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Estimating Nonlinear Dynamic Equilibrium Economies: A Likelihood Approach

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Abstract: This paper presents a framework to undertake likelihood-based inference in nonlinear dynamic equilibrium economies. The authors develop a sequential Monte Carlo algorithm that delivers an estimate of the likelihood function of the model using simulation methods. This likelihood can be used for parameter estimation and for model comparison. The algorithm can deal both with nonlinearities of the economy and with the presence of non-normal shocks. The authors show consistency of the estimate and its good performance in finite simulations. This new algorithm is important because the existing empirical literature that wanted to follow a likelihood approach was limited to the estimation of linear models with Gaussian innovations. The authors apply their procedure to estimate the structural parameters of the neoclassical growth model.

JEL classification: C63, C68, E37

Key words: dynamic equilibrium economies, likelihood function, nonlinear solution methods

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Estimating Nonlinear Dynamic Equilibrium Economies: A Likelihood Approach

1. Introduction

This paper presents a method to undertake likelihood based inference in nonlinear dynamic equilibrium models. We show how we can use sequential monte carlo methods to estimate the structural parameters of the model, those describing preferences and technology, and to compare different economies. Both tasks can be implemented from either a Bayesian or a classical perspective.

Economist now routinely use dynamic general equilibrium economies to answer quantitative questions. However they employ much less often formal econometrics to take these models to the data. Part of the reason might have been the shortcomings of existing tools. To estimates these economies, the empirical literature has been forced to use either limited-information moment methods or likelihood techniques on linearized versions of the model. This situation is unsatisfactory. Moment procedures may suffer from strong small samples biases and may not use efficiently all the existing information. Linearization techniques depend crucially on the shape of the true policy function being accurately approximated by a linear relation and on the presence of gaussian shocks.

The main obstacle for a more standard likelihood-based inference is the difficulty in evaluating the likelihood function implied by a nonlinear dynamic equilibrium economy. Beyond a few particular cases,[1] it is not possible to evaluate this function. Moment methods avoid the problem by moving away from full information approaches to inference. Linearization renounces evaluating the true likelihood function of the model and concentrates instead on the likelihood of an associated, more tractable, linear approximation to the economy.

We propose a Sequential Monte Carlo method to solve this problem. We describe how this technique can be applied to evaluate the likelihood function implied by the nonlinear solution of a dynamic equilibrium economy even if the driving shocks of the model are non-normal

[1] Some of these cases are, however, quite important. For example there is a wide literature on the estimation of dynamic discrete choice models that uses maximum likelihood methods. See Rust (1994) for a survey.

(although the algorithm is general enough that it can also deal with linear models with or without normal shocks).

To do so we borrow from a growing literature on nonlinear filtering (see the seminal paper by Gordon, Salmond and Smith, 1993 and the review of the literature in Doucet, de Freitas and Gordon, 2001). We adapt this know-how to deal with the likelihood functions of dynamic equilibrium models and we show how we get accurate and stable evaluations of the likelihood function. With these evaluations available, the door for likelihood-based inference is open, either by searching for a maximum of the function (Quasi-Maximum Likelihood estimation) or by simulating the posterior distribution of the parameters using a Markov Chain Monte Carlo algorithm (Bayesian estimation).

The general idea of the procedure is as follows. First, for a given set of parameter values, we compute the equilibrium policy functions of the model. Since we want to conduct inference in the nonlinear model and not in a linear approximation, we rely on a nonlinear solution method to find the policy functions. With the policy functions we construct the state space representation of the model. Under certain mild conditions, we use this state space form and a Sequential Monte Carlo scheme to evaluate the likelihood function. Plugging this likelihood evaluation algorithm into an optimization routine or a Markov Chain Monte Carlo we search the parameter space to perform likelihood-based inference, either maximizing the likelihood function or, after specifying some priors on the parameters, finding posterior distributions. Finally, if we applied the algorithm to several models, we can compare models using the output of the model either building likelihood ratios (Voung, 1989) or Bayes factors (Geweke, 1998) even if the models are misspecified and nonnested.

To illustrate our method we compute and estimate the benchmark dynamic equilibrium economy, the stochastic neoclassical growth model. After we solve the model nonlinearly, we estimate it using both bayesian and quasi-likelihood methods and we perform monte carlo analysis to evaluate the efficiency of our procedure.

Being able to perform likelihood based inference is important for several reasons. From a theoretical perspective, the likelihood principle states that all the empirical evidence obtained from the data is contained in the likelihood function (Berger and Wolpert, 1988). From an applied position, likelihood-based inference is a simple way to deal with misspecified models (Monfort, 1996). Dynamic equilibrium economies are false by construction, and likelihood-based inference has both attractive asymptotic properties and good small sample

behavior even when models are misspecified (White, 1994 for quasi-maximum likelihood and Fernández-Villaverde and Rubio-Ramírez, 2003a, for Bayesian procedures). Finally, and for us the most compelling reason, it is that likelihood inference lets us to compare models. Of course we do not want to imply that a likelihood approach is always to be preferred. For example we may only care about accounting for one particular dimension of the data, task for which a moment method can be more suitable. We just argue that in numerous contexts, the likelihood function is an informative tool.

Our paper builds on the existent literature dealing with inference on dynamic equilibrium economies. Hansen (1982) pioneered the use of moments methods that have been widely applied.[2] Sargent (1989) uses the Kalman filter to evaluate the likelihood function of linear or linearized dynamic equilibrium economies with normal shocks. Altuğ (1989), also in a linear framework, proposed to estimate the likelihood in the frequency domain. This spectral approach has been followed by Diebold, Ohanian and Berkowitz (1998) for estimation and by Watson (1993) to compare models with data. Christiano, Eichenbaum and Evans (2001) estimate dynamic equilibrium economies using the information in impulse-response functions of linearized solutions. Miranda and Rui (1997) exploit the structure of an asset pricing model to find the nonlinear likelihood function. Their method, however, involves the computation of the Jacobian of a transformation of variables and it is difficult to generalize. From a Bayesian perspective, DeJong, Ingram and Whiteman (2000) pioneered the Bayesian estimation of Real Business Cycles models using importance sampling. Landon-Lane (1999) and Otrok (2001) first applied the Markov Chain Monte Carlo methods. Schorfheide (2000) formulates the impulse-response approach in the Bayesian framework. All those papers though stay within the linear framework. We also build on the contributions of the literature on non-linear filtering. We delay the discussion of that literature until section 2.3.

The rest of the paper is organized as follows. In the next section we describe our general framework for likelihood based inference and shows the different steps involved in the evaluation of the likelihood function of the model for a given set of parameter values. Section 3

[2]Variations include the Simulated Method of Moments (Lee and Ingramm 1991), the Efficient Method of Moments (Gallant and Tauchen, 1996), Indirect Inference (Gourieroux, Monfort and Renault, 1993 and Smith, 1993) and several information-based approaches (Kitamura and Stutzer, 1997 and Imbens, Spady and Johnson, 1998). We refer the reader to the special issue of the *Journal of Business and Economic Statistics* on the Generalized Method of Moments, Ghysels and Hall (2002) for an overview of the literature.

present the stochastic neoclassical growth model and discuss how we can apply our sequential monte carlo procedure to it. Section 4 estimates the model both with simulated and with real data. Section 5 concludes and an appendix discuss computational details.

2. A Framework for Likelihood-Based Inference

In this section we develop a general framework to estimate and compare a large class of nonlinear dynamic equilibrium models using a likelihood approach. Examples of economies in this class are: the stochastic neoclassical growth model (Cooley and Prescott, 1995), sticky prices models (Chari, Kehoe and McGrattan, 2000, Rotemberg and Woodford, 1997 and Woodford, 2003), asset pricing models (Mehra and Prescott, 1985), macro public finance models (Chari, Christiano and Kehoe, 1994) and regime switching models (Quadrini and Jermann, 2003) among many others.

All of these economies imply a different joint probability distribution function for observables given the model's structural parameters- those describing preferences and technology. We call this density the likelihood function of the economy. The likelihood function is useful for two purposes. First, if we want to perform estimation, we can use an optimization routine to find the parameter values that maximize it or, if we specify a prior for the structural parameters, a Markov Chain Monte Carlo to draw from the posterior. Second, if we are considering several models, we can compare them either by building likelihood ratios (Voung, 1989) or Bayes factors (Geweke, 1998).

The literature shows how to write the likelihood function of a dynamic equilibrium economy only in a few special cases. For example we can evaluate the likelihood of a linear model with Gaussian innovations using the Kalman filter. Unfortunately there is no general procedure to write an analytic expression for this likelihood. As we discussed in the introduction, this problem has been a stumbling block to the application of likelihood methods to perform inference in dynamic equilibrium economies.

This section presents a sequential monte carlo method to address the problem of evaluating the likelihood function of a nonlinear dynamic equilibrium economy. The rest of the section is organized as follows. First we define the likelihood function of a dynamic equilibrium economy. Second we present a simulation filter to evaluate that likelihood. Finally we compare our approach with others existing proposals.

2.1. The Likelihood Function of a Dynamic Equilibrium Economy

A large set of dynamic equilibrium models can be written in the following state space form. First the equilibrium of economy is characterized by some states S_t that change over time according to the transition equation:

$$S_t = f\left(S_{t-1}, W_t; \gamma\right) \tag{1}$$

where $\{W_t\}$ is a sequence of exogenous independent random variables and $\gamma \in \Upsilon$ is the vector of structural parameters of the model.

Second the observables y_t are a realization of the random variable Y_t governed by the measurement equation:

$$Y_t = g\left(S_t, V_t; \gamma\right) \tag{2}$$

where $\{V_t\}$ is a sequence of exogenous independent random variables. The sequences $\{W_t\}$ and $\{V_t\}$ are independent of each other.[3] Along some dimension the function g can be the identity mapping if a state is directly observed without noise.

To summarize our notation: S_t are the states of the economy, W_t are the exogenous shocks that affect the states law of motion, Y_t are the observables, and V_t are the exogenous perturbation that affect the observables but not the states.

The functions f and g come from the equations that describe the equilibrium of the model: policy functions, laws of motion for variables, resource and budget constraints and so on. In general dynamic equilibrium economies do not admit closed-form solutions for those functions. We only require to have a numerical procedure to approximate them.

In order to fix ideas, we now map $\{S_t\}$, $\{W_t\}$, $\{Y_t\}$, $\{V_t\}$, f and g into some dynamic equilibrium economies examples. Consider first the example of the stochastic neoclassical growth model with leisure choice. The states of this economy are capital and the productivity level. Assume that our observables are output and labor supply but that labor supply is measured with some noise. Then S_t will be capital and productivity, W_t the shock to productivity, Y_t output and observed labor supply, V_t the measurement error of labor, f the policy function for capital and the law of motion for technology and g the production function plus the

[3] Assuming independence of $\{W_t\}$ and $\{V_t\}$ is only for notational convenience. Generalization to more involved structures for those stochastic processes is achieved by increasing the dimension of the state space.

policy function for labor augmented by the measurement error. Consider also an economy with nominal rigidities in the form of overlapping contracts. This economy experiences both productivity and money growth shocks and we observe output and inflation. Now the states S_t are the distribution of prices, capital, money and the productivity level, W_t includes the shocks to technology and money growth, Y_t is output and inflation, V_t is a degenerate distribution with mass at zero, f collects the policy functions for capital and prices as well as the law of motions for technology and money growth and g the aggregate supply function and the Phillips curve. Many more examples of dynamic economies can be fitted into this state space formulation.

To continue our analysis we make the following assumptions.

Assumption 1: $\dim(W_t) + \dim(V_t) \geq \dim(Y_t)$.

This assumption ensures that the model is not stochastically singular. We do not impose any restrictions on how those degrees of stochasticity are achieved.[4]

Assumption 2: We can partition $\{W_t\}$ into two sequences $\{W_{1,t}\}$ and $\{W_{2,t}\}$, such that $W_t = (W_{1,t}, W_{2,t})$ and $\dim(W_{2,t}) + \dim(V_t) = \dim(Y_t)$. If $\dim(V_t) = \dim(Y_t)$ we set $W_{1,t} = W_t \ \forall t$, i.e. $\{W_{2,t}\}$ to be a zero-dimensional sequence.[5]

Note that the assumption 2 is in some sense implied by assumption 1 because with a slight abuse of notation we allow the dimension of any of the sequences $\{W_{1,t}\}$, $\{W_{2,t}\}$ and (V_t) to be zero.

Assumption 3: We set $W_{2,t} = W_t \ \forall t$, i.e. $\{W_{1,t}\}$ to be a zero-dimensional sequence, only if $\dim(W_t) + \dim(V_t) = \dim(Y_t)$.

[4]This paper does not contribute to the literature on how to solve the problem of stochastic singularity of dynamic equilibrium economies. Two routes are commonly used to fix this problem. One is to reduce the observables accounted for to the number of stochastic shocks present. This likelihood can be studied to evaluate the model (Landon-Lane, 1999) or to find posteriors for parameters or impulse response functions (Schorfheide, 2000). The second route, increasingly popular, is to fully specify a model rich in stochastic dynamics (for example Smets and Wouters, 2003a and 2003b). This alternative is attractive to address practical policy questions like those of concern for Central Banks.

[5]Alternatively we could consider this more general alternative **Assumption 2'**: We can partition $\{W_t\}$ into two sequences $\{W_{1,t}\}$ and $\{W_{2,t}\}$, such that $W_t = (W_{1,t}, W_{2,t})$ and $\dim(W_{2,t}) + \dim(V_t) \geq \dim(Y_t)$. If $\dim(V_t) \geq \dim(Y_t)$ we set $W_{1,t} = W_t \ \forall t$, i.e. $\{W_{2,t}\}$ to be a zero-dimensional sequence.
The main structure of the algorithm would not change but it would make it much more cumbersome.

Assumption 3 is not really necessary but it makes the implementation of the algorithm easier. As in the case of assumption 2 we can dispense with it at the price of heavier notation.

Now we make some definitions that will be useful in the rest of the paper. First, let $W_i^t = \{W_{i,m}\}_{m=1}^t$ and let w_i^t be a realization of the random variable W_i^t for $i = 1, 2$ and $\forall t$. Let $V^t = \{V_m\}_{m=1}^t$ and let v^t be a realization of the random variable V^t for $\forall t$. Let $S^t = \{S_m\}_{m=0}^t$ and let s^t be a realization of the random variable S^t for $\forall t$. Let $Y^t = \{Y_m\}_{m=1}^t$ and let y^t be a realization of the random variable Y^t for $\forall t$. We also define $W_i^0 = \{\emptyset\}$ and $y^0 = \{\emptyset\}$.

Our goal is to evaluate the likelihood function of the a sequence of realizations of the observable y^T at a particular parameter value γ:

$$L\left(y^T;\gamma\right) = p\left(y^T;\gamma\right). \tag{3}$$

Our fist step is to factor the likelihood function as:

$$\begin{aligned} p\left(y^T;\gamma\right) &= \prod_{t=1}^{T} p\left(y_t|y^{t-1};\gamma\right) \\ &= \int \left(\prod_{t=1}^{T} \int p\left(y_t|W_1^t, y^{t-1}, S_0;\gamma\right) p\left(W_1^t|y^{t-1}, S_0;\gamma\right) dW_1^t \right) p\left(S_0;\gamma\right) dS_0 \end{aligned} \tag{4}$$

where S_0 is the initial state of the model, W_1^t is the history up to date t of $W_{1,t}$, the p's are the relevant densities.[6] To save on notation we assume herein that all the relevant Radon-Nykodim derivatives exist. Extending the exposition to the more general case is straightforward but cumbersome.

In general the factorized likelihood function (4) cannot be computed analytically. The sequential monte carlo algorithm that we propose in the next subsection allows us to use simulation methods to estimate it. The basic idea of the our approach is as follows. First, we present a sequential monte carlo algorithm to draw from $p\left(W_1^t|y^{t-1}, S_0;\gamma\right)$. Second, we use those draws to estimate (4) by monte carlo integration.

Before introducing the algorithm we need to make two additional technical assumptions.

[6] Where we understant that in the trivial case that $\{W_{1t}\}$ has zero dimensions $\int p\left(y_t|W_1^t, y^{t-1}, S_1;\gamma\right) p\left(W_1^t|y^{t-1}, S_1;\gamma\right) dW_1^t = p\left(y_t|y^{t-1}, S_1;\gamma\right)$, for all t.

Assumption 4: For any $\gamma \in \Upsilon$ and any y^t, we can evaluate the conditional densities $p\left(y_t|W_1^t, y^{t-1}, S_0; \gamma\right)$ for $\forall t$.

Assumption 4 implies that for any realizations s_0, w_1^t and y^t of the random variables S_0, W_1^t and Y^t, we can evaluate the probability of the model described by (1) and (2) of generating the observables. In other words, assumption 4 implies that for any s_0, w_1^t and y^t the following system of equations

$$\begin{aligned} S_1 &= f\left(s_0, \left(w_{1,1}, W_{2,1}\right); \gamma\right) \\ y_m &= g\left(S_m, V_m; \gamma\right) \qquad \text{for } m = 1, 2, ...t \\ S_m &= f\left(S_{m-1}, \left(w_{1,m}, W_{2,m}\right); \gamma\right) \qquad \text{for } m = 2, 3, ...t \end{aligned}$$

has a unique solution (v^t, s^t, w_2^t) and that we can evaluate the probabilities $p\left(v^t\right)$ and $p\left(w_2^t\right)$, where $p\left(y_t|W_1^t, y^{t-1}, S_0; \gamma\right) = p\left(v_t\right) p\left(w_{2,t}\right)$ $\forall t$.

Assumption 4 rules out the possibilities of sunspots or indeterminacy of equilibrium. To deal with sunspots and indeterminacy we could extend the results of Lubick and Schorfheide (2003) indexing the multiple solutions that appear under indeterminacy through additional parameters.

Define the set $\Omega\left(S_0\right) = \{w_1^t : \exists w_2^t,\ v^t \text{ s.t. } p\left(y_t|w_1^t, y^{t-1}, S_0\right) > 0 \text{ for all } t\}$. Given some initial state S_0 this set defines the realizations, w_1^t, of the random variable W_1^t for which the model assigns positive probability to the data. To deal with an interesting problem we make now the assumption that this set is not empty.

Assumption 5: $\exists$ *some initial state* S_0 *for which* $\Omega\left(S_0\right) \neq \{\emptyset\}$. If $\dim\left(W_{1,t}\right) = 0$ the assumption holds if $p\left(y_t|y^{t-1}, S_0; \gamma\right) > 0$.

Therefore, if assumptions 1-5 hold, conditional on having N draws of $\{s_0^i\}_{i=1}^N$ from the density $p\left(S_0; \gamma\right)$ and N draws $\left\{\left\{w_1^{t|t-1,i}\right\}_{i=1}^N\right\}_{t=1}^T$ from the corresponding sequence of densities $\{p\left(W_1^t|y^{t-1}, S_0; \gamma\right)\}_{t=1}^T$, using a law of large numbers the likelihood function (4) can be approximated by:

$$p\left(y^T; \gamma\right) \simeq \frac{1}{N}\left(\prod_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} p\left(y_t|w_1^{t|t-1,i}, y^{t-1}, s_0^i; \gamma\right)\right)$$

This shows that the problem of evaluating the likelihood of a dynamic equilibrium economy is equivalent to the problem of drawing from $\{p(W_1^t|y^{t-1}, S_0; \gamma)\}_{t=1}^T$. We now propose a sequential monte carlo algorithm to accomplish this objective.

2.2. A Sequential Monte Carlo Filter

We first fix some further notation. Let $\left\{w_1^{t-1,i}\right\}_{i=1}^N$ be a sequence of N i.i.d. draws from $p\left(W_1^{t-1}|y^{t-1}, S_0; \gamma\right)$. Let $\left\{w_1^{t|t-1,i}\right\}_{i=1}^N$ be a sequence of N i.i.d. draws from $p(W_1^t|y^{t-1}, S_0; \gamma)$. We call each draw $W_1^{t,i}$ a particle and the sequence $\left\{w_1^{t-1,i}\right\}_{i=1}^N$ a swarm of particles. Let also $h(S_t)$ be any measurable functions for which the expectation:

$$E_{p\left(W_1^t|y^t,S_0;\gamma\right)}\left(h\left(W_1^t\right)\right) = \int h\left(W_1^t\right) p\left(W_1^t|y^t, S_0; \gamma\right) dW_1^t$$

exists and is finite.

We now present a proposition that is close to previous results in importance sampling:

Proposition 1. *Let* $\left\{w_1^{t|t-1,i}\right\}_{i=1}^N$ *be a draw from* $p(W_1^t|y^{t-1}, S_0; \gamma)$ *and the weights:*

$$q_t^i = \frac{p\left(y_t|w_1^{t|t-1,i}, y^{t-1}, S_0; \gamma\right)}{\sum_{i=1}^N p\left(y_t|w_1^{t|t-1,i}, y^{t-1}, S_0; \gamma\right)}.$$

Then:

$$E_{p\left(W_1^t|y^t,S_0;\gamma\right)}\left(h\left(W_1^{t|t-1,i}\right)\right) \simeq \sum_{i=1}^N q_t^i h\left(w_1^{t|t-1,i}\right).$$

Proof. By Bayes theorem:

$$p\left(W_1^t|y^t, S_0; \gamma\right) \propto p\left(W_1^t|y^{t-1}, S_0; \gamma\right) p\left(y_t|W_1^t, y^{t-1}, S_0; \gamma\right)$$

Therefore if we use $p(W_1^t|y^{t-1}, S_0; \gamma)$ as a important sampling function to draw from the density $p(W_1^t|y^t, S_0; \gamma)$, the result is a direct consequence of the law of large numbers (e.g. Geweke, 1989, Theorem 1). ■

Proposition 1 shows how we can use $p\left(W_1^t|y^{t-1},S_0;\gamma\right)$ as an important sampling density to draw from $p\left(W_1^t|y^t,S_0;\gamma\right)$ in the following way:

Corollary 2. *Let* $\left\{w_1^{t|t-1,i}\right\}_{i=1}^N$ *be a draw from* $p\left(W_1^t|y^{t-1},S_0;\gamma\right)$. *Let the sequence* $\{\widetilde{w}^i\}_{i=1}^N$ *be a draw with replacement from* $\left\{w_1^{t|t-1,i}\right\}_{i=1}^N$ *where* q_t^i *is the probability of* $w_1^{t|t-1,i}$ *being drawn* $\forall i$ *. Then* $\{\widetilde{w}^i\}_{i=1}^N$ *is a draw from* $p\left(W_1^t|y^t,S_0;\gamma\right)$.

Corollary 2 shows how a draw $\left\{w_1^{t|t-1,i}\right\}_{i=1}^N$ from $p\left(W_1^t|y^{t-1},S_0;\gamma\right)$ can be used to get a draw $\left\{w_1^{t,i}\right\}_{i=1}^N$ form $p\left(W_1^t|y^t,S_0;\gamma\right)$. This corollary is key in the following sequential monte carlo algorithm that generates draws $\left\{\left\{w_1^{t|t-1,i}\right\}_{i=1}^N\right\}_{t=1}^T$ from the sequence of densities $\left\{p\left(W_1^t|y^{t-1},S_0;\gamma\right)\right\}_{t=1}^T$:

Step 0, Initialization: Set $t \rightsquigarrow 1$ and generate N i.i.d. initial states $\{s_0^i\}_{i=1}^N$ from $p\left(S_0;\gamma\right)$ such that $\Omega\left(s_0^i\right) \neq \{\emptyset\}$. Initialize $p\left(W_1^{t-1}|y^{t-1},S_0;\gamma\right)=1$.

Step 1, Prediction: Sample N values $\left\{w_1^{t|t-1,i}\right\}_{i=1}^N$ from the conditional density $p\left(W_1^t|y^{t-1},S_0;\gamma\right)=p\left(W_{1,t};\gamma\right)p\left(W_1^{t-1}|y^{t-1},S_0;\gamma\right)$.

Step 2, Filtering: Assign to each draw $w_1^{t|t-1,i}$ the weight q_t^i as defined above in proposition 1.

Step 3, Sampling: Sample N times with replacement from the set $\left\{w_1^{t|t-1,i}\right\}_{i=1}^N$ with probabilities $\{q_t^i\}_{i=1}^N$. Call each draw $w_1^{t,i}$. If $t<T$ set $t \rightsquigarrow t+1$ and go to step 1. Otherwise stop.

The intuition of the algorithm is as follows. Given a swarm of particles up to period $t-1$, $\left\{w_1^{t-1,i}\right\}_{i=1}^N$, distributed according to $p\left(W_1^{t-1}|y^{t-1},S_0;\gamma\right)$, step 1 generates draws $\left\{w_1^{t|t-1,i}\right\}_{i=1}^N$ from $p\left(W_1^t|y^{t-1},S_0;\gamma\right)$. Then step 3 takes advantage of corollary 2 and resamples from $\left\{w_1^{t|t-1,i}\right\}_{i=1}^N$ using the weights $\{q_t^i\}_{i=1}^N$ to draw a new swarm of particles up to period t,

$\left\{w_1^{t,i}\right\}_{i=1}^N$distributed according to $p\left(W_1^t|y^t, S_0;\gamma\right)$. Notice that we use the output of the algorithm $\{s_0^i\}_{i=1}^N$ and $\left\{\left\{w_1^{t|t-1,i}\right\}_{i=1}^N\right\}_{t=2}^T$ to compute the likelihood:

$$p\left(y^T;\gamma\right) \simeq \frac{1}{N}\left(\prod_{t=1}^{T}\frac{1}{N}\sum_{i=1}^{N} p\left(y_t|w_1^{t|t-1,i}, y^{t-1}, s_0^i;\gamma\right)\right)$$

We emphasize that in the case where the $\dim\left(W_{1,t}\right) = 0$, the algorithm collapse to iterating over step 2.

This algorithm derives from (but is not equal to) sequential monte carlo algorithms for nonlinear filtering. See Fearhead (1998) for an elegant review of the relevant literature. We modify existing procedures to deal with more general classes of state space representations than the ones addressed in the literature. In particular we can handle those cases, common in economics, where $\dim\left(V_t\right) < \dim\left(Y_t\right)$. We consider this more general applicability of our procedure an important advance.

Step 3 is the heart of the algorithm. A naive extension of basic Monte Carlo techniques without this step will diverge as T grows. All the sequences will wander away from the true (unobserved) shocks. To avoid this problem we do not carry over to the next period all the particles generated. We draw with replacement from them, giving a higher probability to those particles that are more likely. The fitting criterion is critical to the convergence of the procedure and very similar to the intuition behind why genetic algorithms work: we allow randomness to generate new simulations but we favor the survival of the particles that are more informative.[7]

Figure 2.1 may help to explain this point. Here we represent six different particles over six periods. We initialize all six particles at different values in period 1 and we move them to period 2. In the second period some of them are sampled (those represented by a green circle) while some are not (those that end with a red circle). As we just explained the sampling favors those draws that are "closer" in probability to the data. In the next period, the particles sampled (like particle 1) give birth to a number of continuations equal to the number of times they have been sampled. We iterate this procedure until the end of our observation sample.

[7]More sophisticated resampling schemes are available for variance reduction. See Doucet, De Freitas and Gordon (2001) and Pitt and Shephard (1999) for a review of these alternatives.

Finally note that the algorithm does not require any assumption on the distribution of the shocks except the ability to evaluate $p\left(W_1^{t-1}|y^{t-1}, S_0; \gamma\right)$ (either analytically or by simulation). This opens the door to dealing with models with a rich specification of non-normal innovations.

2.3. Comparison with Alternative Schemes

The algorithm outlined above is not the only procedure to numerically evaluate the likelihood function of the data implied by nonlinear models. Our previous discussion highlighted how computing the likelihood amounts to solve a nonlinear filtering problem, i.e. to generate estimates of the values of $W_{1,t}$ to evaluate the integral in (4). Since this task is of interest in different fields, several alternative schemes have been proposed to handle this problem.

A first line of research has been in deterministic filtering. Historically the first procedure in this line was the Extended Kalman Filter (Jazwinski, 1973) that linearizes the transition and measurement equations and uses the Kalman Filter to obtain estimates for the states and the shocks to the system. This approach suffers from the approximation error incurred by the linearization and by the inaccuracy incurred by the fact that the posterior estimates of the states are not Gaussian. As the sample size grows those problems accumulate and the filter diverges. Even refinements as the Iterated Extended Kalman Filter or the quadratic Kalman Filter cannot solve these problems.

A second approach in deterministic filtering is the Gaussian Sum approximations (Alspach and Sorenson, 1972) that approximates the different densities required to compute the likelihood with a mixture of normals. Under regularity conditions, as the number of normals increases, we will approximate arbitrarily well the densities. However the approach suffers from an exponential growth in the number of components in the mixture and from the fact that we still need to use the Extended Kalman Filter to approximate the evolution of those different components.

A third alternative in deterministic filtering is the use of grid-based filters, based on deterministic numerical integration as proposed by Bucy and Senne (1974), to compute the different integrals. Their use is limited as those grid-based filters turn out to be very difficult to implement, with a constant need to readjust to small changes in the model or its parameter values, and too computationally expensive to be of any practical use beyond very low

dimensions.[8]

Tanizaki (1996) investigates the performance of all those deterministic filters (Extended Kalman Filter, Gaussian Sum approximations and grid-based filters). He uses monte carlo evidence to document that all those approximations delivered a very poor performance when applied to real economic applications.

A second strategy is to think of the functions f and g as a change of variables of the innovations to the model and use the Jacobian of the transformation to evaluate the likelihood of the observables (Miranda and Rui, 1997). In general this approach is cumbersome and of difficult implementation since we need to approximate the derivatives in the (unknown) Jacobian. These approximations are costly and not very robust. Also technical conditions limit its applicability.

A third line of research is the use of Monte Carlo techniques. This approach was inaugurate by Kitagawa (1987). One of the first lessons from this literature was that it is not straightforward to import basic simulation techniques because of convergence problems. For instance a recursive extension of the Importance Sampling scheme is bound to fail as the number of observations grows (Robert and Casella, 1999).

The key innovation was proposed by Gordon, Salmond and Smith (1993). They pointed out that resampling from the simulated data could be performed using properly chosen weights. With these resampling it is feasible to solve efficient and consistently the filtering problem. Our algorithm is a descendant of the original proposal by Gordon, Salmond and Smith (1993) and the following literature and includes theirs as a particular case when $\{W_{1,t}\} = \{W_t\}$.

Other simulation algorithms include Mariano and Tanizaki (1995) and Geweke and Tanizaki (1999). Mariano and Tanizaki (1995) propose a version of rejection sampling. This method is however difficult because it depends on finding an appropriate density for the rejection test. This search is a time-consuming task that requires substantial work for each particular model. Geweke and Tanizaki (1999) use the whole joint likelihood and draw from the distribution of the whole set of states over the sample using a Metropolis-Hastings algorithm. This approach increases notably the dimensionality of the problem, specially for relatively long samples, and

[8] Another shortcoming of grid-based filters is that the grid points are fixed ex-ante and the results are very dependent on that choice. In comparison we can think about our simulation filter as a grid-based filter where the grid points are chosen endogenously over time based on their ability to account for the data.

also requires good proposal densities and a good initialization of the chain.

3. An Application I: Setup

In this section we present an application of our procedure to a dynamic equilibrium economy. We find it natural to use the stochastic neoclassical growth model for that purpose. First, it is a canonical example of a dynamic equilibrium model and it has been used (either directly or with small variations) to address a large number of questions in macroeconomics. Second, it is a relatively simple model, a fact that facilitates the illustration of the different parts of our procedure. We are more interested in this paper in showing the potential of our approach than in the empirical findings per se and the growth model is the perfect laboratory for that purpose.

We are also aware that, being this a model that for a standard calibration is nearly linear, our procedure may be a bit of an overkill. For example a simpler procedure as using the Kalman filter after linearizing the equilibrium conditions may deliver estimates that are nearly as good as those obtained respecting the nonlinearities of the model. We actually see this fact as an advantage since it may help the reader to notice the differences of our algorithm with other alternatives and allows the comparison of our results to the findings from the Kalman Filter (see Fernández-Villaverde and Rubio-Ramírez, 2003b, for more details). Concurrent research applies our algorithm to models more explicitly nonlinear. For example we investigate, among other examples, models with asset pricing and economies with nominal rigidities. We omit those results to keep the message of this paper focused.

The rest of this section is divided in four parts. First, we present the stochastic neoclassical growth model. Second we briefly describe how we solve the model numerically. Third, we explain how to evaluate the likelihood function. Finally we explain how to introduce our sequential monte carlo algorithm in an estimation procedure. We do this from both a Bayesian and a classical perspective. Later in section 4 we report the results of our estimation for "artificial"and real data.

3.1. The Stochastic Neoclassical Growth Model

As just mentioned we work with the stochastic neoclassical growth model with leisure. This model is well known (see the textbook exposition of Cooley and Prescott, 1995). Consequently

we only go through the minimum exposition required to fix notation.[9]

There is a representative agent in the economy, whose preferences over stochastic sequences of consumption c_t and leisure l_t are representable by the utility function

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t^{\theta} \left(1 - l_t\right)^{1-\theta}\right)^{1-\tau}}{1-\tau}$$

where $\beta \in (0,1)$ is the discount factor, τ determines the elasticity of intertemporal substitution, θ controls labor supply and E_0 is the conditional expectation operator.

There is one good in the economy produced according to the production function $e^{z_t} k_t^{\alpha} l_t^{1-\alpha}$ where k_t is the aggregate capital stock, l_t is the aggregate labor input and z_t is a stochastic process representing random technological progress. The stochastic process z_t follows an AR(1) process $z_t = \rho z_{t-1} + \epsilon_t$ with $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon)$. We restrict ourselves to cases where the process is stationary (i.e. $|\rho| < 1$). Capital's law of motion is $k_{t+1} = i_t + (1-\delta)k_t$ where i_t is investment and the economy must satisfy the resource constrain $c_t + i_t = e^{z_t} k_t^{\alpha} l_t^{1-\alpha}$.

A competitive equilibrium can be defined in an standard way as a sequence of allocations and prices such that both the representative household and the firm maximize and markets clear. However since both welfare theorems hold in this economy, we can instead solve the equivalent and simpler social planner's problem that maximizes the utility of the representative household subject to the economy resource constraint, the law of motion for capital, the stochastic process and some initial conditions k_0 and z_0.

The solution to this problem is fully characterized by the following two stochastic partial differential equations, an Euler intertemporal condition

$$\frac{\left(c_t^{\theta} \left(1 - l_t\right)^{1-\theta}\right)^{1-\tau}}{c_t} = \beta E_t \left\{ \frac{\left(c_{t+1}^{\theta} \left(1 - l_{t+1}\right)^{1-\theta}\right)^{1-\tau}}{c_{t+1}} \left(1 + \alpha e^{z_t+1} k_{t+1}^{\alpha-1} l_t^{\alpha} - \delta\right) \right\} \tag{5}$$

[9] We avoid the case of the model with full depreciation and no leisure choice. Even if in this case the model has a closed-form solution, this form is loglinear and as consequence suited for estimation using the Kalman Filter. We want to deal with an explicitly nonlinear case to illustrate the generality of our procedure.

and a static optimality condition

$$\frac{1-\theta}{\theta}\frac{c_t}{1-l_t} = (1-\alpha)\, e^{z_t} k_t^{\alpha} l_t^{-\alpha} \tag{6}$$

plus the stochastic process for productivity, the law of motion for capital, the economy resource constraint and the boundary condition $c(0, z_t; \theta) = 0$.

We can think about this problem as finding policy functions for consumption $c(\cdot,\cdot)$, labor $l(\cdot,\cdot)$ and next period's capital $k'(\cdot,\cdot)$ that deliver the optimal choices as functions of the two state variables, capital and the technology level. In practice, however, the problem is simpler because we only search for the solution $l(\cdot,\cdot)$ and find $c(\cdot,\cdot)$ using the static optimally condition and $k'(\cdot,\cdot)$ using the resource constraint of the economy.

3.2. Solving the Model

The previous system of equations does not have a known analytical solution and we need to use a numerical method to solve it. In a recent paper, Aruoba, Fernández-Villaverde and Rubio-Ramírez (2003) have documented that the Finite Element Method delivers a highly accurate, fast and numerically stable solution for a wide range of parameter values in a model exactly like the one consider here. In addition theoretical results ensure the convergence of the approximation to the true (but unknown) nonlinear solution of the economy. Details of how to implement the Finite Element Method in our application are provided in the appendix.

We want to emphasize however that nothing in the sequential monte carlo filter stops us from using any other nonlinear solution method for the system of equations as perturbations (Guu and Judd, 1997), Chebyshev polynomials (Judd, 1992) or value function iteration. The appropriate choice of solution method should be dictated by the details of the particular model to be estimated.

3.3. The Likelihood Function

We assume that we have observed the following time series $y^T \in \times_{t=1}^{T} R^3$, where, for each t, the first component is output, gdp_t, the second is hours, $hours_t$ and the third is investment, inv_t. We make this assumption out of pure convenience. On the one hand we want to capture some of the main empirical predictions of the model. On the other hand, and again only

for illustration purposes, we want to keep the dimensionality of the problem low. However the empirical analysis can be performed with very different combinations of data. Our choice should be understood just as an example of how to evaluate the likelihood function associated with a vector of observations.

Let $\gamma^1 \equiv (\theta, \rho, \tau, \alpha, \delta, \beta, \sigma_\epsilon) \in \Upsilon^1 \subset R^7$ be the structural parameters that describe the preferences and technology of the model. Also, as described in the appendix, our implementation of the Finite Element Method requires shocks bounded between -1 and 1. To achieve that goal we transform the productivity shock in the following way: $\lambda_t = \tanh(z_t)$. Let $S_t = (k_t, \lambda_t)$ be the states of the model and set $W_t = \epsilon_t$. Let also $S_{ss} = (k_{ss}, \tanh(0))$, the value of the states variables at the steady state of the model.

Define $V_t \sim \mathcal{N}(0, \Sigma)$ as a vector of measurement errors for our three observables. To economize on parameters we assume that Σ is diagonal with diagonal elements σ_1^2, σ_2^2 and σ_3^2. Define $\gamma^2 = (\sigma_1^2, \sigma_2^2, \sigma_3^2) \in \Upsilon^2 \subset R_+^3$ and $\gamma = (\gamma^1, \gamma^2) \in \Upsilon$. Finally call the approximated labor policy function $l_{fem}(\cdot, \cdot; \gamma)$ where we make the dependence from the structural parameter values explicit.

The transition equation for this model is:

$$\begin{aligned}
k_t &= f_1(S_{t-1}, W_t; \gamma) = e^{\tanh^{-1}(\lambda_{t-1})} k_{t-1}^{\alpha} l_{fem}\left(k_{t-1}, \tanh^{-1}(\lambda_{t-1}); \gamma\right)^{1-\alpha} * \\
&\quad * \left(1 - \frac{\theta}{1-\theta}(1-\alpha)\frac{\left(1 - l_{fem}\left(k_{t-1}, \tanh^{-1}(\lambda_{t-1}); \gamma\right)\right)}{l_{fem}\left(k_{t-1}, \tanh^{-1}(\lambda_{t-1}); \gamma\right)}\right) + (1-\delta) k_{t-1} \\
\lambda_t &= f_2(S_{t-1}, W_t; \gamma) = \tanh(\rho \tanh^{-1}(\lambda_{t-1}) + \epsilon_t)
\end{aligned}$$

and the measurement equation is:

$$\begin{aligned}
gdp_t &= g_1(S_t, V_t; \gamma) = e^{\tanh^{-1}(\lambda_t)} k_t^{\alpha} l_{fem}\left(k_t, \tanh^{-1}(\lambda_t); \gamma\right)^{1-\alpha} + V_{1,t} \\
hours_t &= g_2(S_t, V_t; \gamma) = l_{fem}\left(k_t, \tanh^{-1}(\lambda_t); \gamma\right) + V_{2,t} \\
inv_t &= g_3(S_t, V_t; \gamma) = e^{\tanh^{-1}(\lambda_t)} k_t^{\alpha} l_{fem}\left(k_t, \tanh^{-1}(\lambda_t); \gamma\right)^{1-\alpha} * \\
&\quad * \left(1 - \frac{\theta}{1-\theta}(1-\alpha)\frac{\left(1 - l_{fem}\left(k_t, \tanh^{-1}(\lambda_t); \gamma\right)\right)}{l_{fem}\left(k_t, \tanh^{-1}(\lambda_t); \gamma\right)}\right) + V_{3,t}
\end{aligned}$$

It would be useful below to define the vector $x(S_t; \gamma)$ of predictions of the model regarding

observables. Those are equal to:

$$\begin{aligned}
x_1(S_t;\gamma) &= e^{\tanh^{-1}(\lambda_t)} k_t^{\alpha} l_{fem}\left(k_t, \tanh^{-1}(\lambda_t);\gamma\right)^{1-\alpha} \\
x_3(S_t;\gamma) &= l_{fem}\left(k_t, \tanh^{-1}(\lambda_t);\gamma\right) \\
x_3(S_t;\gamma) &= g_3(S_t, V_t;\gamma) = e^{\tanh^{-1}(\lambda_t)} k_t^{\alpha} l_{fem}\left(k_t, \tanh^{-1}(\lambda_t);\gamma\right)^{1-\alpha} \\
&\quad * \left(1 - \frac{\theta}{1-\theta}(1-\alpha)\frac{\left(1 - l_{fem}\left(k_t, \tanh^{-1}(\lambda_t);\gamma\right)\right)}{l_{fem}\left(k_t, \tanh^{-1}(\lambda_t);\gamma\right)}\right)
\end{aligned}$$

We introduce measurement errors as the easiest way to avoid stochastic singularity (see assumption 1). Nothing in our procedure depends on the presence of measurement errors. We could for example write a version of the model where in addition to shocks to technology we would have shocks to preferences and shocks to depreciation. This alternative model might be more empirically relevant but it would make the solution of the model much more cumbersome. As we have reiterated several times, since our goal here is merely to illustrate how to use our sequential monte carlo filter to evaluate the likelihood of the model in an example as simple as possible, we prefer the "trick" of using measurement errors. We feel, however, than in a large number of empirical application, more structured alternatives to measurement errors are required.

Given the fact that we have four sources of uncertainty, and $\dim(V_t) = \dim(Y_t)$, we follow assumption 2 and set $\dim(W_{2,t}) = 0$ and $W_{1,t} = W_t = \epsilon_t$. Let $L\left(y^T;\gamma\right)$ be the likelihood function of the data. Remember that the likelihood was given by:

$$L\left(y^T;\gamma\right) = \int \left(\prod_{t=1}^{T} \int p\left(y_t|W_1^t, y^{t-1}, S_0;\gamma\right) p\left(W_1^t|y^{t-1}, S_0;\gamma\right) dW_1^t\right) p\left(S_0;\gamma\right) dS_0 \qquad (7)$$

Since $\dim(W_{2,t}) = 0$, $W_{1,t} = W_t$ and $S_t = g\left(S_{t-1}, W_t;\gamma\right)$ observe, first, that:

$$p\left(y_t|W_1^t, y^{t-1}, S_0;\gamma\right) = p\left(y_t|W^t, y^{t-1}, S_0;\gamma\right) = p\left(y_t|S_t;\gamma\right),$$

and second, that to draw from $p\left(W_1^t|y^{t-1}, S_0;\gamma\right)$ is equivalent to draw from $p\left(S_t|y^{t-1}, S_0;\gamma\right)$.

This allow us to write the likelihood function (7) as:

$$L\left(y^T;\gamma\right)=\int\left(\prod_{t=1}^{T}\int p\left(y_t|S_t;\gamma\right)p\left(S_t|y^{t-1},S_0;\gamma\right)dS_t\right)p\left(S_0;\gamma\right)dS_0 \tag{8}$$

But since our measurement equation implies that

$$p\left(y_t|S_t;\gamma\right)=(2\pi)^{-\frac{3}{2}}\left|\Sigma\right|^{-\frac{1}{2}}e^{-\frac{\omega(S_t;\gamma)}{2}}$$

where we define the prediction errors to be $\omega(S_t;\gamma)=\left(y_t-x(S_t;\gamma)\right))'\Sigma^{-1}\left(y_t-x(S_t;\gamma)\right)\ \forall t$, we can rewrite (8) as

$$L\left(y^T;\gamma\right)=(2\pi)^{-\frac{3T}{2}}\left|\Sigma\right|^{\frac{-T}{2}}\int\left(\prod_{t=1}^{T}\int e^{-\frac{\omega(S_t;\gamma)}{2}}p\left(S_t|y^{t-1},S_0;\gamma\right)dS_t\right)p\left(S_0;\gamma\right)dS_1 \tag{9}$$

This last expression is simple to handle. Given particles $\left\{\left\{w_1^{t|t-1,i}\right\}_{i=1}^{N}\right\}_{t=1}^{T}$ and $\{s_0^i\}_{i=1}^{N}$ coming from our sequential monte carlo filter, we can build the states $\left\{\{s_t^i\}_{i=1}^{N}\right\}_{t=1}^{T}$ and the prediction error $\left\{\{\omega(s_t^i;\gamma)\}_{i=1}^{N}\right\}_{t=1}^{T}$ implied by them. We set $s_0^i=S_{ss}\ \forall i$. Therefore, the likelihood function is approximated by:

$$L\left(y^T;\gamma\right)\simeq(2\pi)^{-\frac{3T}{2}}\left|\Sigma\right|^{\frac{-T}{2}}\prod_{t=1}^{T}\frac{1}{N}\sum_{i=1}^{N}e^{-\frac{\omega(s_t^i;\gamma)}{2}} \tag{10}$$

Note that equation (10) is nearly identical to the likelihood function implied by the Kalman Filter (see for example equation 3.4.5 in Harvey, 1989) when applied to a linear model. The difference is that in the Kalman Filter the prediction errors $\omega(s_t^i;\gamma)$ come directly from the output of the Riccati equation while in our filter those come from the output of the simulation.

3.4. The Estimation Algorithms

We now explain how to use to the approximated likelihood function (10) to perform nonlinear likelihood-based estimation from both a Bayesian and a classical perspective. First we will describe the Bayesian approach, then the classical.

In a Bayesian approach the main inference tool is the posterior distribution of the parameters given the data, $\pi\left(\gamma|y^T\right)$. Once the posterior distribution is obtained, we can define a loss function and obtain a point estimate. The Bayes theorem tells us that the posterior density is proportional to the likelihood times the prior. Therefore, we need both to specify priors on the parameters, $\pi(\gamma)$, and to evaluate the likelihood function. We specify our priors in section 4.1, and the likelihood function of the model is approximated by (10). The next step in Bayesian inference is to draw from the posterior. In general the posterior does not have a closed-form, therefore we use a Metropolis-Hasting algorithm to draw from it.[10] The algorithm to draw a chain $\{\gamma_i\}_{i=1}^M$ from $\pi\left(\gamma|y^T\right)$ is as follows:

Step 0, Initialization: Set $i \rightsquigarrow 0$ and an initial γ_i. Solve the model for γ_i and compute $f(\cdot,\cdot;\gamma_i)$ and $g(\cdot,\cdot;\gamma_i)$. Evaluate $\pi(\gamma_i)$ and $L\left(y^T;\gamma_i\right)$ using (10). Set $i \rightsquigarrow i+1$.

Step 1, Proposal draw: Get a proposal draw $\gamma_i^* = \gamma_{i-1} + \varepsilon_i$, where $\varepsilon_i \sim N(0, \sigma_\varepsilon)$.

Step 2, Solving the Model: Solve the model for γ_i^* and compute $f(\cdot,\cdot;\gamma_i^*)$ and $g(\cdot,\cdot;\gamma_i^*)$.

Step 3, Evaluating the proposal: Evaluate $\pi(\gamma_i^*)$ and $L\left(y^T;\gamma_i^*\right)$ using (10).

Step 4, Accept/Reject: Draw $\chi_i \sim U(0,1)$. If $\chi_i \leq \frac{L\left(y^T;\gamma_i^*\right)\pi\left(\gamma_i^*\right)}{L\left(y^T;\gamma_{i-1}\right)\pi\left(\gamma_{i-1}\right)}$ set $\gamma_i = \gamma_i^*$, otherwise $\gamma_i = \gamma_{i-1}$. If $i < M$ set $i \rightsquigarrow i+1$ and go to step 1. Otherwise stop.

Once $\{\gamma_i\}_{i=1}^M$ is obtained through this algorithm any moments of interest of the posterior can be computed as well as the marginal likelihood of the model.

[10] In other examples we could exploit the structure of the problem and use another, more efficient Markov Chain Monte Carlo procedure.

On the classical side the main inference tool is the likelihood function and its global maximum. Once the likelihood is obtained using (10), we can introduce it inside a maximization loop suitable for the model being studied in the following way:

Step 0, Initialization: `Set` $i \rightsquigarrow 0$ `and an initial` γ_i`.` `Set` $i \rightsquigarrow i+1$

Step 1, Solving the Model: `Solve the model for` γ_i `and compute` $f(\cdot,\cdot;\gamma_i)$ `and` $g(\cdot,\cdot;\gamma_i)$`.`

Step 2, Evaluating the Likelihood: `Evaluate` $L\left(y^T;\gamma_i\right)$ `using (10) and get` γ_{i+1} `from a maximization routine.`

Step 3, Stopping Rule: `If` $\left\|\gamma_i - \gamma_{i+1}\right\| > \varepsilon$`, where` $\varepsilon > 0$ `is the accuracy level goal, set` $i \rightsquigarrow i+1$ `and go to step 1. Otherwise stop.`

The output of the algorithm, γ_i, is maximum likelihood point estimate and we can build standard errors in the conventional way. The value of the likelihood function at its maximum is also useful to build likelihood ratios for model comparison purposes.

4. An Application II: Inference

In this section we conduct likelihood based inference on our model. We undertake two exercises. First we simulate data from the model for a particular choice of values of γ. Then with these data, we compute the likelihood and estimate the parameters of the model using our sequential monte carlo algorithm. This exercise documents how our filter delivers good estimates of the "true" parameter values. In this way we address two critical questions. First, since our procedure only produces an estimate of the likelihood function, we want to know if the numerical error incurred stops the filter from finding accurate parameter estimates. Working with simulated data avoids the problem of estimates being affected by model misspecification. Second we can determine how many particles we need to obtain an accurate estimation. The theoretical arguments presented above rely on asymptotic arguments and they cast little light on the number of particles required in a particular application.

The second exercise takes the model to real data. We estimate it using real output per

capita, average hours worked and real gross fixed investment per capita in the U.S. from 1964:1 to 2003:1. This exercises proves how the filter can be brought to "real life" applications and how delivers sensible results.

We perform both exercises from a Bayesian approach. We specify prior distributions over the parameters, evaluate the likelihood using our filter and draw from the posterior using a Metropolis-Hastings algorithm. However, since we specify flat priors, the posterior mean can be interpreted as the maximum likelihood estimate. In that sense we offer answers both from a classical and from a Bayesian approach.

We divide our exposition in four parts. First we specify the priors for the parameters. Second we present results from the "artificial" data experiment. Third, we present the results of the estimation with real data. Finally, we analyze some convergence issues of the sequential monte carlo algorithm.

4.1. Specifying the Priors

The first step is to specify prior distributions for the different parameters of the model $\gamma \equiv (\theta, \rho, \tau, \alpha, \delta, \beta, \sigma_\epsilon, \sigma_1, \sigma_2, \sigma_3) \in \Upsilon$. We write $\pi(\gamma) : \Upsilon \to R^+$ when we denote the product of all the different priors.

We adopt flat priors for all ten parameters subject only to some boundary constraints to make the priors proper and to rule out parameter values that are either incompatible with the model (i.e. a negative value for a variance) or extremely implausible (the parameter governing the elasticity of substitution being bigger than 100). The looseness of those last constraints is shown by the fact that the simulation never travel even close to those bounds.

Our choice of flat priors is motivated by two reasons. First, since we are going to undertake estimation on simulated data generated by some known parameter values, we do not want to bias the results in favor of our procedure by a careful choice of priors. Second, with a flat prior the posterior is proportional to the likelihood function.[11] As consequence our experiment can be interpreted as classical exercise where the model of the likelihood function is the maximum likelihood estimate. Also a researcher that prefers to use more informative

[11] Except for the small issue of the bounded support of the priors. If we think about those bounds as frontiers of admissible parameter values in a classical perspective, the argument equating the posterior and likelihood holds exactly. Otherwise, it holds nearly exactly because the likelihood puts an negligible mass outside the support of the priors.

priors can always reweight the draws from the posterior to accommodate his favorite priors (see Geweke, 1998).[12]

We now describe the priors in more detail. The parameter governing labor supply, θ, follows a uniform distribution between 0 and 1. That range captures all the possible values for which leisure has positive marginal utility. The persistence of the technology shock, ρ, follows a uniform distribution between 0 and 1. This region implies a stationary distribution of the variables of the model[13] with a lower bound on no persistence. The parameter governing the elasticity of substitution, τ, follows a uniform between 0 (linear preferences) and 100. That choice encompasses all range of empirical estimates of the parameter and only rules out risk loving behavior and risk aversions that will predict differences in interest rates orders of magnitude higher than the observed ones.[14] The prior for the technology parameter, α, is uniform between 0 and 1, including all values for which marginal productivity of capital and labor are positive. The prior on the depreciation rate ranges between 0 and 0.05, covering all national accounts estimates of quarterly depreciation. The discount factor, β, ranges between 0.75 and 1, implying steady state annual interest rates between 0% and 316%. The standard deviation of the innovation of productivity, σ_ϵ, follows a uniform between 0 and 0.1, a bound 15 times higher than the usual estimates. For the three variances of the measurement errors we choose the same prior than for the productivity shock. Table 3.1 summarizes the previous discussion.

[12] Note that we do not argue that our flat priors are uninformative. After a reparametrization of the model, a flat prior may become highly curved.

[13] This prior rules out almost surely the presence of a unit root in the output process. One attractive point of Bayesian inference is that, in contrast with classical methods, it is not necessary to use special tools to deal with unit roots (Sims and Uhlig, 1991). In the same way our filter can deal with these unit roots paying the cost of a somehow lower efficiency. As a consequence our prior choice is not motivated by any technical reason but out of our view of what is a reasonable characteristic of the data. We are using a version of the neoclassical growth model without long-run technological progress. As described below, we filter our data using a H-P filter before feeding them into the likelihood function. Since the H-P filter removes up to two unit roots (King and Rebelo, 1993), we are only ruling out the presence of three unit roots in output, a highly implausible hypothesis.

[14] As Lucas (1987) pointed out, in the steady state of the model the product τ and the rate of growth of output is equal to a constant plus the interest rate: from the deterministic Euler condition, $(1+g)^\tau = \beta(1+r)$ and then $\tau g = \log\beta + r$.

Table 3.1: Priors for the Parameters of the Model

Parameters	Distribution	Hyperparameters
θ	Uniform	0,1
ρ	Uniform	0,1
τ	Uniform	0,100
α	Uniform	0,1
δ	Uniform	0,0.05
β	Uniform	0.75,1
σ_ϵ	Uniform	0,0.1
σ_1	Uniform	0,0.1
σ_2	Uniform	0,0.1
σ_3	Uniform	0,0.1

4.2. Results with "Artificial" Data

As a first step to test our procedure we simulate observations from our model to use them as data for the estimation. We will generate data from two different calibrations.

First we select the benchmark calibration values for the stochastic neoclassical growth model according to the standard practice (Cooley and Prescott , 1995) to make our experiment as relevant as possible. The discount factor $\beta = 0.9896$ matches an annual interest rate of 4.27% (see McGrattan and Prescott, 2000 for a justification of this number based on their measure of the return on capital and on the risk-free rate of inflation-protected U.S. Treasury bonds). The risk aversion $\tau = 2$ is a common choice in the literature. $\theta = 0.357$ matches the microeconomic evidence of labor supply. We set $\alpha = 0.4$ to match labor share of national income. The depreciation rate $\delta = 0.02$ fixes the investment/output ratio and $\rho = 0.95$ and $\sigma = 0.007$ match the stochastic properties of the Solow residual of the U.S. economy. With respect to the standard deviations of the measurement errors we set them equal to a 0.01% the steady state value of output, 0.35% of the steady state value hours and 0.2% of the steady state of investment based on our priors regarding the relative importance of measurement errors in National and Income Product Accounts. The chosen values are summarized in table 3.2.

Table 3.2: Calibrated Parameters

Parameter	θ	ρ	τ	α	δ	β	σ_ϵ	σ_1	σ_2	σ_3
Value	0.357	0.95	2.0	0.4	0.02	0.99	0.007	0.0011	$1.58*10^{-4}$	$8.66*10^{-4}$

The second calibration keeps the same values for all the parameters except for τ and σ_ϵ. We increase τ to a value of 50 (implying a relative risk aversion of 24.5) and σ_ϵ to 0.035. The interaction between high risk aversion and high variance introduce a strong nonlinearity in the model that will help us to assess how the procedure does in a much more challenging environment. Our value for risk aversion is an order of magnitude higher than the usual values used in macroeconomics but not too far away from some numbers implied by practitioners in finance (see Cochrane and Hansen, 1992). However we do not justify our choice based on empirical relevance but on our desire to assess the performance of our algorithm.

We solve the model using our finite element method and we draw a sample of size 100 for each of the two calibrations. We use our priors and our likelihood evaluation algorithm with 40000 particles to get 50000 draws from the posterior distribution using the Metropolis-Hastings algorithm.

We discuss first the results for the "standard calibration". We graph our 10 empirical distributions in figure 3.1 and report the mean and standard deviations of these distributions in table 3.3. Under a quadratic loss function, the mean of the posterior distribution is the optimal point estimate of the parameter. Also, given our flat priors, the modes in figure 3.1 will be our maximum likelihood point estimate.

Table 3.3: Posterior Distributions Benchmark Calibration

Parameters	Mean	s.d.
θ	0.357	6.72×10^{-5}
ρ	0.950	3.40×10^{-4}
τ	2.000	6.78×10^{-4}
α	0.400	8.60×10^{-5}
δ	0.020	1.34×10^{-5}
β	0.989	1.54×10^{-5}
σ_ϵ	0.007	9.29×10^{-6}
σ_1	1.58×10^{-4}	5.75×10^{-8}
σ_2	1.12×10^{-2}	6.44×10^{-7}
σ_3	8.64×10^{-4}	6.49×10^{-7}

Inspecting table 3.3, our method does an excellent job of pinning down the values of the parameters, specially considering the low number of iterations. All the structural parameters except the standard deviation of the measurement error on output are estimated in an unbiased and tight way.

However it is important to remember that this simulation may be strongly biased in favor of our technique since we initialize the Metropolis-Hastings close to the true parameter values. The problem of the initial values for Markov Chain Monte Carlo is well known but in our case it is specially relevant since we know the "true" parameter values (although of course even this biased result has a positive spin: we can argue that the procedure stays when it needs to stay if we begin at the right point of the parameter space).

There are two alternatives to check the robustness of the finding. One is to initiate the chain at the mean of the priors. Since our priors are flat over a large range, this choice implies initial values very far away from the true parameter values. The second alternative is to begin at a middle distance from the parameter values (for example 20% off). We investigated both alternatives. We found that the algorithm quickly moves in the right direction searching for higher likelihoods. The drawback is that we need a large burn-in period until the likelihood stabilizes. This observation is similar to the one of the main practical teachings from the literature in Markov Chain Monte Carlo: a careful exploration of the parameters space for a good initial value is key to achieve a good performance in reasonable time. To illustrate

how the algorithm searches for the right region, figure 3.2 plots the evolution of values of parameters in the simulation and how they converge to the true value, represented by the red line (again the measurement errors seem to have more difficulties).

Summarizing we interpret these results from different chains as follows. First, if we begin around the true parameter values, we stay in that neighborhood. Second, if we begin far away, after a long burn-in period we converge to the right region.

The results of the second calibration are reported in table 3.4. Again the algorithm delivers tight and consistent estimates in a highly nonlinear model.

Table 3.4: Posterior Distributions Extreme Calibration

Parameters	Mean	s.d.
θ	0.357	7.19×10^{-4}
ρ	0.950	1.88×10^{-4}
τ	50.00	7.12×10^{-3}
α	0.400	4.80×10^{-5}
δ	0.020	3.52×10^{-6}
β	0.989	8.69×10^{-6}
σ_ϵ	0.035	4.47×10^{-6}
σ_1	1.58×10^{-4}	1.87×10^{-8}
σ_2	1.12×10^{-2}	2.14×10^{-7}
σ_3	8.65×10^{-4}	2.33×10^{-7}

An alternative to bayesian inference is to perform pure maximum likelihood inference. Given our previous exposition, such task is relatively simple. We only need to plug-in our maximization algorithm inside a maximization routine and let the procedure find a maximum of the function. However our simulation procedure makes difficult to use a simple Newton-Raphson update scheme. Since we cannot compute derivatives analytically, we approximate them numerically. The sampling error associated with the likelihood function evaluation makes these numerical derivatives very unstable and the procedure faces extraordinary difficulties to converge. We find, however, that using a simulated annealing scheme we get

successful estimates of the parameter value.[15]

Finally let us mention that model comparison in this framework is straightforward. From the output of the algorithm we can either find the maximum value of the likelihood (to build likelihood ratios) or compute the marginal likelihood. The appendix shows how to perform that computation following Geweke (1998). Fernández-Villaverde and Rubio-Ramírez (2003b) use the marginal likelihoods of the stochastic growth model to suggest that the evidence in the data in favor of the model is much stronger than when we built marginal likelihoods from linearized versions of the economy.

4.3. Results with U.S. Data

Now we apply our procedure to estimate the stochastic neoclassical growth model with U.S. quarterly data. We use real output per capita, average hours worked and real gross fixed investment per capita from 1964:1 to 2003:1. We first remove a trend from the data using a H-P filter. In this way we do not need to model explicitly the presence of a trend and possible changes to it.

Table 3.5 presents the results from the posterior distribution from 50.000 draws and figure 3.4 displays the posteriors. In this case we used to initialize the chain the mean of the posterior computed from a linearized version of the model and the Kalman filter after 400 million iterations.[16]

[15] That result should not be surprising given the similarity in spirit of the Metropolis-Hastings and simulated annealing.

[16] Such large of draws allows for an "overkill" in terms of convergence of the Metropolis-Hastings and thus for the elimination of the influence of initial guesses.

Table 3.5: Posterior Distributions Real Data

Parameters	Mean	s.d.
θ	0.390	0.003
ρ	0.953	0.011
τ	1.733	0.024
α	0.325	0.001
δ	0.006	9.53×10^{-5}
β	0.997	1.29×10^{-4}
σ_ϵ	0.022	2.56×10^{-4}
σ_1	0.041	4.16×10^{-4}
σ_2	0.034	2.69×10^{-3}
σ_3	0.037	1.00×10^{-3}

We briefly discuss some of the parameters. The discount factor, β, goes very close to 1, a common finding wen dynamic equilibrium economies are estimated. The parameter controlling the elasticity of substitution, τ, has a value of 1.733 and θ of 0.39. These two parameters imply an elasticity of substitution of 1,29. The estimated depreciation factor is very low, 0.006 since the estimation tries to compensate for the high desire of accumulation of capital implied by the very high discount factor. The parameter α is close to the canonical value of one third. Finally the autoregressive component, ρ, is estimated to be 0.953.

These numbers are close to the ones coming from a standard calibration exercise and to those generally accepted as reasonable after the accumulation of empirical evidence over the last two decades by very different empirical methods. Nearly as important, the standard deviations of the posterior are very low, indicating tight estimates. We interpret this findings as another strong endorsement of the ability of the procedure to uncover sensible values for the structural parameters of dynamic equilibrium economies.

The estimation delivers numbers a bit more problematic regarding the standard deviation of the productivity shock. In particular this shock seems to be much more variable than the number estimated directly from the Solow residual. At the same time the values for the standard deviations of the measurement errors seem high. The combination of these two results may be an indication of the lack of identification of the stochastic growth model along the dimension of the different shocks.

4.4. Convergence of the Sequential Monte Carlo

An important question to answer in practical applications is how many particles to use to evaluate the likelihood function. The theory only provides us with a convergence result as the number of particles goes to infinity but little guidance regarding finite behavior.

To explore this issue we change the number of particles and compute 50 different times the likelihood of the model for each number of particles (i.e. we compute 50 estimations of the likelihood with 10.000 particles, 50 with 20.000 and so on).

Tables 3.6 to 3.8 report the mean and the standard deviation of the estimated loglikelihood at a particular parameter choice for the three different calibrations. For the Benchmark case we estimate the loglikehood at the mean of the posterior, while we choose different parameters values for the other two cases. We make this choice because of the different behavior of the estimated log likelihood at those points. While the standard deviation for parameter choice different from the mean of the posterior is very low for any number of particles, the standard deviation increases when the loglikelihood is estimated around it. The reason for this increase in the variance is that at the mean of the posterior the slope of the loglikelihood goes from positive to negative very quickly, making this estimation more difficult.

In any case, the results justify our choice of $N = 40000$ because even in the worse case the standard deviation is less than 0.2% of the value of the loglikelihood. Efficiency could be improved if we deal properly with the tails of the distribution but in the interest of simplicity we leave this issue for future research.

Table 3.6: Convergence Benchmark Calibration

N	Mean	s.d.
10000	1459.163	6.4107
20000	1461.928	2.8298
30000	1462.078	1.5415
40000	1462.031	0.9900
50000	1462.636	0.7168
60000	1462.696	0.6353

Table 3.7: Convergence Extreme Calibration		
N	Mean	s.d.
10000	831.493	0.1954
20000	831.471	0.1347
30000	831.489	0.0971
40000	831.508	0.0836
50000	831.509	0.0882
60000	831.532	0.0607

Table 3.8: Convergence Real Data		
N	Mean	s.d.
10000	1014.558	0.3296
20000	1014.600	0.2595
30000	1014.653	0.1829
40000	1014.666	0.1604
50000	1014.688	0.1465
60000	1014.664	0.1347

This different behavior is also reflected in figures 3.4 to 3.6. These figures represent the C.D.F. for the weights q_t^i as defined in proposition 1 for a particular t and the three models. Figure 3.4 draws the C.D.F. for the Benchmark case, figure 3.5 for the Extreme Calibration and figure 3.5 for the real data case. The optimal behavior in terms of informational content of the different paths will be $q_t^i = q_t^j$ for t, i and j. This optimal behavior will imply a straight C.D.F. with slope $\frac{1}{N}$ and equal weight for all particles. The further away from this straight line the higher the weight on a small set of particles and the higher the standard deviation of the estimated loglikelihood. As the Tables reflect both for the Extreme Calibration and the real data the actual C.D.F. almost matches the straight line, while for the Benchmark case the actual C.D.F. is someway further away from the straight line.

5. Conclusions

We have presented a simple, general and efficient algorithm to perform likelihood-based inference in nonlinear dynamic equilibrium economies. We have shown how parameter estimation

and model comparison can be undertaken, either from a classical or from a bayesian perspective, when we work with the whole, nonlinear solution of the model. Also we can perform this inference regardless of whether or not we have normal innovations to the model. The key result has been the development of an algorithm to evaluate the likelihood function of the dynamic model. To achieve this goal we have worked on the tradition of nonlinear filtering theory to develop a sequential monte carlo algorithm that can be applied to a large class of economies. The intuition of the procedure is to simulate different paths for the states of the model but to resample from them using some appropriately built weights.

Our simulation results and the application to real data of the stochastic neoclassical growth model suggest that the procedure works superbly in delivering accurate and consistent estimates. In a companion paper (Fernández-Villaverde and Rubio-Ramírez, 2003b) we show how the procedure compares in terms of efficiency with a linear approach. Future research will apply the algorithm to fully nonlinear models of asset pricing and nominal rigidities, study further issues like the importance of nongaussian innovations to models (see Geweke 1994 for some suggestive evidence), regime-switching models and economies with multiplicity of equilibria.

6. Appendix

This appendix presents further details about the computational details of the paper. First, it explains in some detail the Finite Element method. Second, it offers further details of the Metropolis-Hastings algorithm used and its convergence. Finally, it discusses the computing language and the software used.

6.1. The Finite Element Method

We provide a brief exposition of the finite elements method as applied in the paper. For more detailed explanation the interested reader should consult the expositions in McGrattan (1999) and Auroba, Fernández-Villaverde and Rubio-Ramírez (2003).

The first step in the Finite Elements method is to note that we can rewrite the Euler equation for consumption as

$$U_c(k_t, z_t) = \frac{\beta}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \left[U_c(k_{t+1}, z_{t+1})(1 + \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} l(k_{t+1}, z_{t+1})^{1-\alpha} - \delta)\right] \exp(-\frac{\epsilon_{t+1}^2}{2\sigma^2}) d\epsilon_{t+1} \tag{11}$$

where $U_c(t) = U_c(k_t, z_t)$, $k_{t+1} = e^{z_{t+1}} k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t - c(k_t, z_t)$ and $z_{t+1} = \rho z_t + \epsilon_{t+1}$.

The problem is to find two policy functions $c(k, z) : R^+ \times [0, \infty] \to R^+$ and $l(k, z) : R^+ \times [0, \infty] \to [0, 1]$ that satisfy the model equilibrium conditions. Since the static first order condition gives a relation between the two policy functions, we only need to solve for one of them. For the rest of the exposition we will assume that we actually solve for $l(k, z)$ and then we find $c\left(l(k, z)\right)$.

First we bound the domain of the state variables to partition it in nonintersecting elements. To bound the productivity level of the economy define $\lambda_t = \tanh(z_t)$. Since $\lambda_t \in [-1, 1]$ we can write the stochastic process as $\lambda_t = \tanh(\rho \tanh^{-1}(z_{t-1}) + \sqrt{2}\sigma v_t)$ where $v_t = \frac{\epsilon_t}{\sqrt{2}\sigma}$. Now, since $\exp(\tanh^{-1}(z_{t-1})) = \frac{\sqrt{1+\lambda_{t+1}}}{\sqrt{1-\lambda_{t+1}}} = \widehat{\lambda}_{t+1}$, we rewrite (11) as

$$U_c(t) = \frac{\beta}{\sqrt{\pi}} \int_{-1}^{1} \left[U_c(k_{t+1}, z_{t+1}) \left(1 + \alpha \widehat{\lambda}_{t+1} k_{t+1}^{\alpha-1} l(k_{t+1}, z_{t+1})^{1-\alpha} + \delta\right)\right] \exp(-v_{t+1}^2) dv_{t+1} \tag{12}$$

where $k_{t+1} = \widehat{\lambda}_{t+1} k_t^{\alpha} l\left(k_t, z_t\right)^{1-\alpha} + (1-\delta)k_t - c\left(l(k_t, z_t)\right)$ and $z_{t+1} = \tanh(\rho \tanh^{-1}(z_t) + \sqrt{2}\sigma v_{t+1})$. For convenience we use the same notation for $l\left(\cdot\right)$ in both (11) and (12) although they are not the same function since their domain is different. To bound the capital we fix an ex-ante upper bound $\overline{k}$, picked sufficiently high that it will only bind with an extremely low probability.

Then define $\Omega = \left[0, \overline{k}\right] \times [-1, 1]$ as the domain of $l_{fe}(k, z; \overline{\theta})$ and divide Ω into nonover-

lapping rectangles $[k_i, k_{i+1}] \times [z_j, z_{j+1}]$, where k_i is the ith grid point for capital and z_j is jth grid point for the technology shock. Clearly $\Omega = \cup_{i,j} [k_i, k_{i+1}] \times [z_j, z_{j+1}]$. These elements may be of unequal size. In our computations we define 14 unequal elements in the capital dimension and 10 on the λ axis. We have small elements in the areas of Ω where the economy spends most of the time while just a few, big size elements cover wide areas of the state space infrequently visited (see figure a.1 for our partition). Note that we define the elements in relation with the steady state of the model for each particular value of the parameters in the estimation. In that sense our mesh is endogenous to the estimation procedure, increasing efficiency and accuracy.

Next we set $l_{fe}\left(k, z; \overline{\theta}\right) = \sum_{i,j} \overline{\theta}_{ij} \Psi_{ij}(k, z) = \sum_{i,j} \overline{\theta}_{ij} \widehat{\Psi}_i(k) \widetilde{\Psi}_j(z)$ where

$$\widehat{\Psi}_i(k) = \begin{cases} \frac{k-k_i}{k_{i+1}-k_i} & \text{if } k \in [k_{i-1}, k_i] \\ \frac{k_{i+1}-k}{k_{i+1}-k_i} & \text{if } k \in [k_i, k_{i+1}] \\ 0 \text{ elsewhere} \end{cases} \qquad \widetilde{\Psi}_j(z) = \begin{cases} \frac{z-z_j}{z_{j+1}-z_j} & \text{if } z \in [z_{j-1}, z_j] \\ \frac{z_{j+1}-z}{z_{j+1}-z_j} & \text{if } z \in [z_j, z_{j+1}] \\ 0 \text{ elsewhere} \end{cases}$$

First, note that $\Psi_{ij}(k, z) = 0$ if $(k, z) \notin [k_{i-1}, k_i] \times [z_{j-1}, z_j] \cup [k_i, k_{i+1}] \times [z_j, z_{j+1}]$ $\forall i, j$, i.e. the function is 0 everywhere except inside two elements. Second $l_{fe}(k_i, z_j; \overline{\theta}) = \overline{\theta}_{ij}$ $\forall i, j$, i.e. the values of $\overline{\theta}$ specify the values of c_{fe} at the corners of each subinterval $[k_i, k_{i+1}] \times [z_j, z_{j+1}]$.

Let us define $U_c(k_{t+1}, z_{t+1})_{fe}$ be the marginal utility of consumption evaluated at the finite element approximation values of consumption and leisure. In this case, from the Euler equation we have a residual equation:

$$R(k_t, z_t; \theta) = \frac{\beta}{\sqrt{\pi}} \int_{-1}^{1} \left[\frac{U_c(k_{t+1}, z_{t+1})_{fe}}{U_c(k_{t+1}, z_{t+1})_{fe}} \left(1 + \alpha \widehat{\lambda}_{t+1} k_{t+1}^{\alpha-1} l_{fe}^{1-\alpha} - \delta \right) \right] \exp(-v_{t+1}^2) dv_{t+1} - 1 \quad (13)$$

A Galerkin scheme implies that we weight the residual function by the basis functions and solve the system of $\overline{\theta}$ equations

$$\int_{\left[0, \overline{k}\right] \times [-1,1]} \Psi_{i,j}(k, z) R(k, z; \overline{\theta}) dz dk = 0 \quad \forall i, j \quad (14)$$

on the $\overline{\theta}$ unknowns.

Since $\Psi_{ij}(k, z) = 0$ if $(k, z) \notin [k_{i-1}, k_i] \times [z_{j-1}, z_j] \cup [k_i, k_{i+1}] \times [z_j, z_{j+1}]$ $\forall i, j$ we can rewrite (14) as

$$\int_{[k_{i-1}, k_i] \times [z_{j-1}, z_j] \cup [k_i, k_{i+1}] \times [z_j, z_{j+1}]} \Psi_{i,j}(k, z) R(k, z; \overline{\theta}) dz dk = 0 \quad \forall i, j \quad (15)$$

We evaluate the integral in the residual equation with a Gauss-Hermite method and the integrals in (15) with a Gauss-Legendre procedure. Finally we solve the associated system

of nonlinear equations with a Quasi-Newton algorithm with a conservative update to avoid numerical instabilities.

6.2. The Metropolis-Hastings Algorithm

Given the prior and the likelihood function we need to find the posterior distribution, $p(\gamma|y^T)$. As described in the main body of the paper, since we do not have a closed-form solution for the posterior, we draw a sample of size of M, $\{\gamma_i\}_{i=1}^{M}$, using a Random Walk Metropolis-Hastings algorithm.

The success of the algorithm depends on the fulfillment of a number of technical conditions. In practice, it is extremely important to adjust the variance of the innovation of the proposal density to get an appropriate acceptance rate.[17] If the rate is small, the chain does not visit the tails of the posterior. If the acceptance rate is high, the chain does not stay enough time at the high probability regions. Gelman, Roberts and Gilks (1996) suggest that a 20% acceptance rate tends to give the best performance. We found that rates between 25 and 30% outperformed different alternatives.

We monitored convergence using standard techniques. A complete guide to convergence can be found in Mengersen, Robert and Guihenneuc-Jouyaux (1999).

6.3. Computation of the Marginal Likelihood

An interesting by-product of the simulation output is that it can be used to build the marginal likelihood of the model. This marginal likelihood determines the probability that the model assigns to the observations and serves to compare models (see Geweke, 1998 and Fernández-Villaverde and Rubio-Ramírez, 2003a).

Following Gelfand and Dey (1994) note that for any 10-dimensional probability density $h(\cdot)$ with support contained in Υ:

$$E\left[\left.\frac{h(\gamma)}{L(y^T;\gamma)\pi(\gamma)}\right| y^T\right] = \int_{\Upsilon} \frac{h(\gamma)}{L(y^T;\gamma)\pi(\gamma)} dP(\gamma|y^T) =$$

$$= \int_{\Upsilon} \frac{h(\gamma)}{L(y^T;\gamma)\pi(\gamma)} \frac{L\left(y^T;\gamma\right)\pi(\gamma)}{\int_{\Upsilon} L(y^T;\gamma)\pi(\gamma)\,d\gamma} d\gamma = \frac{\int_{\Upsilon} h(\gamma)\,d\gamma}{\int_{\Upsilon} L(y^T;\gamma)\pi(\gamma)\,d\gamma} = p\left(y^T\right)^{-1} \tag{16}$$

This expression is an unbiased and consistent estimator of the marginal likelihood and satisfies a Central Limit Theorem if $\frac{\int_{\Upsilon} h^2(\gamma)d\gamma}{\int_{\Upsilon} L(y^T;\gamma)\pi(\gamma)d\gamma} < \infty$.

[17] The acceptance rate is equal to the number of times when the chain changes position divided by the number of iterations.

Then, from the M draws of the simulation and applying a Strong Law of Large Numbers, we can compute:

$$p\left(y^T\right)^{-1} = \frac{1}{M}\sum_{i=1}^{M}\frac{h\left(\gamma\right)}{L\left(y^T;\gamma\right)\pi\left(\gamma\right)} \tag{17}$$

As a choice of h we modify Geweke's (1998) proposal. First, from the output of the simulation define $\widehat{\gamma} = \frac{1}{M}\sum_{i=1}^{M}\gamma$ and

$$\widehat{\Sigma_M} = \frac{1}{M}\sum_{i=1}^{M}\left(\gamma_i - \widehat{\gamma}\right)\left(\gamma_i - \widehat{\gamma}\right)'$$

Then, for a given $p \in (0,1)$ define the set $\Upsilon_M = \left\{\gamma : \left(\gamma - \widehat{\gamma}\right)\widehat{\Sigma_m}^{-1}\left(\gamma - \widehat{\gamma}\right)' \leq \chi^2_{1-p}\left(10\right)\right\}$ where $\chi^2_{1-p}\left(\cdot\right)$ is a chi-squared distribution with degrees of freedom equal to the number of parameters. Letting $I_{\Upsilon_M\cap\Upsilon}\left(\cdot\right)$ be the indicator function of a vector of parameters belonging to the intersection $\Upsilon_M \cap \Upsilon$, we can take a truncated multivariate normal as our h function:

$$h\left(\gamma\right) = \frac{1}{\widehat{p}\left(2\pi\right)^{\frac{k}{2}}}\left|\widehat{\Sigma_m}\right|^{\frac{1}{2}} e^{-0.5*\left(\gamma-\widehat{\gamma}\right)\widehat{\Sigma_m}^{-1}\left(\gamma-\widehat{\gamma}\right)'} I_{\Upsilon_M\cap\Upsilon}\left(\gamma\right) \tag{18}$$

where $\widehat{p}$ is an appropriate normalizing constant. With this choice, if the posterior density is uniformly bounded away from zero on every compact subset of Υ, our computation approximates the marginal likelihood.

With the output of the Markov chain Monte Carlo, the estimation of the marginal likelihood is then rather direct: we use the computed values of $L\left(y^T;\gamma\right)\pi\left(\gamma\right)$ for each point in the Markov chain and we find its harmonic mean using the function h as a weight.

6.4. Computational Details

All programs needed for the computation of the model were programed in Fortran 95 and compiled in Compaq Visual Fortran 6.6 to run on Windows based machines. On a Pentium 4 at 3.00 GHz each draw from the posterior with 20.000 particles takes around 3.1 seconds. That implies a total of around 44 hours for each simulation of 50.000 draws. To put this number in perspective note that the linearized version of the model runs 400 million draws in 12 hours. Versions paralellized with MPI directives to be run in the IBM-SP facilities at the Minnesota Supercomputer Institute were also prepared but they were not used in the final computations of the paper. All the code, both in serial and parallel versions, is available upon request.

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Figure 2.1: Particles evolving over time

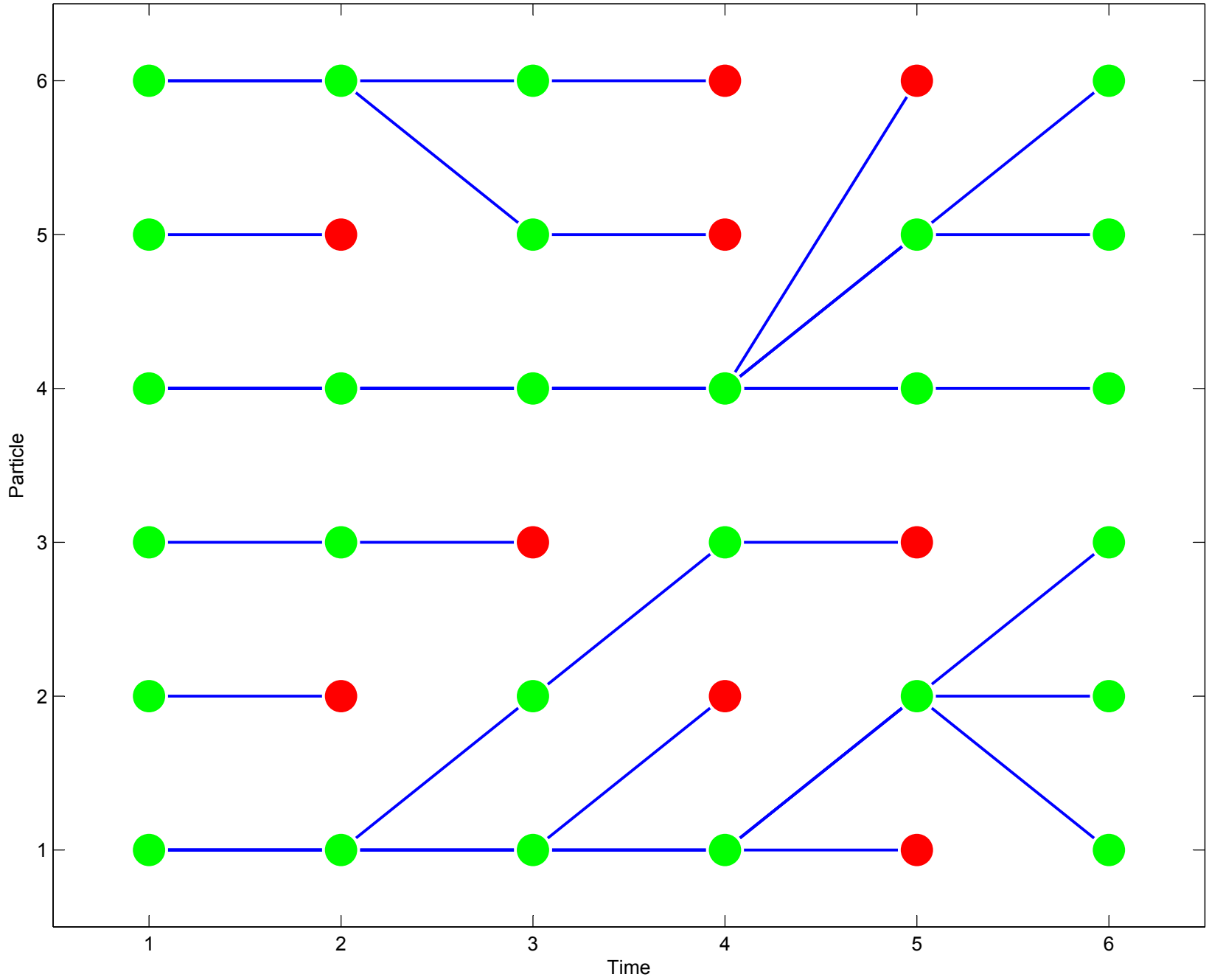
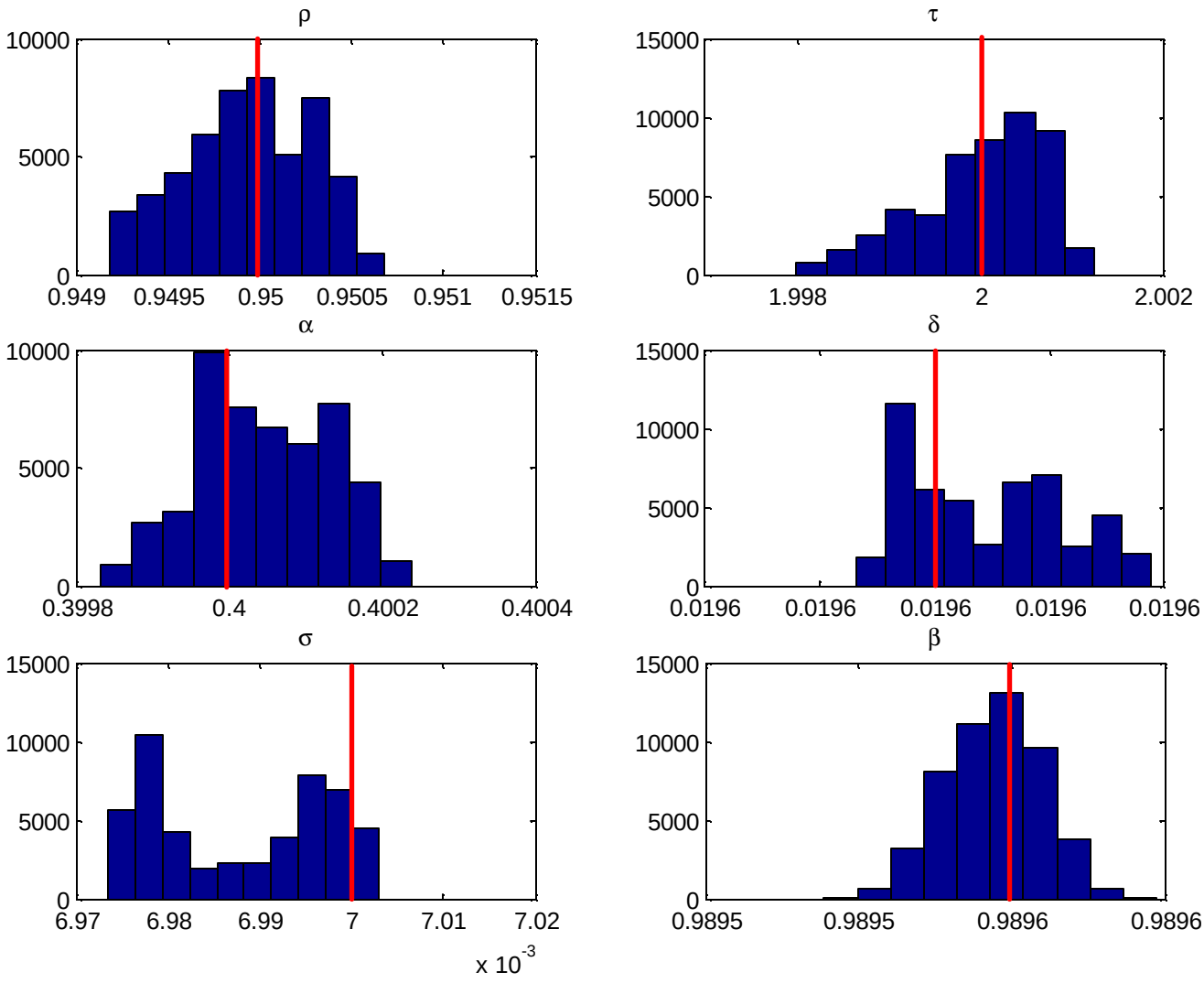


Figure 3.1: Posteriors Benchmark Calibration



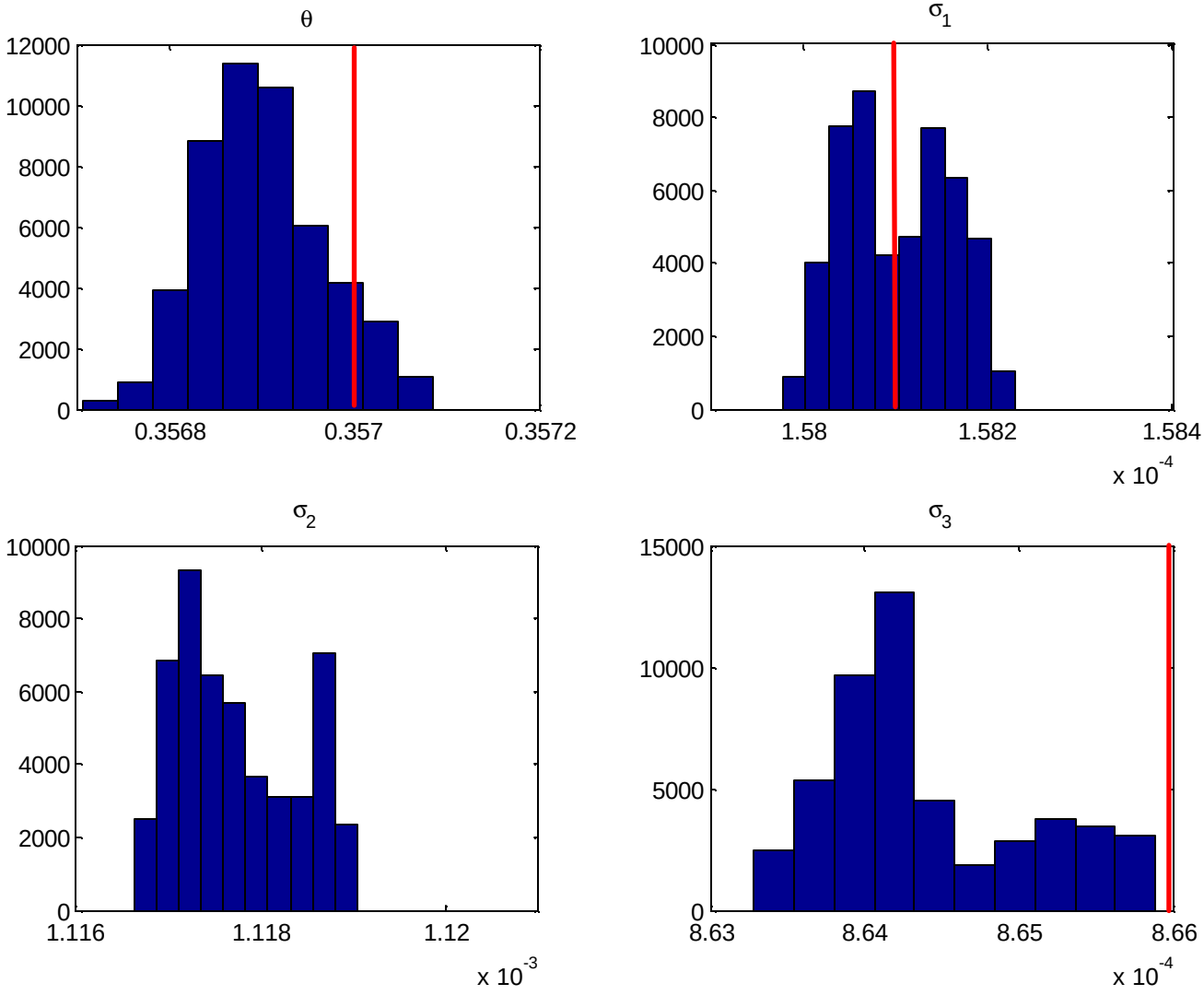
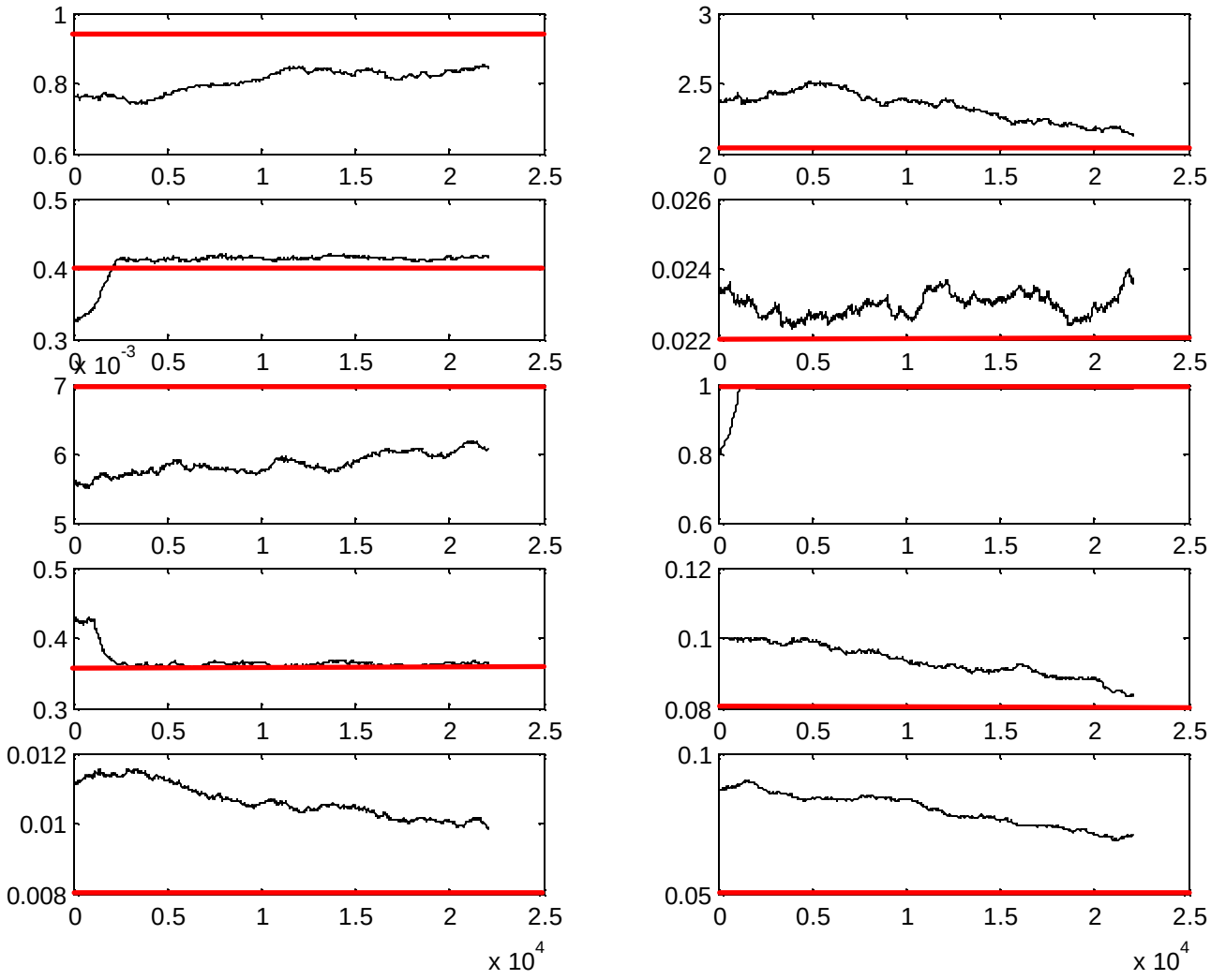

θ
12000
10000
8000
6000
4000
2000
0
0.3568
0.357
0.3572
σ1
10000
8000
6000
4000
2000
0
1.58
1.582
1.584
x 10-4
σ2
10000
8000
6000
4000
2000
0
1.116
1.118
1.12
x 10-3
σ3
15000
10000
5000
0
8.63
8.64
8.65
8.66
x 10-4

Figure 3.2: Convergence of Estimates



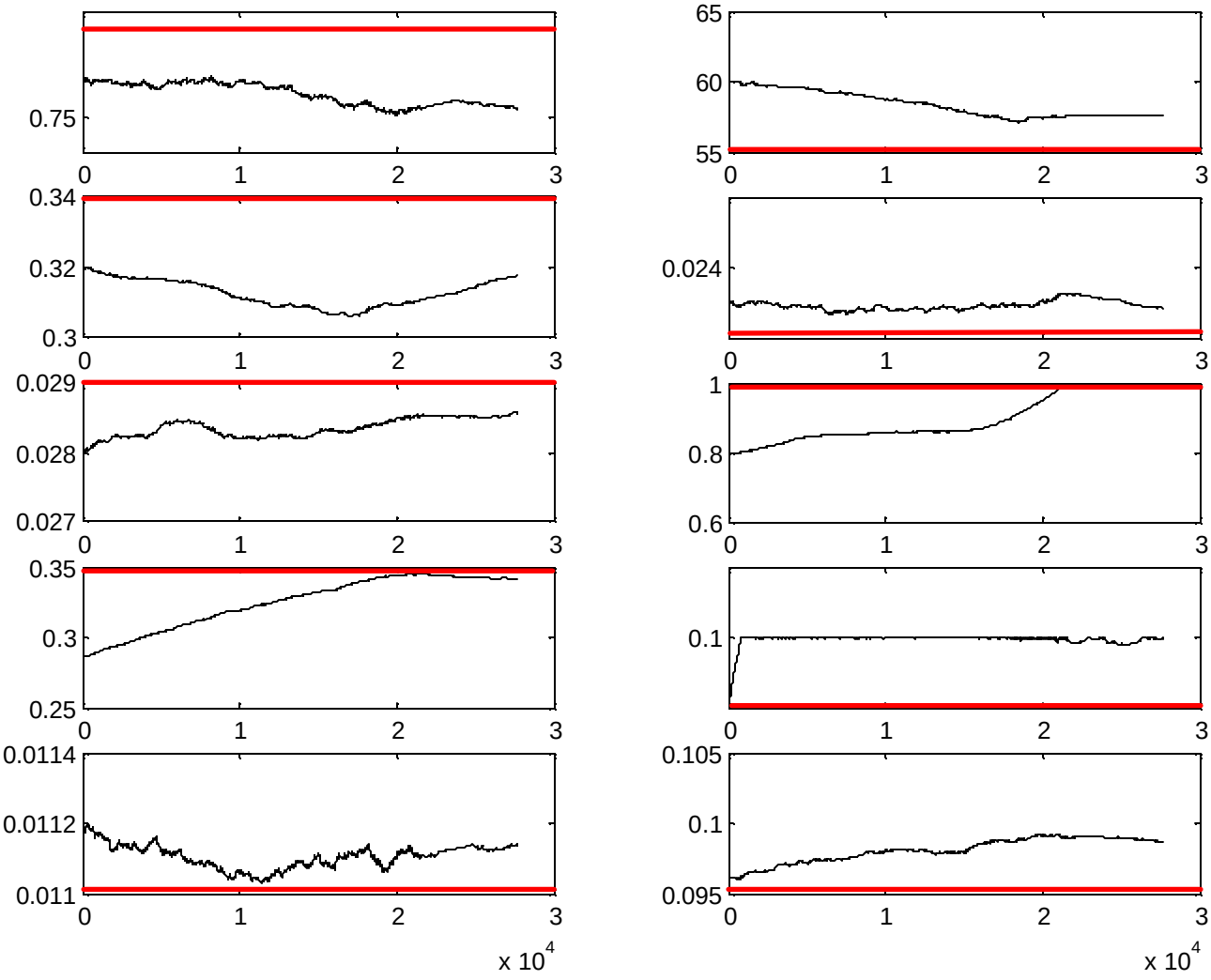
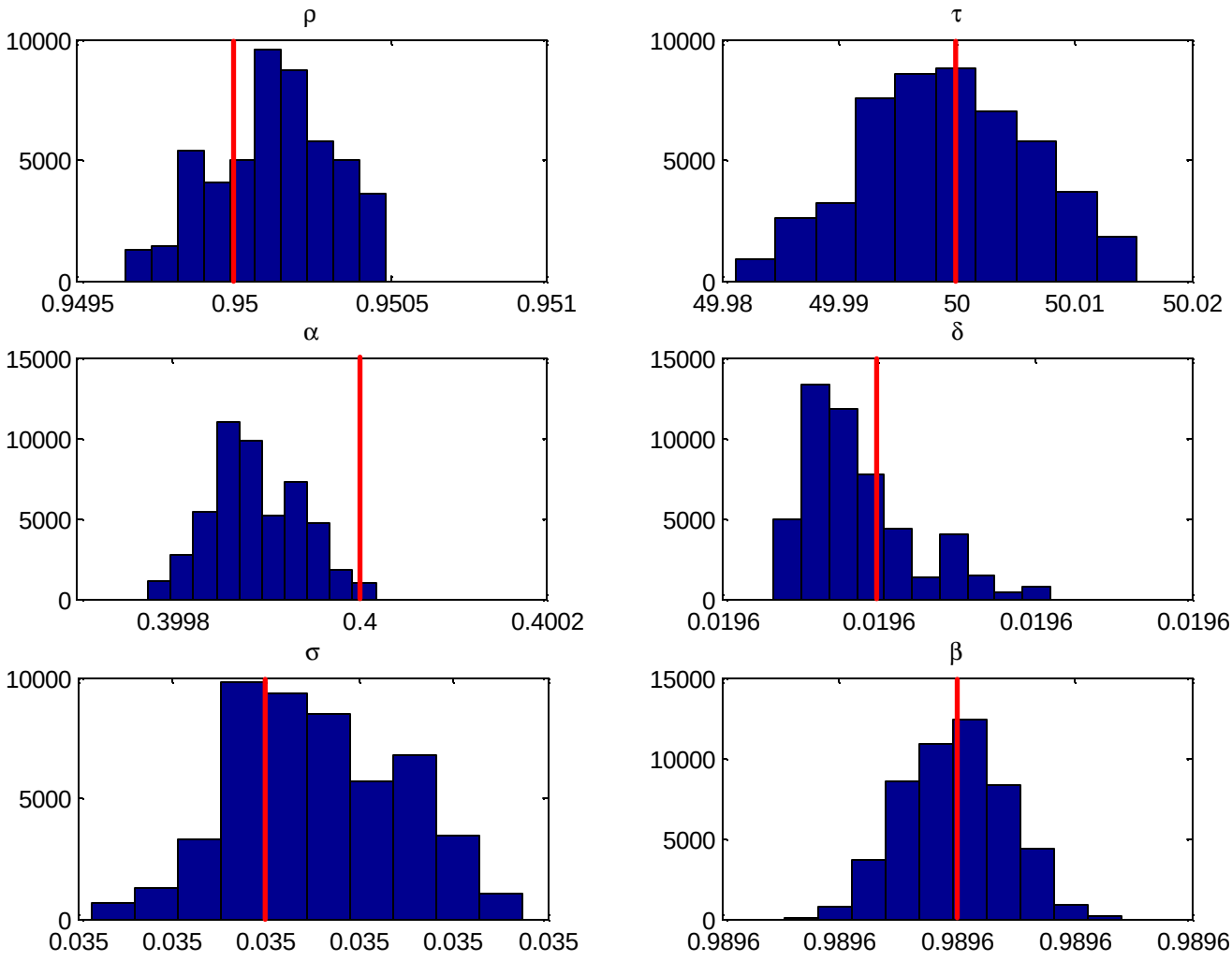

0.75
0
1
2
3
0.34
0.32
0.3
0.029
0.028
0.027
0.35
0.3
0.25
0.0114
0.0112
0.011
x 10^4
65
60
55
0.024
1
0.8
0.6
0.1
0.105
0.1
0.095
x 10^4

Figure 3.3: Posteriors Extreme Calibration



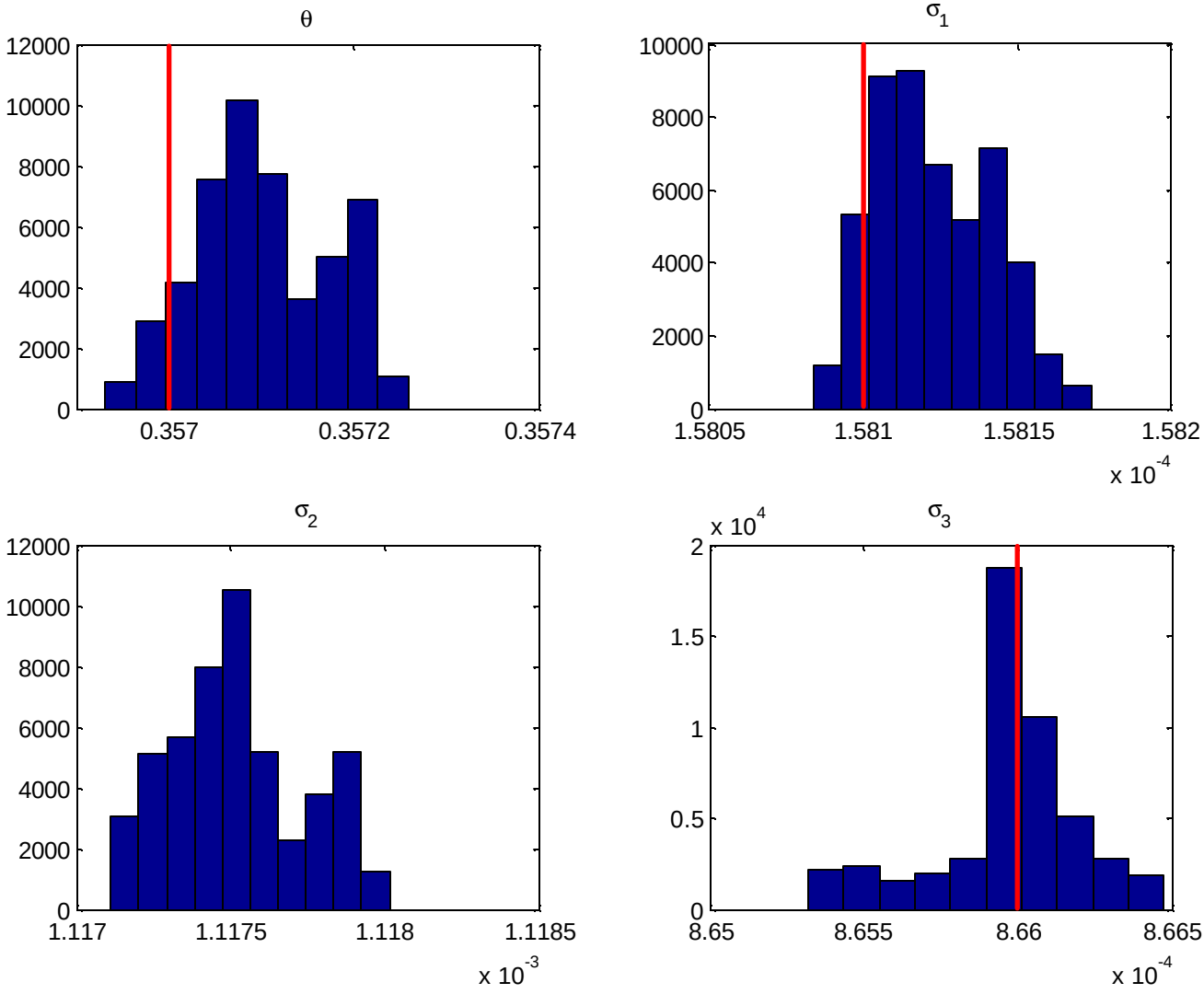
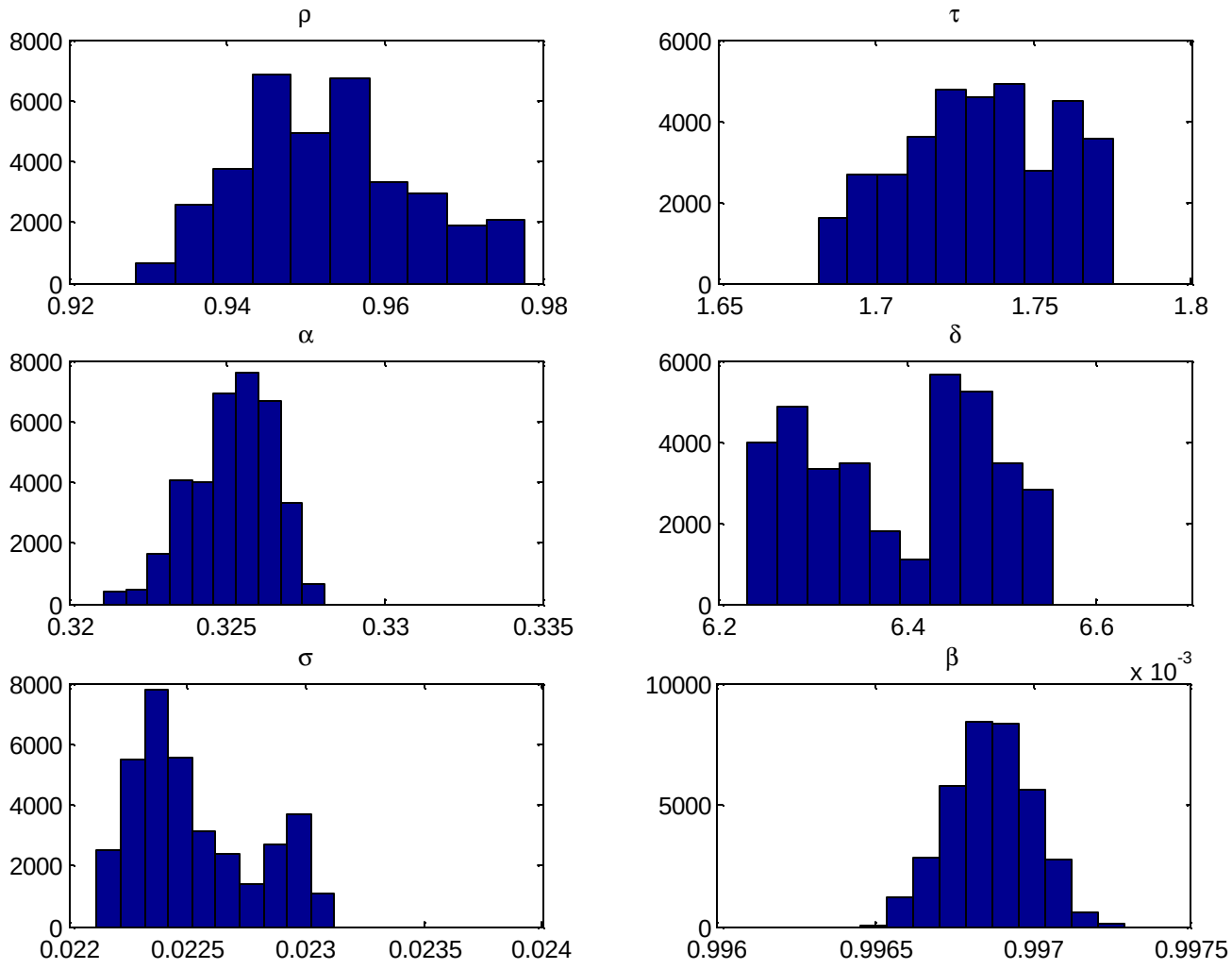
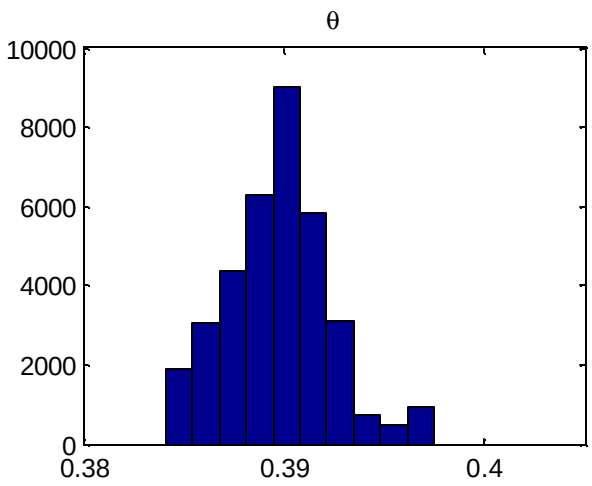

θ
12000
10000
8000
6000
4000
2000
0
0.357
0.3572
0.3574
σ1
10000
8000
6000
4000
2000
0
1.5805
1.581
1.5815
1.582
x 10-4
σ2
12000
10000
8000
6000
4000
2000
0
1.117
1.1175
1.118
1.1185
x 10-3
σ3
x 10^4
2
1.5
1
0.5
0
8.65
8.655
8.66
8.665
x 10-4

Figure 3.4: Posteriors Real Data

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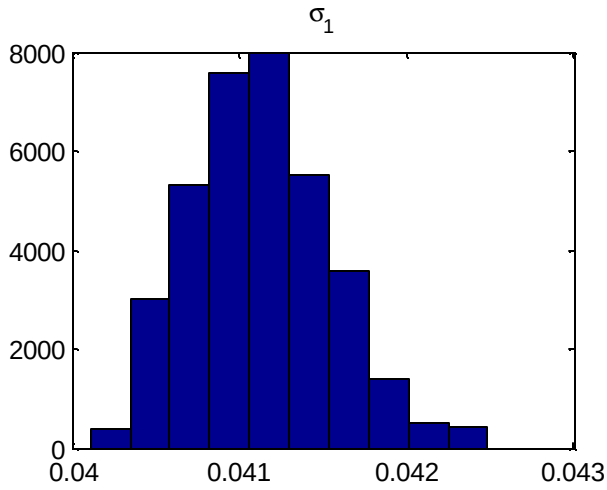
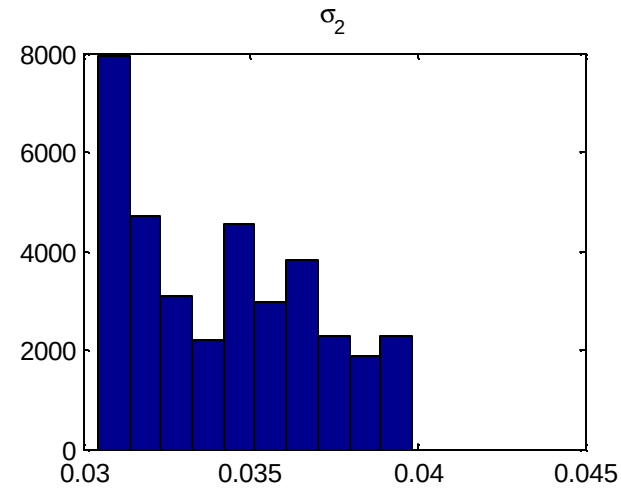
8000
6000
4000
2000
0
0.38
0.39
0.4


σ1
8000
6000
4000
2000
0
0.04
0.041
0.042
0.043


σ2
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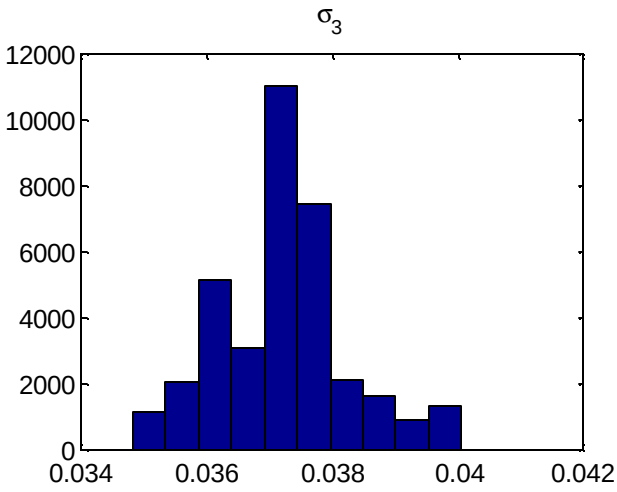
6000
4000
2000
0
0.03
0.035
0.04
0.045


σ3
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10000
8000
6000
4000
2000
0
0.034
0.036
0.038
0.04
0.042

Figure 3.5: CDF Benchmark Calibration

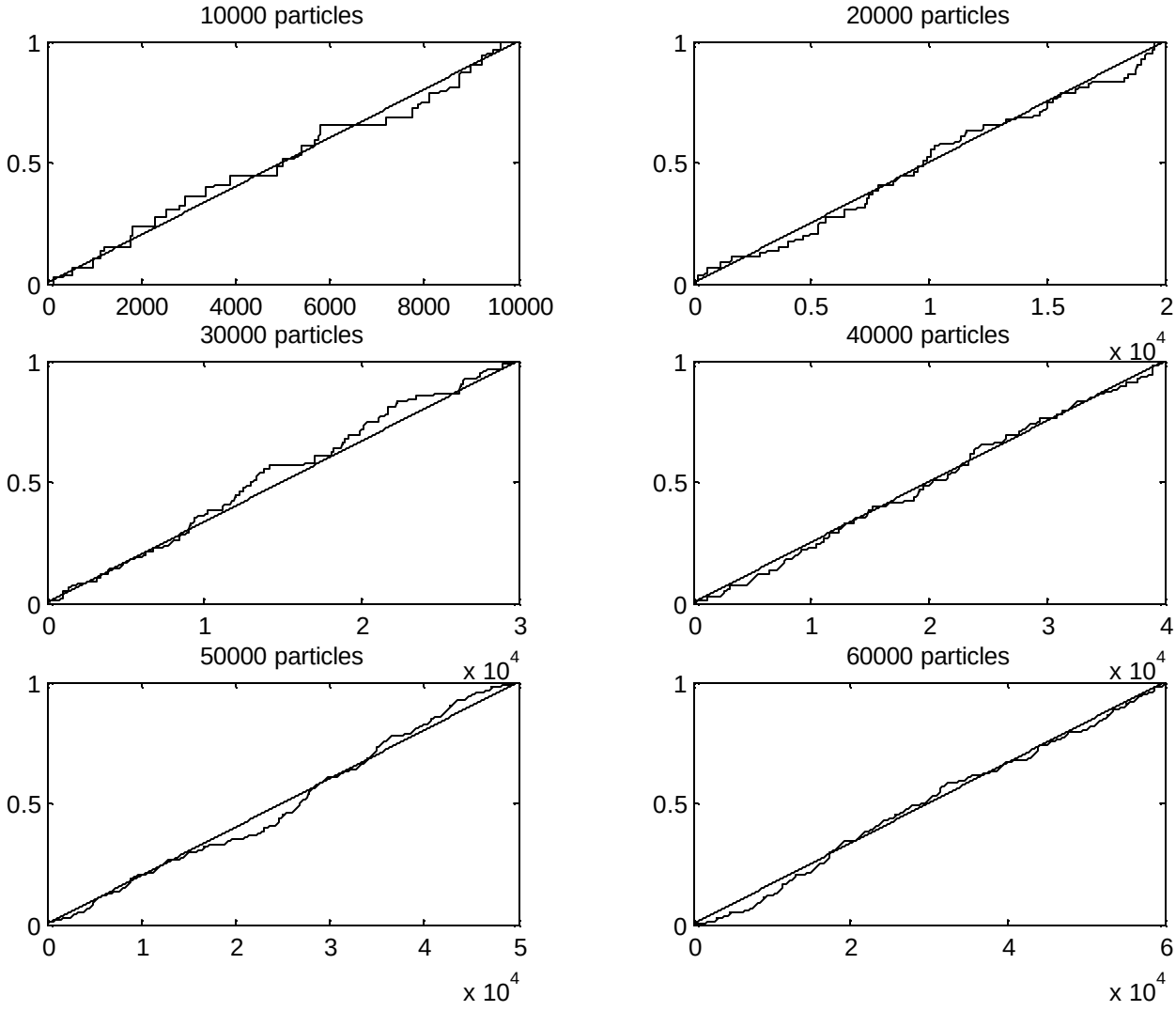

Figure 3.6: CDF Extreme Calibration

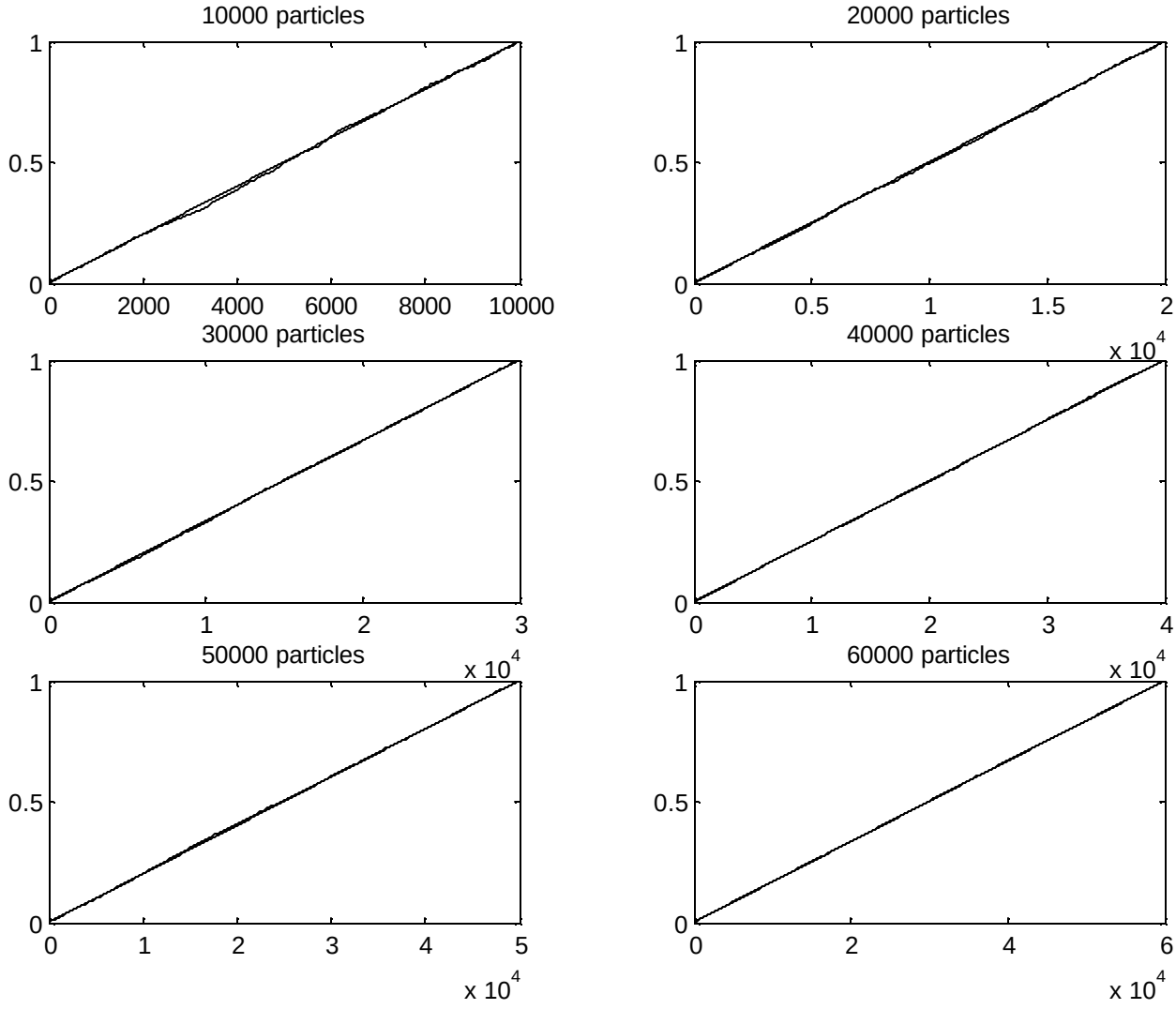

Figure 3.7: CDF Real Data

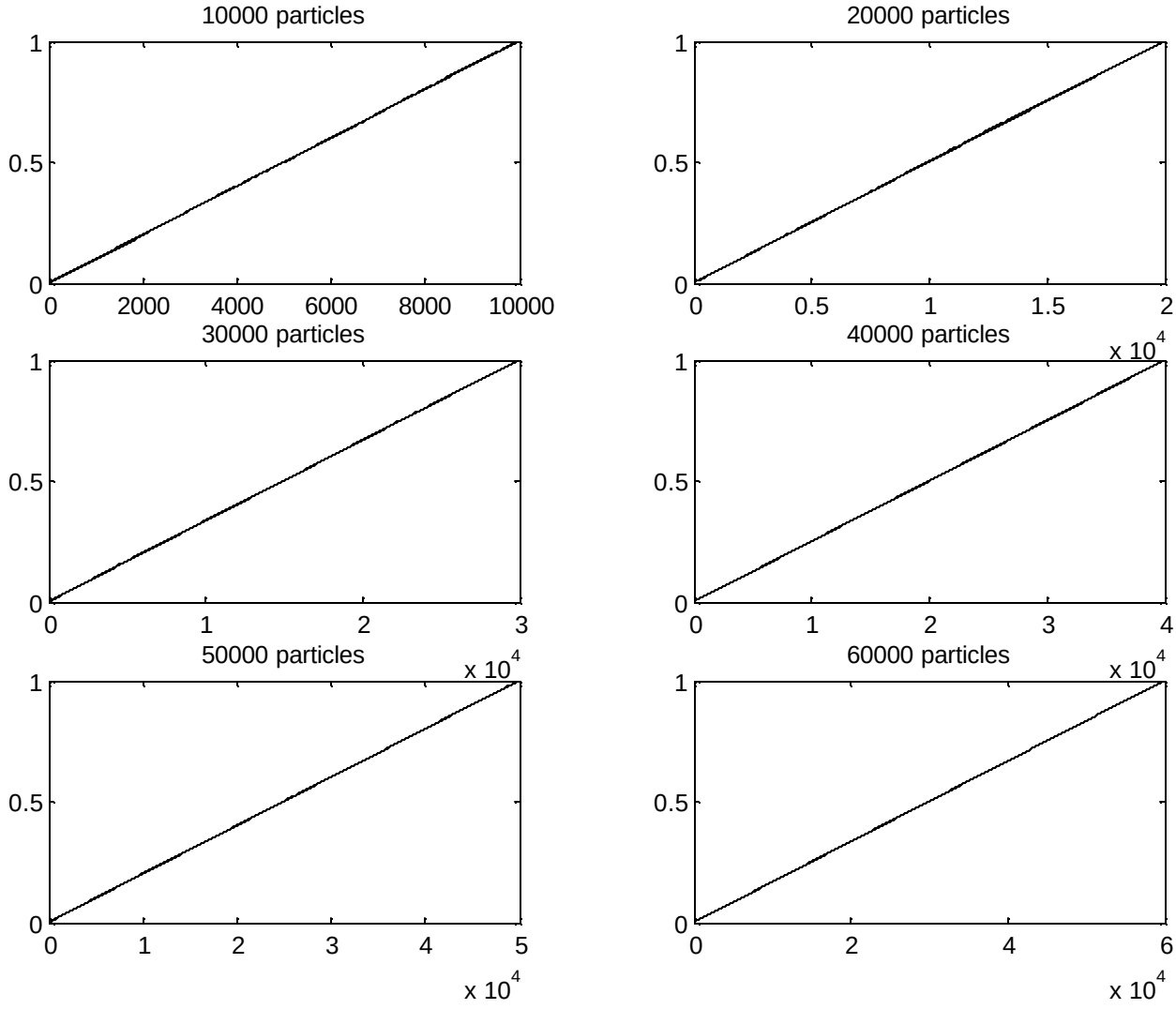

Figure A1: Finete Element Partition