost unchanged only at the price of a significant drop in trading volume: figure 7 shows the extent to which standardized margins have a detrimental effect on risk sharing.

Suppose now that the counterparties are too-big-to-fail, and any shortfall of a defaulting party (not only the first one to default) would have to be covered by the government. The results depicted in figures 6 and 7 would then suggest that a social planner maximizing the agents' expected utilities, while minimizing the expected shortfall costs, may be worse off for parameter configurations resulting in a large enough reduction in risk sharing. As an example, we compute the total expected shortfalls,

$$\sum_{i \in \{A, B\}} E [1_{\tau^i \leq T} (W_T^i + Z_T^i)^-] = 2E [1_{\tau^i \leq T} (W_T^i + Z_T^i)^-], \quad (4.2)$$

for the case of optimal proportional CSA and standardized margins. We depict the results in figure 8, for different time horizons and for the case of agents having initial capital equal to a 99% Value-at-Risk of the illiquid exposure over a one-year horizon. We see that expected shortfalls for the standardized collateral case dominate those arising in the case of an optimal proportional CSA, due to lower risk sharing. In particular, we note that variation margins can be bad in single default states when the defaulting



party is out-of-the-money, and hence fewer resources are available to cover its liabilities. On the other hand, collateral is overall good in the most severe state of the world, when both parties default over the trading horizon, but initial margins are the only thing that really matters, as variation margins have zero net effect on the liability shortfall. While stylized, the results suggest that proposals aimed at standardizing collateral rules for non-centrally cleared derivatives should weigh carefully the potential costs of a lower demand for hedging instruments, for which bespoke CSAs are currently the norm. Co-existence of initial margins and customized CSAs may be preferable to standardized variation margins that would require regulators to take a stance on mark-to-market/model procedures to properly implement full collateralization.

< FIGURE 6 ABOUT HERE >

< FIGURE 7 ABOUT HERE >

< FIGURE 8 ABOUT HERE >

## **5 Extensions and further discussion**

### **5.1 Re-hypothecation vs. segregation of collateral**

In sections 2 and 3, we have always assumed that collateral amounts received from a counterparty become part of the trading account and can therefore be invested in the financial market. More generally, collateral could be re-pledged for other purposes, such as trades difficult to unwind, exposing the other party to close-out risk, should the position turn in her favor at default. For this reason collateral may be segregated and

its re-hypothecation forbidden. This situation can be accommodated as follows in our setting.

Assume that collateral is transferred to an external account held by a custodian. Recalling that interest on collateral is rebated, and allowing for a fee  $\eta \geq 0$  charged by the custodian, the budget constraint takes now the form

$$\begin{aligned} dW_t^i &= (1 - N_{t-}^i) [(rW_t^i + \pi_t^i(\mu - r)) dt + \pi_t^i \sigma_S dB_t^S] + N_t^i W_t^i r dt \\ &\quad + (1 - N_{t-}^i) \left[ - (dC_t(k) - (r - \eta) C_t(k) dt) 1_{Z_t^i \geq 0} \right. \\ &\quad \left. + (R_{t-}^i(k))^+ dN_t^i + \left( C_{t-}(k) - (R_{t-}^i(k))^- \right) dN_t^j \right], \end{aligned} \quad (5.1)$$

with  $W_0^i = w^i$ . Here,  $C(k)$  denotes the collateral collected by the custodian during the life of the trade in  $k$  units of the forward contract. For example, under the formulation of section 3.2, at each time  $t < \tau \wedge T$  we have

$$C_t(k) = (C_t^A(k))^- + (C_t^B(k))^+ = c_k(t) k e^{-r(T-t)} |Z_t|. \quad (5.2)$$

The term  $1_{Z_t^i \geq 0}$  in (5.1) reflects the fact that agents incur collateral outflows only when they are out-of-the-money. Collateral inflows only materialize at maturity or default, as collateral is held by the custodian.

Although traded wealth evolves quite differently from the non-segregated case, the problem can be solved along the same lines as the baseline problem, and we have the following result.

**Proposition 5.1.** *For  $\eta = 0$ , the optimal CSA and trading position are the same as in the case when collateral can be re-hypothecated. For  $\eta > 0$ , collateral levels are uniformly lower than in the re-hypothecation case.*

When interest is rebated in full, it makes no difference to a borrowing unconstrained CARA agent whether collateral is held by a custodian or available for trade. When segregation is costly, self-insurance becomes more appealing, and collateral levels are lower. A similar result holds with borrowing costs, as we discuss in the following section.

## 5.2 Borrowing costs

In our setting, it would be natural to assume a borrowing rate of  $r + \lambda$  for the agents, reflecting their credit quality. More generally, we could think of a spread  $\tilde{\lambda}$  above the risk-free rate charged by a bank to the trading desk entering the OTC contract. In any case, the introduction of asymmetry between borrowing and lending rates would make our problem hard to solve. It is not difficult to see, however, that our main predictions would be strengthened. The reason is that borrowing costs introduce a further wedge between states where an agent is in-the-money and states where she is out-of-the-money. With rehypotecation, receiving collateral is a source of cheap funding: as interest is rebated on cash collateral, the agents can fund trading positions at rate  $r$  instead of  $r + \lambda$ . For the same reason, however, the agents lose a spread  $\lambda$  whenever they are forced to borrow to post collateral. In line with our findings for the case of no borrowing costs, we expect this asymmetry to magnify collateral costs and, *ceteris paribus*, to result in lower collateral levels. When collateral is segregated, agents cannot use collateral to support other trades. The cheap funding channel is shut off, while collateral outflows are more expensive relative to the case of riskless borrowing, making collateralization even less desirable.

### 5.3 The meaning of ‘full collateralization’

In the previous sections we have seen that full collateralization is in general suboptimal. The fact that such strong requirement is common in some OTC markets (e.g., in interest rate swaps; see ISDA, 2010a), should be interpreted in the light of frictions affecting the collateralization process in real-world transactions. In practice the collateralization process is discrete and, depending on the specific asset class or products considered, collateral revisions may be relatively infrequent. In these situations, full collateralization may provide a simple way to provide a buffer against gaps in the protection against counterparty risk originating from discrete collateralization. These aspects are clearly absent from our model.

There is an additional important issue to note, however. The term ‘full collateralization’ is often a misnomer in OTC transactions, as counterparty risk mitigation is ‘full’ only to the extent of the replacement cost defined by the CSA.<sup>22</sup> The definition is not unique, and is often ambiguous, as there may be cases when counterparties are given the opportunity to choose between a default-risk-free close-out, such as the one considered in (3.2), and a credit risky close-out, for example estimated from a range of market quotes obtained after the default event. Even if the definition were unambiguous, it is clear that collateralization would be full only if the replacement cost were to coincide with the market-value of the OTC contract being considered. This is often not the case, as CSAs typically specify proxies and models to determine collateral amounts, which may be only partially correlated with the value of the contract under consideration.

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<sup>22</sup>A different situation is when the maximum payout from an instrument can be defined, and an equal amount of collateral is posted at inception and segregated (this is the case for catastrophe bonds and other asset backed securities; e.g., Lakdawalla and Zanjani, 2012). Even there, however, collateralization is not entirely full, as cash collateral is too expensive and high-yield (but credit-risky) fixed-income instruments are often used.

## 6 Conclusion

In this work we have considered OTC transactions with bilateral default risk. In our baseline model, agents face an illiquid source of risk that can be hedged by entering a forward contract. As agents are credit risky, the hedging instrument gives rise to counterparty risk. The latter can be mitigated by suitably designing a CSA indicating when and how much collateral to post during the life of the transaction, as well as the close-out convention to apply at default. We have determined optimal collateral rules over different admissible strategies, and obtained results that are consistent with CSAs commonly observed in practice, and aimed at economizing on collateral. At the same time, we have discussed some interesting features of non-standardized OTC transactions, such as re-hypothecation and segregation of collateral, funding costs, and close-out conventions, which have important implications for optimal collateral rules and trading volume. Finally, we have used our framework to analyze standardized margin requirements (such as those proposed by the recent Dodd-Frank and EMIR regulation) for non-centrally cleared OTC transactions. Our results show that standardized collateral rules have a detrimental impact on risk sharing. We argue that the benefits of standardized collateralization should be carefully weighted against the associated potential reduction in demand for (and supply of) customized hedging instruments, for which bespoke CSAs are currently the norm. We further argue that full collateralization is not particularly meaningful in the absence of a precise stance on the models required to implement the marking-to-market procedure and of a consistent definition of close-out conventions.

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# Appendix

## A Tables and figures

	ITM	OTM
$\tau = \tau^A$	$R_\tau^A = (R_\tau^A - C_\tau^A) + C_\tau^A \geq 0$	$C_\tau^A < 0$
$\tau = \tau^B$	$C_\tau^A \geq 0$	$R_\tau^A = (R_\tau^A - C_\tau^A) + C_\tau^A < 0$

Table 1: Payments from the forward contract (with a CSA) to agent A at the default time  $\tau$ , in case the agent is in-the-money (ITM) or out-of-the-money (OTM).

Exposure	Instrument	Insurance	Retention	Solution
$-Z$	forward	$k$	$1 - k$	$k^* = 1$
$-Z, \tau$	forward ISDA CSA	$k$ $c$	$1 - k$ $1 - c$	$k^* > 1$ $c^* < 1$
$-Z, \tau$	forward EMIR/Dodd-Frank $C^i = \tilde{c} + cR^i$	$k$ $c = 1$ $\tilde{c}$	$1 - k$ 0 tail risk	$k^* \leq 1$

Table 2: Optimal CSA and forward position; agent A's viewpoint.

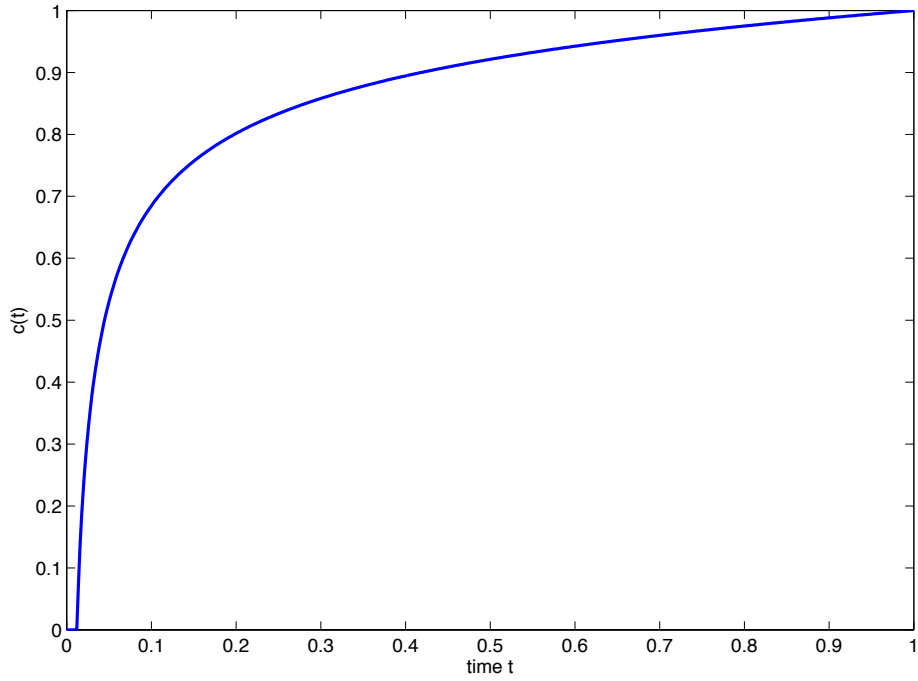


Figure 1: Optimal deterministic collateral fraction  $c(\cdot)$ , for the baseline parameters  $T = 1, \gamma = 0.2, \lambda = 5\%, \sigma_Z = 20\%, r = 3\%, \mu = 6\%, \sigma_S = 20\%$ .

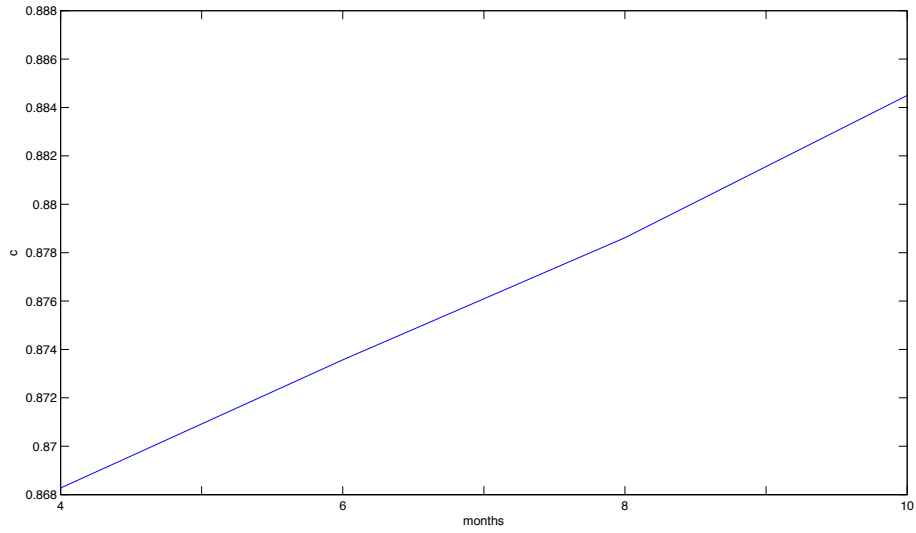


Figure 2: Optimal collateral fraction for the average pair  $(Z^A, C^A)$  and for different times to maturity. Parameter values:  $\gamma = 0.2, T = 1, \lambda = 5\%, k = 0.9, \sigma_Z = 20\%, r = 3\%, \mu = 6\%, \sigma_S = 20\%$ .

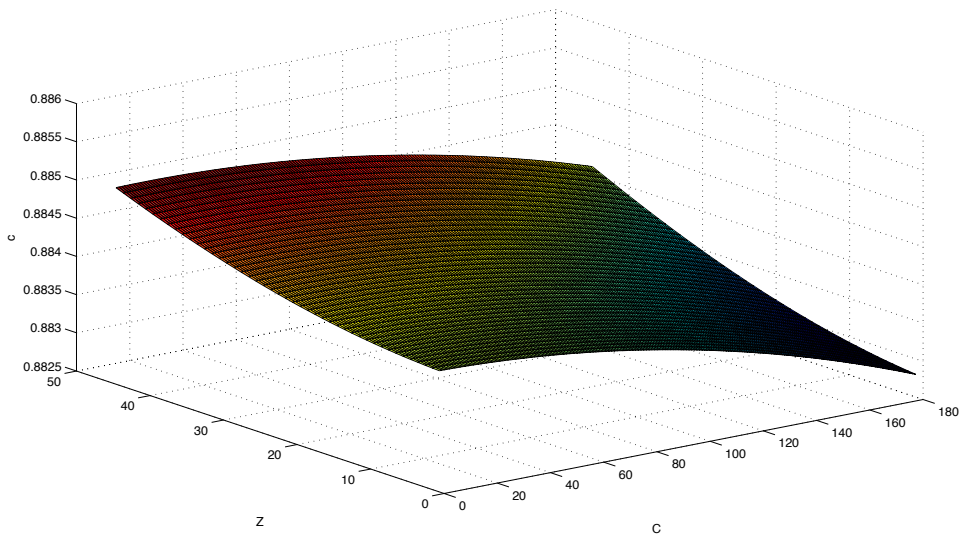


Figure 3: Optimal collateral fraction  $\tilde{c}^i$  as a function of the collateral posted to date,  $C^i$ , and realization of  $Z^i$ . The parameter values are as in figure 2.

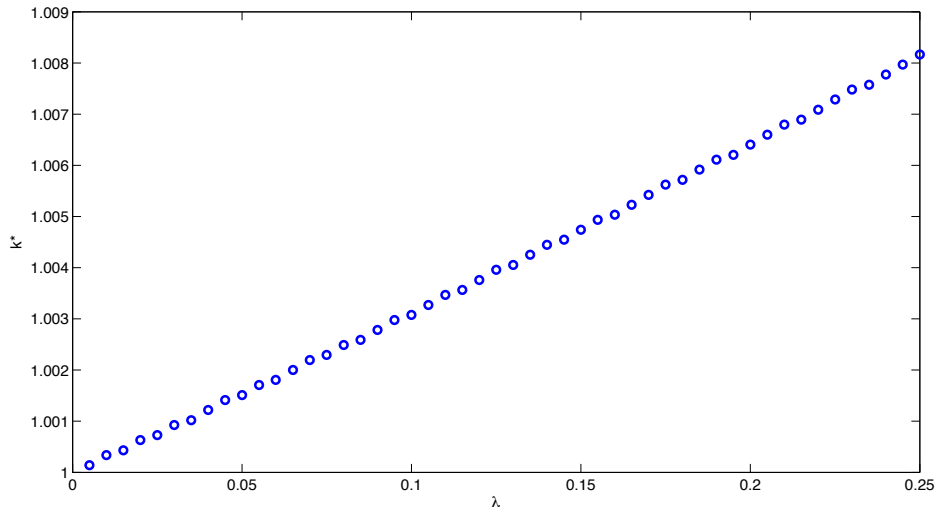
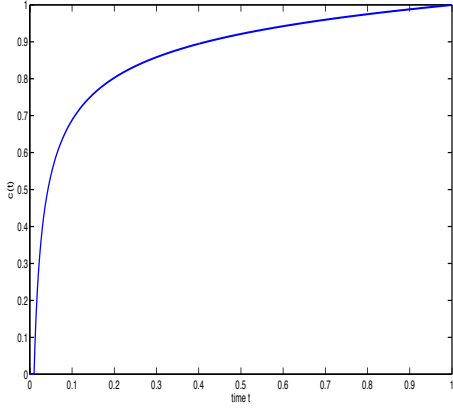
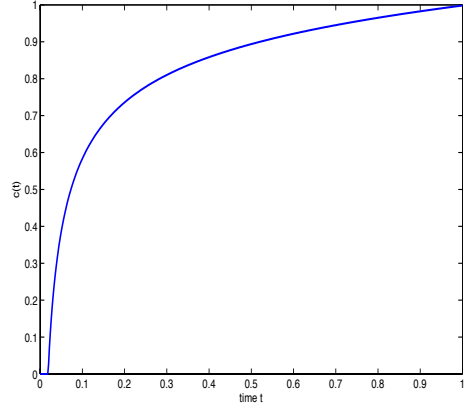


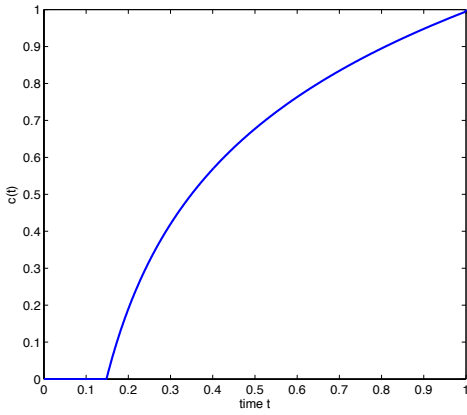
Figure 4: Optimal risk sharing level  $k^*$  as a function of the default rate  $\lambda$ , for the baseline parameters  $T = 1, \gamma = 0.2, \sigma_Z = 20\%, r = 3\%, \mu = 6\%, \sigma_S = 20\%$ .



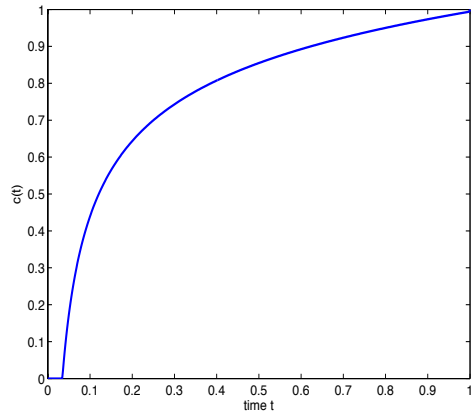
(a) Higher risk aversion ( $\gamma = 0.5$ ),  $k^* = 1.0006$



(b) Lower volatility of  $Z$  ( $\sigma_Z = 15\%$ ),  $k^* = 1.0021$



(c) Higher expected return ( $\mu = 9\%$ ),  $k^* = 1.0062$



(d) Lower volatility of  $S$  ( $\sigma_S = 15\%$ ),  $k^* = 1.0027$

Figure 5: Optimal deterministic collateral fraction and trading volume for different key parameter values. For the baseline parameters  $T = 1, \gamma = 0.2, \lambda = 5\%, \sigma_Z = 20\%, r = 3\%, \mu = 6\%, \sigma_S = 20\%$ , the optimal trading volume is  $k^* = 1.0013$ .

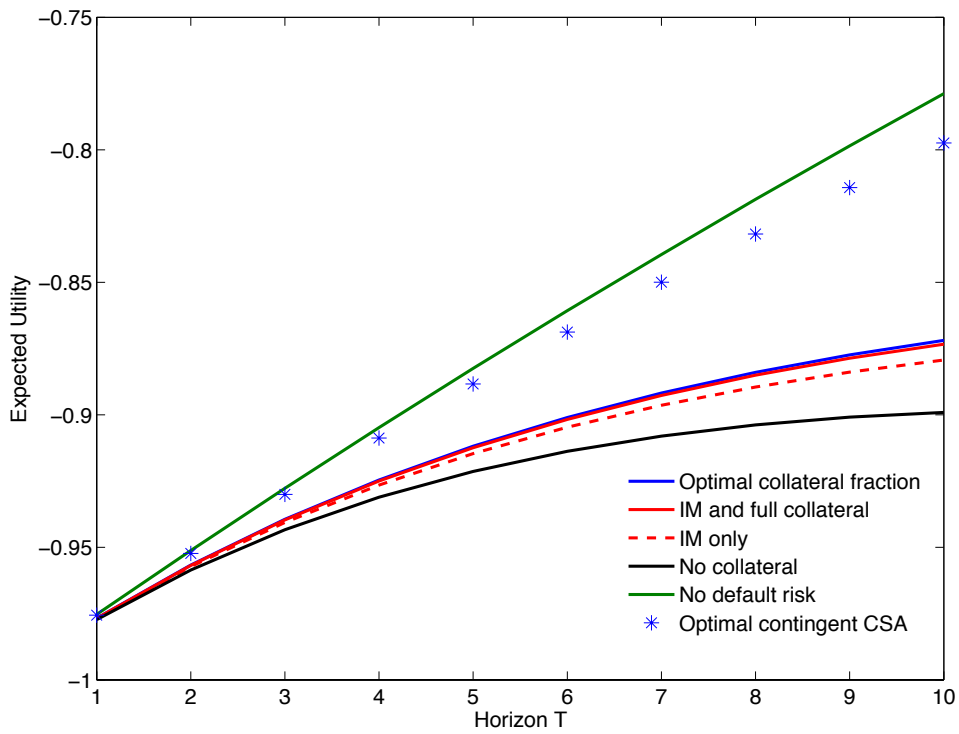


Figure 6: Expected utilities for different cases: no counterparty risk, counterparty risk and no collateral, initial margin (IM) only, IM and variation margin, optimal collateral  $\sigma_{k^*}^*$ , and optimal contingent collateral (optimal CSA), for the baseline parameters  $\gamma = 0.2, \lambda = 5\%, \sigma_Z = 20\%, r = 3\%, \mu = 6\%, \sigma_S = 20\%$  and different maturities.

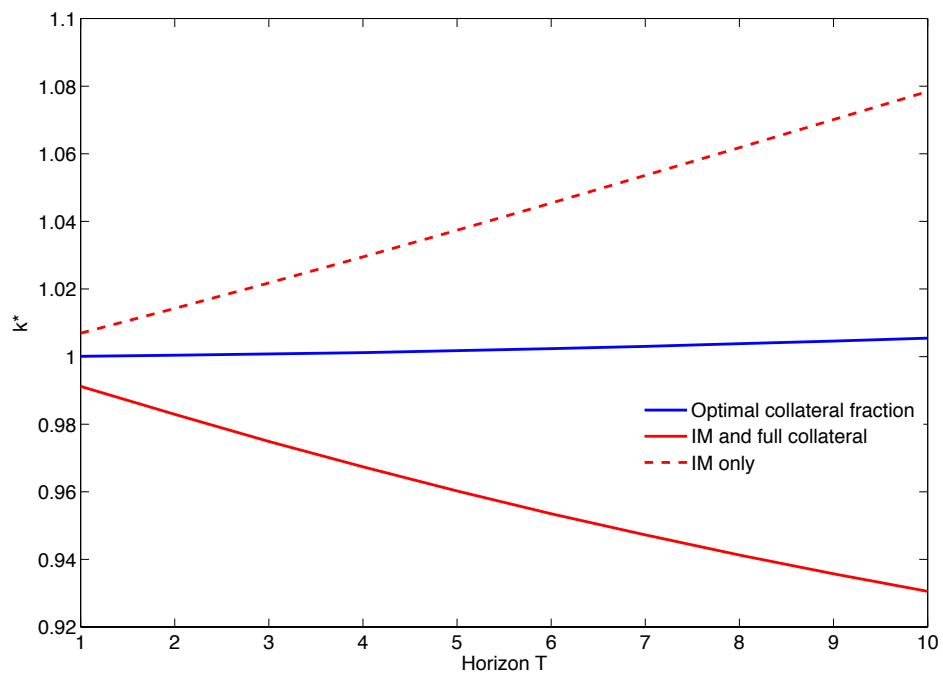


Figure 7: Optimal trading volume in the case of IM and variation margin, IM only, and optimal collateral, for the baseline parameters  $\gamma = 0.2$ ,  $\lambda = 5\%$ ,  $\sigma_Z = 20\%$ ,  $r = 3\%$ ,  $\mu = 6\%$ ,  $\sigma_S = 20\%$  and different maturities.



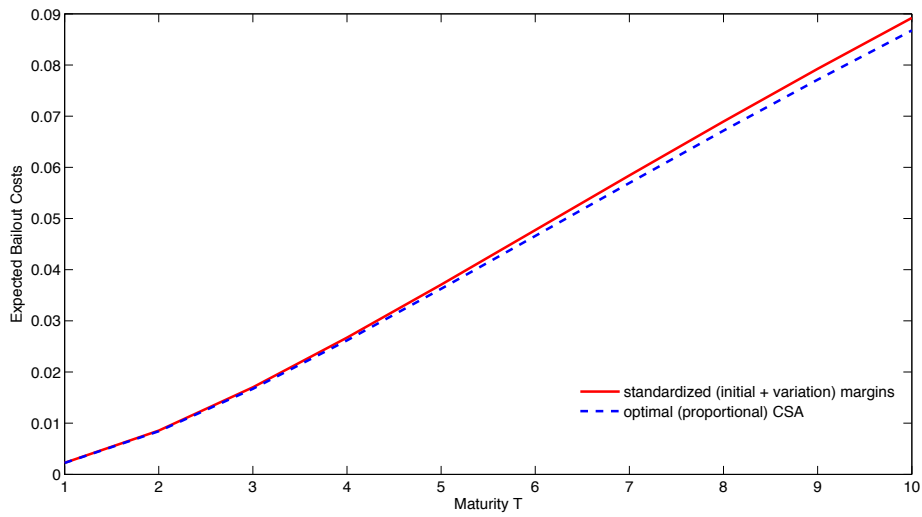


Figure 8: Expected bailout costs (4.2): the case of optimal (proportional) CSA and standardized (initial and variation) margins, for the baseline parameters  $\gamma = 0.2$ ,  $\lambda = 5\%$ ,  $\sigma_Z = 20\%$ ,  $r = 3\%$ ,  $\mu = 6\%$ ,  $\sigma_S = 20\%$  and different maturities.

## B Technical Appendix

### B.1 Results without risk sharing

*Proof of Proposition 2.1.* Consider the position of agent A:

$$\begin{aligned}
v(t, W_t^A, Z_t) &= \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} \lambda U(W_s^A e^{r(T-s)} - Z_T) ds \right] + \mathbb{E}_t \left[ e^{-\lambda(T-t)} U(W_T^A - Z_T) \right] \\
&= -\mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} \frac{\lambda}{\gamma} e^{-\gamma W_s^A e^{r(T-s)}} e^{\gamma Z_T} ds \right] - \frac{1}{\gamma} \mathbb{E}_t \left[ e^{-\lambda(T-t)} e^{-\gamma W_T^A} e^{\gamma Z_T} \right] \\
&= -\frac{\lambda}{\gamma} e^{\frac{1}{2}\gamma^2 \sigma_Z^2 (T-t) + \gamma Z_t} \mathbb{E}_t \left[ \int_t^T e^{-\lambda(s-t)} e^{-\gamma W_s^A e^{r(T-s)}} ds \right] \\
&\quad - \frac{1}{\gamma} e^{\frac{1}{2}\gamma^2 \sigma_Z^2 (T-t) + \gamma Z_t} \mathbb{E}_t \left[ e^{-\lambda(T-t)} e^{-\gamma W_T^A} \right],
\end{aligned}$$

where  $v(T, x, z) = -\frac{1}{\gamma} e^{-\gamma(x-z)}$ . Define

$$\begin{aligned}
\tilde{v}(t, W_t^A) &= e^{-\frac{1}{2}\gamma^2 \sigma_Z^2 (T-t) - \gamma Z_t - \lambda t} v(t, W_t^A, Z_t) \\
&= -\frac{\lambda}{\gamma} \mathbb{E}_t \left[ \int_t^T e^{-\lambda s - \gamma W_s^A e^{r(T-s)}} ds \right] - \frac{1}{\gamma} \mathbb{E}_t \left[ e^{-\lambda T - \gamma W_T^A} \right],
\end{aligned}$$

where  $\tilde{v}(T, x) = -\frac{1}{\gamma} e^{-\lambda T - \gamma x}$ . Hence, the HJB equation for problem (2.2) is

$$\tilde{v}_t + \sup_{\pi} \left\{ -\frac{\lambda}{\gamma} e^{-\lambda t - \gamma x e^{r(T-t)}} + r x \tilde{v}_x + x \pi (\mu - r) \tilde{v}_x + \frac{1}{2} x^2 \pi^2 \sigma_S^2 \tilde{v}_{xx} \right\} = 0.$$

From the first order condition, we obtain

$$\pi^* = -\frac{(\mu - r)}{\sigma_S^2} \frac{\tilde{v}_x}{x \tilde{v}_{xx}}.$$

Therefore, the HJB equation becomes

$$\begin{cases} \tilde{v}_t - \frac{\lambda}{\gamma} e^{-\lambda t - \gamma x e^{r(T-t)}} + r x \tilde{v}_x - \frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} \frac{\tilde{v}_x^2}{\tilde{v}_{xx}} = 0, \\ \tilde{v}(T, x) = -\frac{1}{\gamma} e^{-\lambda T - \gamma x}. \end{cases} \quad (\text{B.1})$$

The solution for (B.1) is:

$$\tilde{v}(t, x) = -\frac{1}{\gamma} e^{-\gamma x e^{r(T-t)}} \left( e^{-\frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} (T-t)} (1 - \alpha) e^{-\lambda T} + \alpha e^{-\lambda t} \right), \quad (\text{B.2})$$

where

$$\alpha = \frac{\lambda}{\lambda + \frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2}}.$$

And therefore,

$$\pi_t^* X_t = \frac{\mu - r}{\gamma \sigma_S^2} e^{-r(T-t)}. \quad (\text{B.3})$$

Then,

$$v(t, x, z) = e^{\frac{1}{2} \gamma^2 \sigma_Z^2 (T-t) + \gamma z + \lambda t} \left( -\frac{1}{\gamma} e^{-\gamma x e^{r(T-t)}} \left( (1 - \alpha) e^{-\frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2} (T-t) - \lambda T} + \alpha e^{-\lambda t} \right) \right).$$

Thus,

$$v(t, x, z) = -\frac{1}{\gamma} \exp \left\{ -\gamma x e^{r(T-t)} + \gamma z + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-t) \right\} \times \left( (1 - \alpha) e^{-\frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2} (T-t) - \lambda(T-t)} + \alpha \right), \quad (\text{B.4})$$

where  $\alpha = \frac{\lambda}{\lambda + \frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2}}$ , which after rearranging yields (2.4).  $\square$

## B.2 Results with risk sharing and default

We need the following standard lemma, which is easily verified by solving the corresponding integrals:

**Lemma B.1.** *For  $Z \sim N(m, \psi^2)$ , we have:*

$$\mathbb{E} [\exp\{a Z^+ - b Z^-\}] = e^{m a + \frac{1}{2} a^2 \psi^2} \Phi \left( \frac{m}{\psi} + a \psi \right) + e^{m b + \frac{1}{2} b^2 \psi^2} \left( 1 - \Phi \left( \frac{m}{\psi} + b \psi \right) \right),$$

where  $\Phi(\cdot)$  is the standard Normal cumulative distribution function.

We immediately focus on the situation with  $k$  forward contracts. Assume the collateral amount  $C_t = -c_t k e^{-r(T-t)} Z_t^i$ , where  $c_t$  is a differentiable function. It follows—just like in the more general case treated in Appendix B.3—that the optimal investment strategy is given by

$$\pi_t^* = \frac{(\mu - r)}{\sigma_S^2 \gamma} e^{-r(T-t)},$$

and we have for agent  $i$ 's wealth on  $t < \tau = \min\{\tau^A, \tau^B\}$ :

$$dW_t^i = r W_t^i dt + \pi_t^* [(\mu - r) dt + \sigma_S dB_t^S] - \overbrace{(c_t' k e^{-r(T-t)} Z_t^i dt + c_t k e^{-r(T-t)} dZ_t^i)}^{+dC_t^i - r C_t^i dt}.$$

We obtain:

**Lemma B.2.** For  $u \geq t \geq 0$ :

$$W_t^i = \exp\{rt\} \left[ W_0 + \frac{(\mu - r)^2}{\sigma_S^2 \gamma} e^{-rT} t + \frac{(\mu - r)}{\sigma_S \gamma} e^{-rT} B_t^S - e^{-rT} k c_t Z_t^i \right]$$

and  $W_u^i = \exp\{r(u - t)\} \left[ W_t^i + \frac{(\mu - r)^2}{\sigma_S^2 \gamma} e^{-r(T-t)} (u - t) + \frac{(\mu - r)}{\sigma_S \gamma} e^{-r(T-t)} (B_u^S - B_t^S) - e^{-r(T-t)} k (c_s Z_s^i - c_t Z_t^i) \right].$

Without loss of generality, we focus on agent A. Then:

$$\begin{aligned} & v(t, W_t^A, Z_t) \\ &= \mathbb{E}_t [u (W_T^A + (k \mathbf{1}_{\tau > T} - 1) Z_T)] \\ &= -\mathbb{E}_t \left[ \int_t^T e^{-2\lambda(s-t)} \frac{\lambda}{\gamma} \mathbb{E}_s \left[ \exp \left\{ -\gamma \left( W_s^A + k e^{-r(T-s)} (Z_s^+ - c_s Z_s^+) \right) e^{r(T-s)} + \gamma Z_T \right\} \right] ds \right. \\ &\quad \left. - \int_t^T e^{-2\lambda(s-t)} \frac{\lambda}{\gamma} \exp \left\{ -\gamma \left( W_s^A - k e^{-r(T-s)} (Z_s^- - c_s Z_s^-) \right) e^{r(T-s)} + \gamma Z_s + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s) \right\} \right. \\ &\quad \left. \times \left\{ \alpha + (1 - \alpha) \exp \left\{ -\left( \frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} + \lambda \right) (T-s) \right\} \right\} ds \right] \\ &\quad - \frac{1}{\gamma} \mathbb{E}_t \left[ e^{-2\lambda(T-t)} \exp \left\{ -\gamma W_T^A - \gamma (k - 1) Z_T + \gamma k c_T Z_T \right\} \right], \end{aligned}$$

where we used Proposition 2.1. Therefore, with Lemma B.2:

$$\begin{aligned} & v(t, W_t^A, Z_t) \\ &= - \int_t^T e^{-2\lambda(s-t)} \frac{\lambda}{\gamma} \mathbb{E}_t \left[ \exp \left\{ -\gamma \left( W_t^A e^{r(T-t)} + \frac{(\mu - r)^2}{\sigma_S^2 \gamma} (s-t) + \frac{(\mu - r)}{\sigma_S \gamma} (B_s^S - B_t^S) \right) \right. \right. \\ &\quad \left. \left. - \gamma k Z_s^+ - \gamma k c_s (Z_s - Z_s^+) + \gamma k c_t Z_t + \gamma Z_s + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s) \right\} \right. \\ &\quad \left. + \exp \left\{ -\gamma \left( W_t^A e^{r(T-t)} + \frac{(\mu - r)^2}{\sigma_S^2 \gamma} (s-t) + \frac{(\mu - r)}{\sigma_S \gamma} (B_s^S - B_t^S) \right) \right. \right. \\ &\quad \left. \left. + \gamma k Z_s^- - \gamma k c_s (Z_s + Z_s^-) + \gamma k c_t Z_t + \gamma Z_s + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s) \right\} \right. \\ &\quad \left. \times \left\{ \alpha + (1 - \alpha) \exp \left\{ -\left( \frac{1}{2} \left( \frac{(\mu - r)^2}{\sigma_S^2} + \lambda \right) (T-s) \right) \right\} \right\} ds \right] \\ &\quad - \frac{1}{\gamma} e^{-2\lambda(T-t)} \mathbb{E} \left[ \exp \left\{ -\gamma W_t^A e^{r(T-t)} - \frac{(\mu - r)^2}{\sigma_S^2} (T-t) - \frac{(\mu - r)}{\sigma_S} (B_T^S - B_t^S) \right. \right. \\ &\quad \left. \left. - \gamma k c_T Z_T + \gamma k c_t Z_t - \gamma (k - 1) Z_T + \gamma k c_T Z_T \right\} \right] \end{aligned}$$

$$\begin{aligned}
&= -\frac{\exp\{-\gamma W_t e^{r(T-t)} + \gamma k c_t Z_t\}}{\gamma} \times \\
&\quad \left[ \int_t^T e^{-2\lambda(s-t)} \lambda \mathbb{E} \left[ \exp \left\{ -\frac{(\mu-r)^2}{\sigma_S^2} (s-t) + \frac{(\mu-r)}{\sigma_S} (B_s^S - B_t^S) \right\} \right] \right. \\
&\quad \times \mathbb{E} \left[ \exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s) \right\} \times \left( \exp \{-\gamma k Z_s^+ + \gamma Z_s + \gamma k c_s Z_s^-\} \right. \right. \\
&\quad \quad \left. \left. + \exp \{\gamma k Z_s^- + \gamma Z_s - \gamma k c_s Z_s^+\} \right) \right. \\
&\quad \quad \left. \times \left( \alpha + (1-\alpha) \exp \left\{ -\left( \frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2} + \lambda \right) (T-s) \right\} \right) \right] ds \\
&\quad + e^{-2\lambda(T-t)} \mathbb{E} \left[ \exp \left\{ -\frac{(\mu-r)^2}{\sigma_S^2} (T-t) - \frac{(\mu-r)}{\sigma_S} (B_T^S - B_t^S) - \gamma(k-1) Z_T \right\} \right] \Big] \\
&= -\frac{\exp\{-\gamma W_t e^{r(T-t)} + \gamma k c_t Z_t\}}{\gamma} \times \\
&\quad \left[ \int_t^T e^{-2\lambda(s-t)} \lambda \exp \left\{ -\frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2} (s-t) + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s) \right\} \right. \\
&\quad \quad \left( \mathbb{E} [\exp \{\gamma(1-k) Z_s^+ - \gamma(1-k c_s) Z_s^-\}] \right. \\
&\quad \quad \left. + \mathbb{E} [\exp \{\gamma(1-k c_s) Z_s^+ - \gamma(1-k) Z_s^-\}] \times h(s) \right) ds \\
&\quad \left. + e^{-2\lambda(T-t)} \exp \left\{ -\frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2} (T-t) \right\} \mathbb{E} [\exp \{-\gamma(k-1) Z_T\}] \right],
\end{aligned}$$

where

$$h(s) = \alpha + (1-\alpha) \exp \left( -\left( \frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2} + \lambda \right) (T-s) \right).$$

Now  $Z_s|Z_t \sim N(Z_t, \sigma_Z^2(s-t))$  and  $Z_T|Z_t \sim N(Z_t, \sigma_Z^2(T-t))$  so that we obtain with Lemma B.1:

$$\begin{aligned}
&v(t, W_t^A, Z_t) \tag{B.5} \\
&= -\frac{\exp\{-\gamma W_t e^{r(T-t)} + \gamma k c_t Z_t\}}{\gamma} \left[ \int_t^T e^{-2\lambda(s-t)} \lambda \exp \left\{ -\frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2} (s-t) + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s) \right\} \right. \\
&\quad \left( \exp \{\gamma(1-k) Z_t + 1/2 \gamma^2 (1-k)^2 \sigma_Z^2 (s-t)\} \Phi \left( \frac{Z_t}{\sigma_Z \sqrt{s-t}} + \gamma(1-k) \sqrt{s-t} \right) \right. \\
&\quad \quad \left. + \exp \{\gamma(1-k c_s) Z_t + 1/2 \gamma^2 (1-c_s k)^2 \sigma_Z^2 (s-t)\} \left( 1 - \Phi \left( \frac{Z_t}{\sigma_Z \sqrt{s-t}} + \gamma(1-c_s k) \sqrt{s-t} \right) \right) \right) \\
&\quad + \left( \exp \{\gamma(1-k c_s) Z_t + 1/2 \gamma^2 (1-k c_s)^2 \sigma_Z^2 (s-t)\} \Phi \left( \frac{Z_t}{\sigma_Z \sqrt{s-t}} + \gamma(1-k c_s) \sqrt{s-t} \right) \right. \\
&\quad \quad \left. + \exp \{\gamma(1-k) Z_t + 1/2 \gamma^2 (1-k)^2 \sigma_Z^2 (s-t)\} \left( 1 - \Phi \left( \frac{Z_t}{\sigma_Z \sqrt{s-t}} + \gamma(1-k) \sqrt{s-t} \right) \right) \right) h(s) ds \\
&\quad \left. + e^{-2\lambda(T-t)} \exp \left\{ -\frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2} (T-t) \right\} \mathbb{E} [\exp \{-\gamma(k-1) Z_t + 1/2 \gamma^2 (k-1)^2 \sigma_Z^2 (T-t)\}] \right].
\end{aligned}$$

*Proof of Proposition 3.1.* Using Equation (B.5), setting  $c = 0$ , and reordering yields the result.  $\square$

## B.2.1 Optimal Fixed Collateralization

With  $t = 0$  in Equation (B.5), we obtain:

$$\begin{aligned} V_0 = & -\frac{1}{\gamma} \exp\left(-\gamma W_0 e^{rT} + \frac{1}{2}\gamma^2 \sigma_Z^2 T\right) \left[ \int_0^T e^{-2\lambda s} \lambda \exp\left\{-\frac{1}{2}\frac{(\mu-r)^2}{\sigma_S^2} s\right\} \right. \\ & \times \left[ \exp\left\{\gamma^2 \sigma_Z^2 \left(\frac{1}{2}s k^2 - k s\right)\right\} \times [h(s) + [1-h(s)] \Phi(\gamma \sigma_Z \sqrt{s} - k \gamma \sigma_Z \sqrt{s})] \right. \\ & + \exp\left\{\gamma^2 \sigma_Z^2 \left(\frac{1}{2}k^2 c_s^2 s - k s c_s\right)\right\} \times [h(s) + [1-h(s)] \Phi(k \gamma \sigma_Z c_s \sqrt{s} - \gamma \sigma_Z \sqrt{s})] \left. \right] ds \\ & \left. + e^{-2\lambda T} \exp\left\{-\frac{1}{2}\frac{(\mu-r)^2}{\sigma_S^2} T + \frac{1}{2}\gamma^2 \sigma_Z^2 T (k^2 - 2k)\right\} \right], \end{aligned}$$

and the optimization problem is maximizing this expression in  $c$ .

Ignoring constant factors, we can represent the optimization problem as:

$$\min_c \int_0^T g(t, c_t) dt + V(T), \quad (\text{B.6})$$

where

$$\begin{aligned} g(t, c_t) = & e^{-2\lambda t} \lambda \exp\left\{-\frac{1}{2}\frac{(\mu-r)^2}{\sigma_S^2} t\right\} \\ & \times \left( \exp\left\{\gamma^2 \sigma_Z^2 \left(\frac{1}{2}t k^2 - k t\right)\right\} \times [h(t) + [1-h(t)] \Phi(\gamma \sigma_Z \sqrt{t} - k \gamma \sigma_Z \sqrt{t})] \right. \\ & \left. + \exp\left\{\gamma^2 \sigma_Z^2 \left(\frac{1}{2}k^2 c_t^2 t - k t c_t\right)\right\} \times [h(t) + [1-h(t)] \Phi(k \gamma \sigma_Z c_t \sqrt{t} - \gamma \sigma_Z \sqrt{t})] \right) \end{aligned}$$

and

$$V(T) = e^{-2\lambda T} \exp\left\{-\frac{1}{2}\frac{(\mu-r)^2}{\sigma_S^2} T + \frac{1}{2}\gamma^2 \sigma_Z^2 T (k^2 - 2k)\right\}.$$

The Hamilton-Jacobi-Bellman Equation for the optimal value function reads:

$$0 = \frac{\partial}{\partial t} V(t) + \min_{c_t} \{g(t, c_t)\}. \quad (\text{B.7})$$

*Proof of Proposition 3.2.* Again we treat the more general case with  $k$  forward contracts. The necessary condition for the optimal collateral rule at time  $t$  amounts to optimizing the function

$g$  where:

$$g(t, c_t) = e^{-2\lambda t} \lambda \exp \left\{ -\frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} t \right\} \times \left[ \exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 t (k^2 - 2k) \right\} \right. \\ \left. \times \left[ h(t) + [1 - h(t)] \Phi \left( \gamma \sigma_Z \sqrt{t} (1 - k) \right) \right] \right. \\ \left. + \exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 t (k^2 c_t^2 - 2k c_t) \right\} \left[ h(t) + [1 - h(t)] \Phi \left( \gamma \sigma_Z \sqrt{t} (k c_t - 1) \right) \right] \right].$$

Taking first order condition yields:

$$0 = \frac{1}{2} \gamma^2 \sigma_Z^2 t (k^2 2c_t - 2k) \exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 t (k^2 c_t^2 - 2k c_t) \right\} \\ \times \left[ h(t) + [1 - h(t)] \Phi \left( \gamma \sigma_Z \sqrt{t} (k c_t - 1) \right) \right] \\ + \exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 t (k^2 c_t^2 - 2k c_t) \right\} \gamma \sigma_Z \sqrt{t} k \phi \left( \gamma \sigma_Z \sqrt{t} (k c_t - 1) \right) [1 - h(t)] \\ \Leftrightarrow 0 = \gamma \sigma_Z (k c_t - 1) \left[ h(t) + [1 - h(t)] \Phi \left( \gamma \sigma_Z \sqrt{t} (k c_t - 1) \right) \right] \\ + (1 - h(t)) \phi \left( \gamma \sigma_Z \sqrt{t} (k c_t - 1) \right) \\ \Leftrightarrow \frac{-h(t)}{1 - h(t)} = g \left( \gamma \sigma_Z \sqrt{t} (k c_t - 1) \right),$$

where  $g(x) = \Phi(x) + \frac{\phi(x)}{x}$ . Therefore,

$$g^{-1} \left( \frac{-h(t)}{1 - h(t)} \right) = \gamma \sigma_Z \sqrt{t} (k c_t - 1) \\ \Leftrightarrow c_t = \frac{1}{k} + \frac{1}{\gamma \sigma_Z k \sqrt{t}} g^{-1} \left( -\frac{\alpha + (1 - \alpha) \exp \left\{ -\left( \frac{1}{2} \frac{(\mu - r)^2}{\sigma_Z^2} + \lambda \right) (T - t) \right\}}{(1 - \alpha) \left( 1 - \exp \left\{ -\left( \frac{1}{2} \frac{(\mu - r)^2}{\sigma_Z^2} + \lambda \right) (T - t) \right\} \right)} \right) \\ \Leftrightarrow c_t = \frac{1}{k} + \frac{1}{\gamma \sigma_Z k \sqrt{t}} g^{-1} \left( -\frac{\lambda + \frac{1}{2} s^2 \exp \left\{ -\left( \frac{1}{2} s^2 + \lambda \right) (T - t) \right\}}{\frac{1}{2} s^2 \left( 1 - \exp \left\{ -\left( \frac{1}{2} s^2 + \lambda \right) (T - t) \right\} \right)} \right), \quad (\text{B.8})$$

which after rearranging yields (3.6) completing the proof.  $\square$

*Proof of Proposition 3.3.* Since  $g(x) < x$  for  $x < 0$ , the sensitivities to  $\sigma_Z$  and  $\gamma$  follow immediately from (B.9).

For the sensitivity with respect to  $\lambda$ , note that:

$$\frac{\partial c_t}{\partial \lambda} = \frac{1}{k \sigma_Z \gamma \sqrt{t}} \frac{(g^{-1})^2}{\phi(g^{-1})} \frac{1 - \left( 1 + \left( \frac{1}{2} s^2 + \lambda \right) (T - t) \right) \exp \left\{ -\left( \frac{1}{2} s^2 + \lambda \right) (T - t) \right\}}{\frac{1}{2} s^2 \left( 1 - \exp \left\{ -\left( \frac{1}{2} s^2 + \lambda \right) (T - t) \right\} \right)},$$

which is positive since  $(1 + x)e^{-x} \leq 1$  for  $x \geq 0$ .

Finally, it is easy to verify that  $\frac{\partial c_t}{\partial s}$  is negative.  $\square$

*Proof of Proposition 3.5 and 3.6.* We have:

$$\begin{aligned}
\frac{\partial V_0}{\partial k} &= \int_0^T e^{-2\lambda s} \lambda \exp \left\{ -\frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} s \right\} \\
&\quad \left[ \gamma^2 \sigma_Z^2 s (k - 1) \exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 s (k^2 - 2k) \right\} [h(s) + [1 - h(s)] \Phi(\gamma \sigma_Z \sqrt{s}(1 - k))] \right. \\
&\quad \left. + \gamma^2 \sigma_Z^2 s (k c_s^2 - c_s) \exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 s (k^2 c_s^2 - 2k c_s) \right\} [h(s) + [1 - h(s)] \Phi(\gamma \sigma_Z \sqrt{s}(k c_s - 1))] \right] ds \\
&\quad + (c_s - 1) \gamma \sigma_Z \sqrt{s} \phi(\gamma \sigma_Z \sqrt{s}) [1 - h(s)] \\
&\quad + e^{-2\lambda T} \exp \left\{ -\frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} T \right\} \gamma^2 \sigma_Z^2 T (k - 1) \exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 T (k^2 - 2k) \right\}
\end{aligned}$$

since the other derivatives cancel. Thus, for  $c \equiv 1$ , we have  $k = 1$  is optimal whereas for  $c < 1$ , we have  $k > 1$ . Similarly, the first order condition for fixed  $c$  shows the optimal  $c < 1$ .

Now assume optimal deterministic collateralization. All that is left to prove is that the optimal trading volume  $k > 1$ . We have:

$$\begin{aligned}
&c_s \gamma^2 \sigma_Z^2 s (k c_s - 1) \exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 s (k^2 c_s^2 - 2k c_s) \right\} \\
&\quad \times [h(s) + [1 - h(s)] \Phi(\gamma \sigma_Z \sqrt{s}(k c_s - 1))] \\
&= -c_s \gamma \sigma_Z \sqrt{s} \varphi(\gamma \sigma_Z \sqrt{s}(k c_s - 1)) [1 - h(s)] \exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 s (k^2 c_s^2 - 2k c_s) \right\} \\
&= -c_s \gamma \sigma_Z [1 - h(s)] \phi(\gamma \sigma_Z \sqrt{s})
\end{aligned}$$

so that the first order condition for  $k$  reads:

$$\begin{aligned}
&(k - 1) \left[ \int_0^T e^{-2\lambda s} \lambda \exp \left\{ -\frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} s \right\} s \exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 s (k^2 - 2k) \right\} \right. \\
&\quad \left. \times [h(s) + [1 - h(s)] \Phi(\gamma \sigma_Z \sqrt{s}(k c_s - 1))] \right. \\
&\quad \left. + e^{-2\lambda T} \exp \left\{ -\frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} T \right\} T \exp \left\{ \frac{1}{2} \gamma^2 \sigma_Z^2 T (k^2 - 2k) \right\} \right] \gamma \sigma_Z \\
&= \int_0^T [1 - h(s)] \exp \left\{ -\frac{1}{2} \gamma^2 \sigma_Z^2 s \right\} ds,
\end{aligned}$$

which shows that  $k > 1$ . □

*Proof of Proposition 5.1.* We commence by considering the case  $\eta = 0$ . With the wealth dy-



namics (5.1) and collateral rules (5.2) we obtain for agent  $A$ :

$$\begin{aligned}
dW_t^A &= (1 - N_{t-}^A) [(rW_t^A + \pi_t^A(\mu - r)) dt + \pi_t^A \sigma_S dB_t^S] + N_t^A W_t^A r dt \\
&\quad + (1 - N_{t-}) \left[ - \left( d(C_t^A)^- - r(C_t^A)^- dt \right) + (R_{t-}^A)^+ dN_t^A \right. \\
&\quad \quad \left. + \left( (C_{t-}^A)^+ + (C_{t-}^A)^- - (R_{t-}^A)^- \right) dN_t^B \right] \\
&= (1 - N_{t-}^A) [(rW_t^A + \pi_t^A(\mu - r)) dt + \pi_t^A \sigma_S dB_t^S] + N_t^A W_t^A r dt \\
&\quad + (1 - N_{t-}) \left[ -e^{-r(T-t)} k d(c(t) Z_t)^- + e^{-r(T-t)} k (Z_t)^+ dN_t^A \right. \\
&\quad \quad \left. + e^{-r(T-t)} k (c(t) |Z_t| - (Z_t)^-) dN_t^B \right],
\end{aligned}$$

so that for  $t < \tau$ , we have:

$$W_t^A = e^{rt} \left[ W_0 + \frac{(\mu - r)^2}{\sigma_S^2 \gamma} e^{-rT} t + \frac{(\mu - r)^2}{\sigma_S^2 \gamma} e^{-rT} B_t^S - e^{-rT} k c(t) Z_t^- \right].$$

Hence, we have for the expected utility:

$$\begin{aligned}
V_0 &= \mathbb{E} [u(W_T^A + (k 1_{\tau > T} - 1) Z_T)] \\
&= -\mathbb{E} \left[ \int_0^T e^{-2\lambda s} \frac{\lambda}{\gamma} \mathbb{E}_s \left[ \exp \left\{ -\gamma \left( W_s^A + k e^{-r(T-s)} Z_s^+ \right) e^{r(T-s)} + \gamma Z_T \right\} \right] ds \right. \\
&\quad \left. - \int_0^T e^{-2\lambda s} \frac{\lambda}{\gamma} \exp \left\{ -\gamma \left( W_s^A - k e^{-r(T-s)} Z_s^- + c_s k e^{-r(T-s)} |Z_s| \right) \right\} e^{r(T-s)} \right. \\
&\quad \left. + \gamma Z_s + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s) \right\} h(s) ds \right] \\
&\quad - \frac{1}{\gamma} \mathbb{E} \left[ e^{-2\lambda T} \exp \left\{ -\gamma W_T - \gamma (k-1) Z_T - \gamma k c_T Z_T^- \right\} \right] \\
&= -\int_0^T e^{-2\lambda s} \frac{\lambda}{\gamma} \mathbb{E} \left[ \exp \left\{ -\gamma \left( W_0 e^{rT} + \frac{(\mu - r)^2}{\sigma_S^2 \gamma} s + \frac{(\mu - r)}{\sigma_S \gamma} B_s^S \right) \right. \right. \\
&\quad \left. \left. + \gamma k c_s Z_s^- - \gamma k Z_s^+ - + \gamma Z_s + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s) \right\} \right. \\
&\quad \left. + \exp \left\{ -\gamma \left( W_0 e^{rT} + \frac{(\mu - r)^2}{\sigma_S^2 \gamma} s + \frac{(\mu - r)}{\sigma_S \gamma} B_s^Z \right) \right. \right. \\
&\quad \left. \left. + \gamma k Z_s^- - \gamma k c_s Z_s^+ + \gamma Z_s + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s) \right\} h(s) ds \right. \\
&\quad \left. - \frac{1}{\gamma} e^{-2\lambda T} \mathbb{E} \left[ \exp \left\{ -\gamma W_0 e^{rT} - \frac{(\mu - r)^2}{\sigma_S^2} T - \frac{(\mu - r)}{\sigma_S} B_T^S \right. \right. \right. \\
&\quad \quad \left. \left. - \gamma (k-1) Z_T \right\} \right],
\end{aligned}$$

which is exactly of the same form as in the non-segregated case, which proves the claim for

$\eta = 0$ .

In case  $\eta > 0$ , the optimization problem becomes state-contingent since timing of collateral payments becomes material. In particular, the relevant states are related to the interest accrued on collateral accounts. The solution features less collateral than for  $\eta = 0$  since posting collateral is even less advantageous leading to less collateral. Details are available from the authors upon request.  $\square$

### B.3 Optimal Dynamic Collateralization

Again focussing on agent  $A$ , with any collateral specification such that  $(W_t^A, Z_t, C_t)$  is Markovian, we have:

$$\begin{aligned}
v(t, W_t^A, Z_t) &= \mathbb{E}_t [U^A (W_T^A - Z_T + k Z_T 1_{\{\tau > T\}})] \\
&= \mathbb{E}_t \left[ \int_t^T e^{-2\lambda(s-t)} \lambda \mathbb{E}_s \left[ U^A ((W_s^A + k Z_s^+ - C_s^+) e^{r(T-s)} - Z_T) \right] ds \right] \\
&\quad + \mathbb{E}_t \left[ \int_t^T e^{-2\lambda(s-t)} \lambda \mathbf{v}^{(1)}(s, W_s^A - k Z_s^- + C_s^-, Z_s) ds \right] \\
&\quad + \mathbb{E}_t \left[ e^{-2\lambda(T-t)} U^A (W_T^A + (k-1) Z_T) \right]
\end{aligned}$$

where  $\mathbf{v}^{(1)}$  is defined as in Proposition 3.1. Thus,

$$\begin{aligned}
e^{-2\lambda t} v(t, W_t^A, Z_t, C_t) &= -\frac{\lambda}{\gamma} \mathbb{E}_t \left[ \int_t^T e^{-2\lambda s} e^{-\gamma(W_s^A + k Z_s^+ - C_s^+) e^{r(T-s)} + \gamma Z_s + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s)} ds \right] \\
&\quad - \frac{\lambda}{\gamma} \mathbb{E}_t \left[ \int_t^T e^{-2\lambda s} e^{-\gamma(W_s^A - k Z_s^- + C_s^-) e^{r(T-s)} + \gamma Z_s + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s)} \right. \\
&\quad \quad \quad \left. \times \left( (1-\alpha) e^{-\frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2} (T-s) - \lambda(T-s)} + \alpha \right) ds \right] \\
&\quad - \frac{1}{\gamma} \mathbb{E}_t \left[ e^{-2\lambda T} e^{-\gamma W_T^A - \gamma(k-1) Z_T} \right] \\
&= -\frac{\lambda}{\gamma} \mathbb{E}_t \left[ \int_t^T e^{-2\lambda s} e^{\gamma Z_s + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s)} \right. \\
&\quad \quad \times \left( e^{-\gamma(W_s^A + k Z_s^+ - C_s^+) e^{r(T-s)}} + \alpha e^{-\gamma(W_s^A - k Z_s^- + C_s^-) e^{r(T-s)}} \right. \\
&\quad \quad \quad \left. \left. + (1-\alpha) e^{-\gamma(W_s^A - k Z_s^- + C_s^-) e^{r(T-s)} - \frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2} (T-s) - \lambda(T-s)} \right) ds \right] \\
&\quad - \frac{1}{\gamma} \mathbb{E}_t \left[ e^{-2\lambda T} e^{-\gamma W_T^A - \gamma(k-1) Z_T} \right]. \tag{B.9}
\end{aligned}$$

We now start by considering a more general collateral specification that nests the speci-

cations from Sections 3.2 and 3.3 (again from the perspective from party A):

$$\begin{aligned} C_t^A &= k e^{-r(T-t)} c_t^{(1)} \int_0^t c^{(2)}(s, W_s^A, Z_s, C_s^A) dZ_s \\ \Rightarrow dC_t^A &= r C_t^A dt + \frac{(c_t^{(1)})'}{c_t^{(1)}} C_t^A dt + k e^{-r(T-t)} c_t^{(1)} c^{(2)}(t, W_t^A, Z_t, C_t^A) dZ_t. \end{aligned}$$

Hence,  $\tilde{v}(t, x, z, y) = -\gamma e^{-\lambda t} v(t, x, z, y)$  satisfies the following HJB equation:

$$\begin{aligned} 0 &= \tilde{v}_t + \inf_{\pi, c^1, c^2} \left\{ \lambda \exp \left\{ -2\lambda t + \gamma z + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s) - \gamma(x + k e^{-r(T-s)} z^+ - y^+) e^{r(T-s)} \right\} \right. \\ &\quad \left. + \lambda h(s) \exp \left\{ -2\lambda t + \gamma z + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s) - \gamma(x - k e^{-r(T-s)} z^- + y^-) e^{r(T-s)} \right\} \right. \\ &\quad \left. + r x \tilde{v}_x + \pi (\mu - r) \tilde{v}_x + \frac{1}{2} \pi^2 \sigma_S^2 \tilde{v}_{xx} + \frac{(c_t^{(1)})'}{c_t^{(1)}} y \tilde{v}_x \right. \\ &\quad \left. + \frac{1}{2} k^2 e^{-2r(T-t)} (c_t^{(1)})^2 (c^{(2)}(t, x, z, y))^2 \sigma_z^2 \tilde{v}_{xx} + \frac{1}{2} \sigma_Z^2 \tilde{v}_{zz} + k e^{-r(T-t)} c_t^{(1)} c^{(2)}(t, x, z, y) \sigma_Z^2 \tilde{v}_{zx} \right. \\ &\quad \left. + \frac{1}{2} k^2 e^{-2r(T-t)} c_t^{(1)} c^{(2)}(t, x, z, y) \sigma_Z^2 \tilde{v}_{yy} + k e^{-r(T-t)} c_t^{(1)} c^{(2)}(t, x, z, y) \sigma_Z^2 \tilde{v}_{zy} \right. \\ &\quad \left. + k^2 e^{-2r(T-t)} (c_t^{(1)})^2 (c^{(2)}(t, x, z, y))^2 \sigma_Z^2 \tilde{v}_{xy} \right\}. \end{aligned}$$

Then,  $\tilde{v}$  has the form:

$$\tilde{v} = \lambda \exp \left\{ -2\lambda t - \gamma x e^{r(T-t)} + \frac{1}{2} \gamma^2 k^2 \sigma_Z^2 (T-t) \right\} \Theta(t, z, y)$$

and, thus, we obtain for the optimal investment

$$\pi^* = -\frac{(\mu - r) \tilde{v}_x}{\sigma_S^2 \tilde{v}_{xx}} = \frac{(\mu - r)}{\sigma_S^2 \gamma} e^{-r(T-t)},$$

and we have that the optimal collateral rules are independent of wealth proving the first part of Proposition 3.4.

We now consider the specification from Section 3.3:

$$C_t^A = k e^{-r(T-t)} \underbrace{\int_0^t c(s, W_s, C_s) dZ_s}_{=\tilde{C}_t},$$

which together with the optimal asset allocation rule yields:

$$W_t^A = \exp\{rt\} W_0 + e^{-r(T-t)} \frac{(\mu - r)^2}{\sigma_S^2 \gamma} t + \frac{(\mu - r)}{\sigma_S \gamma} e^{-r(T-t)} B_t^S + C_t^A.$$

Hence, from (B.10) we obtain:

$$\begin{aligned} \gamma e^{-2\lambda t} v(t, W_t^A, Z_t, C_t) &= -e^{\gamma W_0} e^{rT} \mathbb{E}_t \left[ \int_t^T \lambda e^{-2\lambda s + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-s) - \frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2} s} \right. \\ &\quad \times \left( \exp \left\{ \gamma Z_s - k \gamma (Z_s^+ - \tilde{C}_s^-) \right\} + \exp \left\{ \gamma Z_s - k \gamma (\tilde{C}_s^+ - Z_s^-) \right\} h(s) \right) ds \\ &\quad \left. + \exp \left\{ -2\lambda T - \frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2} T - \gamma(k-1) Z_T \right\} \right], \end{aligned}$$

where

$$d\tilde{C}_t = c(t, Z_t, \tilde{C}_t) dZ_t.$$

The corresponding HJB equation for  $\tilde{v}(t, z, y)$  is:

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t} + \inf_c \left\{ \lambda \exp \left\{ -2\lambda t + \frac{1}{2} \gamma^2 \sigma_Z^2 (T-t) - \frac{1}{2} \frac{(\mu-r)^2}{\sigma_S^2} t \right\} \right. \\ &\quad \times \left( e^{\gamma z - k \gamma (z^+ - y^-)} + e^{\gamma z - k \gamma (y^+ - z^-)} h(t) \right) \\ &\quad \left. + \frac{1}{2} c^2(t, z, y) \sigma_Z^2 v_{yy} + \frac{1}{2} \sigma_z^2 v_{zz} + \sigma_z^2 c(t, z, y) v_{zy} \right\}. \end{aligned} \quad (\text{B.10})$$

However, collateral has to evolve in parallel for both agents (in opposite directions). Due to the symmetry between the agents, reflected paths of  $Z$  yield identical collateral requirements (with opposite sign), leading to the restriction:

$$c(t, z, y) = c(t, -z, -y),$$

so that the optimization in the HJB equation considers reflected states simultaneously. Since these paths have identical probability and the agents are identical, positive and negative states are weighted equally. Hence, the joint order condition for the simultaneous optimization is:

$$\begin{aligned} 0 &= c(t, z, y) (\tilde{v}_{yy}(t, z, y) + \tilde{v}_{yy}(t, -z, -y)) + (\tilde{v}_{zy}(t, z, y) + \tilde{v}_{zy}(t, -z, -y)) \\ \Rightarrow c(t, z, y) &= -\frac{\tilde{v}_{zy}(t, z, y) + \tilde{v}_{zy}(t, -z, -y)}{\tilde{v}_{yy}(t, z, y) + \tilde{v}_{yy}(t, -z, -y)}, \end{aligned}$$

proving the second assertion of Proposition 3.4.

Note that then equation (B.11) becomes a *differential equation with deviating arguments* since the optimal collateral fraction interlinks positive and negative arguments. We solve the equation with an explicit finite difference scheme.