FEDERAL RESERVE BANK *of* ATLANTA | **WORKING PAPER SERIES**

Some Unpleasant Properties of Log-Linearized Solutions When the Nominal Rate Is Zero

R. Anton Braun, Lena Mareen Körber, and Yuichiro Waki

Working Paper 2012-5
March 2012

Abstract: A growing body of recent research examines the nonlinearity created by a zero lower bound on the nominal interest rate. It is common practice in the literature to log-linearize the other equilibrium restrictions around a deterministic steady state with a stable price level. This paper shows that the resulting log-linearized equilibria can have some very unpleasant properties. We make this point using a tractable stochastic New Keynesian model that admits an exact solution. We characterize the log-linearized equilibrium. This characterization is highly misleading. Using the log-linearized equilibrium conditions gives incorrect results about existence and uniqueness of equilibrium and provides an incorrect classification of the types of zero-bound equilibria that can arise in the true economy. These problems are severe. For instance, using empirically relevant parameterizations of the model labor falls in response to a tax cut in the log-linearized economy but rises in the true nonlinear economy.

JEL classification: H30, E12, E62, E50

Key words: monetary policy, fiscal policy, zero bound

The authors thank participants from seminars at the Federal Reserve Bank of Atlanta, the University of Tokyo, the Bank of Portugal, and Australian National University for their helpful comments. They also thank Larry Christiano, Isabel Correira, and Pedro Teles for their helpful comments. The views expressed here are the authors' and not necessarily those of the Federal Reserve Bank of Atlanta or the Federal Reserve System. Any remaining errors are the authors' responsibility.

Please address questions regarding content to R. Anton Braun, Research Department, Federal Reserve Bank of Atlanta, 1000 Peachtree Street, N.E., Atlanta, GA 30309, 404-498-8708, r.anton.braun@gmail.com; Lena Mareen Körber, Department of Economics, London School of Economics and Political Science, Houghton Street, London WC2A 2AE, United Kingdom, l.m.koerber@lse.ac.uk; or Yuichiro Waki, School of Economics, the University of Queensland, 613 Colin Clark Building (39), QLD 4072, Australia, 61 7 336 56574, y.waki@uq.edu.au.

1 Introduction

The recent experiences of Japan, the United States, and Europe with zero/near-zero nominal interest rates have raised new questions about the conduct of monetary and fiscal policy in a liquidity trap. These events have produced a large and growing body of new research that explicitly models the zero bound on the nominal interest rate. Findings from this literature have already influenced the thinking and actions of monetary policy makers. One strand of this recent literature analyzes the effectiveness of monetary and/or fiscal policy in New Keynesian models under the assumption that the monetary authority pursues a Taylor rule that has a nonlinearity due to the zero lower bound on the nominal interest rate. Some recent examples include Del Negro, Eggertsson, Ferrero, and Kiyotaki (2010), Bodenstein, Erceg, and Guerrieri (2009), Eggertsson and Krugman (2010), Eggertsson (2011), Woodford (2011), Christiano, Eichenbaum, and Rebelo (2011) and Erceg and Lindé (2010). All of these papers explicitly model the nonlinearity created by a zero bound on the nominal interest rate on the Taylor rule. However, all of these papers use loglinearized versions of the remaining equilibrium conditions when solving the model.

A second strand of this recent literature analyzes optimal monetary policy in the presence of a zero bound on nominal interest rate. Examples include Jung, Teranishi, and Watanabe (2005), Adam and Billi (2006) and Nakov (2008). Global solution methods are used to compute the optimal policy. However, these papers assume that the implementability condition is summarized by a loglinearized Euler equation and New Keynesian Phillips curve, in addition to the non-negativity constraint on the nominal interest rate.

One conclusion that has emerged from models solved using loglinearized equilibrium conditions is that the dynamics of the New Keynesian model are very different when the nominal interest rate is zero: Eggertsson (2011) finds that output and hours worked fall in response to a labor tax cut when the nominal interest rate is zero, a property that is known as the "paradox of toil"; Braun and Waki (2006) and Christiano, Eichenbaum, and Rebelo (2011) find that the output response to a positive technology shock is negative when the nominal interest rate is zero; and Christiano, Eichenbaum, and Rebelo (2011), Woodford (2011) and Erceg and Lindé (2010) find that the size of the government purchase multiplier is substantially larger than one when the nominal interest rate is zero.

It is well known that loglinear solution methods and perturbation methods more generally only work well within a given radius of the point at which the approximation that is taken and that outside of this radius these solutions break down (See e.g. Den Haan and Rendahl (2009) and Aldrich and Kung (2009)).

The objective of this paper is to provide evidence that such a breakdown occurs when one analyzes the zero bound on nominal interest rates using loglinear solution techniques. We demonstrate that the loglinearized economy exhibits a range of unpleasant properties in this setting. The conditions for existence and uniqueness of equilibrium are misleading and it delivers an incorrect classification of the types of equilibria that can arise. We show that these problems are severe. Using the log-linearized equilibrium conditions labor falls in response to a tax cut for a broad range of parameters and shocks. In the true equilibrium the opposite is the case: labor generally increases. More importantly these reversals occur when using the same parameterizations of the model and shocks that have been used to argue that an anomalous response of hours in the zero bound is *empirically relevant*.

To understand the nature of the problem it is instructive to consider a specific example. Christiano, Eichenbaum, and Rebelo (2011), investigate the size of the government purchase multiplier when the nominal interest rate is zero in the following way. They first parameterize a New Keynesian model in a way that implies that the steady state inflation rate is zero and then use loglinearized equations for all of the equilibrium conditions except the Taylor rule. They then drive the interest rate to fall to zero by shocking the preference discount factor. Finally, they compute the government purchase multiplier by comparing the outcome from this impulse to an alternative scenario where government purchases are disturbed at the same time as the preference discount rate.

It takes a large (5 percent) shock in the preference discount rate to induce a binding zero interest rate. This is not specific to the parameterization considered by Christiano, Eichenbaum, and Rebelo (2011). Coenen, Orphanides, and Wieland (2004), for instance, estimate an New Keynesian model on U.S. data from 1980 to 1999. They find that the probability of a shock driving the nominal interest rate to zero is very low, when the long-run inflation target rate is 2%. Only very large shocks produce a binding zero nominal interest rate in their estimated specification.

We wish to emphasize at the outset that the problem we are raising is *not* about how the lower bound on the interest rate is handled. Instead it pertains to how the *remaining* equilibrium conditions are specified. In this literature the convention is to loglinearize them.[1]

We document the unpleasant properties of the loglinearized economy using a stochastic New Keynesian (NK) model with a zero bound constraint that is similar to specifications considered in Eggertsson (2011) and Woodford (2011). We assume Rotemberg (1996) price adjustment costs. This choice is very convenient. The loglinearized equilibrium conditions for our model are identical to those considered by e.g. Eggertsson (2011) with a suitable choice of the price adjustment cost parameter. Moreover, with Rotemberg (1996) price adjustment costs the exact nonlinear dynamics of the model in a liquidity trap can be represented with two equations in the variables labor input and inflation. A distinct advantage of our setup is that we can compute the *exact* equilibrium of the nonlinear economy. This is particularly helpful in situations where we can not derive analytical results. There is no need to approximate the true solution using .e.g projection methods. [2]

We start by providing a characterization of the time-invariant zero bound equilibria that can arise in the log-linearized economy. We find that there are two types of time-invariant zero bound equilibria and that for a given set of parameters and shocks only one of them exists. The two equilibria have very different properties in terms of their implications for the response of the economy to fiscal policy. They also differ in their implications for comparative statics. In one equilibrium a reduction in the amount of price rigidity magnifies the declines in output and hours. In the other equilibrium the opposite occurs.

[1] Wolman (2005) and Mertens and Ravn (2010) are two notable exceptions to this common practice.

[2] This is no longer the case under Calvo price setting because relative price dispersion is an endogenous state variable.

We then go on to characterize equilibrium of the nonlinear economy. Two new types of equilibria arise that are not possible in the loglinear economy. In addition, the two equations that determine hours and inflation are now nonlinear polynomials and this results in multiple zero bound equilibria for some settings of parameters and shocks. These differences between the log-linearized economy and the true economy are not just theoretical curiosa. Inference suffers severe distortions in empirically relevant settings. In an example from the Great Depression we find that using the log-linearized equilibrium conditions to estimate/calibrate the model produces large biases in the degree of price rigidity and identifies the wrong equilibrium! If one calibrates the model using the exact equilibrium conditions instead the resulting equilibrium is of a type that does not exist in the log-linearized economy and it has different implications for the effects of fiscal policy in a liquidity trap.

One of the sources of the breakdown of the loglinearized economy relates to the resource costs of price adjustment. It is very convenient when solving NK models to center the approximation at a steady state with a stable price level. Under this assumption the resource costs of price adjustment are zero and they disappear from the loglinearized aggregate resource constraint. We show that recognizing these resource costs acts to stabilize the response of the price level much as the Taylor rule does when the nominal rate is positive and provide a graphical analysis that illustrates how this occurs.

Our research is most closely related to work by Braun and Waki (2010) who investigate the size of the government purchase multiplier in a NK model with capital accumulation in a liquidity trap. They consider both Calvo (1983) and Rotemberg price setting schemes. The loglinearized solution exhibits large approximation errors under either form of price adjustment. They find that the implied resource costs of price dispersion/price adjustment are as large as 16% of output using the loglinearized solution for a 5% shock to the preference discount factor. Recognizing these resource costs using a global solution method reduces the size of the government purchase multiplier by as much as 50%.

Our research is also related to work by Braun and Körber (2011). They calibrate a NK model with capital formation to Japanese data and confront the model with Japan's experience of zero interest rates. In contrast to the previous papers cited above they find that output increases in response to labor tax cuts and improvements in technology when they solve the model using a global solution method.

Finally, our research is related to Mertens and Ravn (2010). One of the two time-invariant zero bound equilibria that occurs in the log-linearized economy arises under configurations of shocks and parameters that produce sunspot equilibria of the type considered by Mertens and Ravn (2010). That equilibrium has the same qualitative properties documented by Mertens and Ravn (2010). A difference though is that our equilibrium is triggered by a shock to a fundamental and is unique within the class of time invariant zero fundamentals driven zero bound equilibria that we consider here.

The remainder of the paper proceeds as follows. Section 2 proposes an analytically tractable, nonlinear model of a New Keynesian economy with Rotemberg costs of price adjustment when the nominal interest rate is zero. Section 3 provides a characterization of equilibrium in the log-linearized economy. Section 4 compares and contrasts the properties of the loglinearized economy with those of the true nonlinear economy. Section 5 concludes.

2 A New Keynesian model with Rotemberg costs of price adjustment

We use a New Keynesian model with fixed capital in which price stickiness is introduced by Rotemberg (1996) quadratic price adjustment costs. Although Calvo price adjustment is more common in the literature, we defer its analysis to a later section.

Our choice is justified for two reasons. First, when using loglinearized solution methods centered at a stable price level this choice is innocuous: By a suitable choice of parameters Calvo and Rotemberg quadratic adjustment cost specifications deliver the identical loglinearized equilibrium condition and, hence, decision rules. We choose parameters carefully, so that when loglinearized our economy is equivalent to that of Eggertsson (2011).

Second, in the Rotemberg model the equilibrium conditions can be reduced to a two equation system in (current and steadystate) hours and inflation. This makes it very easy to characterize equilibrium of the economy.

2.1 Model

Consider a representative household who chooses consumption c_t, labor h_t and bond holdings b_t to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \prod_{j=0}^{t} d_j \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{h_t^{1+\nu}}{1+\nu} \right\} \tag{1}$$

subject to

$$b_t + c_t = \frac{b_{t-1}(1+R_{t-1})}{1+\pi_t} + w_t h_t (1-\tau_{w,t}) + T_t \tag{2}$$

where ν governs the elasticity of labor supply and σ is the curvature parameter for consumption. The variable d_t is a shock to the preference discount factor. The variable T_t includes transfers from the government and profit distribution from the intermediate producers.

The optimality conditions for the household's problem imply that consumption and labor supply choices satisfy

$$c_t^{\sigma} h_t^{\nu} = w_t (1-\tau_{w,t}) \tag{3}$$

$$1 = \beta E_t \left\{ \frac{d_{t+1}(1+R_t)}{1+\pi_{t+1}} \left(\frac{c_t}{c_{t+1}} \right)^{\sigma} \right\} \tag{4}$$

Perfectly competitive final good firms use a continuum of intermediate goods $i \in [0,1]$ to produce a single final good that can be used for consumption and investment. The final good is produced using the following production technology

$$y_t = \left(\int_0^1 y_t(i)^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}} \tag{5}$$

The profit maximizing input demands of the final good firm are

$$y_t(i)^d = \left(\frac{p_t(i)}{P_t}\right)^{-\theta} y_t \tag{6}$$

where $p_t(i)$ denotes the price of the good produced by firm i. The price of the final good P_t satisfies

$$P_t = \left(\int_0^1 p_t(i)^{1-\theta} di\right)^{1/(1-\theta)} \tag{7}$$

Intermediate goods producer i uses labor to produce output, using the technology: $y_t(i) = h_t(i)$. This production function implies that for all firms their real marginal cost is equal to the real wage w_t. Producer i sets prices to maximize

$$E_0 \sum_{t=0}^{\infty} \lambda_{c,t} \left[p_t(i)y_t(i) - P_t w_t y_t(i) - \frac{\gamma}{2}(\frac{p_t(i)}{p_{t-1}(i)})^2 P_t y_t\right] / P_t \tag{8}$$

subject to the demand function (6). Here $\lambda_{c,t}$ is the stochastic discount factor and is equal to $\beta^t(\prod_{j=0}^t d_j)c_t^{-\sigma}$ in equilibrium.

The first order condition for this problem in a symmetric equilibrium is:

$$0 = \theta w_t + (1-\theta) - \gamma(\pi_t)(1+\pi_t) + \beta E_t \left\{d_{t+1}\left(\frac{c_t}{c_{t+1}}\right)^{\sigma} \frac{y_{t+1}}{y_t}\gamma(\pi_{t+1})(1+\pi_{t+1})\right\} \tag{9}$$

Monetary policy follows a Taylor rule that respects the zero lower bound on the nominal interest rate

$$R_t = max(0, r_t^e + \phi_\pi \pi_t + \phi_y \hat{y}_t) \tag{10}$$

$$r_t^e = -\log(\beta) - \log(d_{t+1})$$

where $\hat{y}_t$ denotes the log deviation of output from its steady state value. The aggregate resource constraint is given by

$$c_t = (1 - \kappa_t - \eta_t) y_t \tag{11}$$

where $\kappa_t \equiv \frac{\gamma}{2}(\pi_t)^2$ is the resource cost of price adjustment and government purchases $g_t = \eta_t y_t$.[3] It follows that gross domestic product in our economy, gdp_t, is given by:

$$gdp_t \equiv (1 - \kappa_t)y_t. \tag{12}$$

This definition assumes that the price adjustment costs are treated as intermediate inputs and thus are subtracted when calculating GDP.

The equilibrium resource constraint is worth looking at closely:

$$c_t + g_t = (1 - \kappa_t)h_t,$$

where $h_t = \int h_t(i)di$. GDP is depressed either by a higher price adjustment costs κ_t, or by lower aggregate labor input h_t, or both. This points to the possibility that there is a wedge between movements in GDP and those in hours. In particular, an increase in GDP and a decrease in hours can coexist. When loglinearized about a steady-state with a stable price level, however, the term κ disappears from the resource constraint and there is a tight link between GDP and hours h_t: if hours increase, that necessarily increases GDP. The premise for the loglinearization method is that κ term is so small that it can be ignored. We will provide examples that illustrate this premise is not correct when one is analyzing an economy in a liquidity trip.

2.2 Aggregate demand and aggregate supply

One of the attractive aspects of our setup is that it is straight forward to summarize the equilibrium restrictions in two equilibrium conditions that are expressed in terms of current and steadystate hours and inflation. In the course of characterizing equilibrium of this economy it will prove to be very helpful to use a short hand to refer to each of these two equations. We choose to use the same terminology as Eggertsson (2011) and Eggertsson and Krugman (2010) and adopt the expressions

[3] We introduce new notation for κ_t because it allows us to isolate the role of omitting the resource costs of price adjustment from the resource constraint.

Aggregate Supply (AS) and Aggregate Demand (AD) to refer to the two equations.[4]

To obtain the AS schedule, we first rewrite the real wage w_t using the labor supply decision of the household, (3), and the resource constraint (11)

$$w_t = \frac{c_t^\sigma h_t^\nu}{(1-\tau_{w,t})} = \frac{(1-\kappa_t-\eta_t)^\sigma h_t^{\sigma+\nu}}{(1-\tau_{w,t})} \tag{13}$$

Next we use expressions (11) and (13) to substitute marginal costs and consumption out of the firm's optimal price setting restriction (9) and obtain the AS schedule

$$0 = \frac{\theta(1-\kappa_t-\eta_t)^\sigma}{(1-\tau_{w,t})} h_t^{\nu+\sigma} + (1-\theta) - \gamma(\pi_t)(1+\pi_t) + \tag{14}$$
$$\beta E_t \left\{ d_{t+1} \left(\frac{1-\kappa_t-\eta_t}{1-\kappa_{t+1}-\eta_{t+1}} \right)^\sigma \left(\frac{h_t}{h_{t+1}} \right)^{\sigma-1} \gamma(\pi_{t+1})(1+\pi_{t+1}) \right\}$$

The second equation is a nonlinear version of the New Keynesian IS curve, or AD curve. It is obtained by substituting consumption out of the household's intertemporal Euler equation (4) using (11) and the production function. The resulting AD schedule is

$$1 = \beta E_t \left\{ \frac{d_{t+1}(1+R_t)}{1+\pi_{t+1}} \left(\frac{1-\kappa_t-\eta_t}{1-\kappa_{t+1}-\eta_{t+1}} \right)^\sigma \left(\frac{h_t}{h_{t+1}} \right)^\sigma \right\} \tag{15}$$

where R_t is given by (10).

We find it convenient to express the aggregate demand and supply schedules in terms of labor input (or gross production) rather than GDP. This choice allows us to highlight the fact that the response of labor differs according to the solution method and to also provide some intuition for why this is the case.

[4] One reason for this terminology is that when interest rates positive under some weak regularity conditions the slope of AS equation is positive and the slope of the AD equation is negative (See e.g. Eggertsson (2011) for more details).

2.3 Stochastic equilibrium with zero interest rates

Following Eggertsson and Woodford (2003) and Eggertsson (2011), we assume that the economy starts off in a perfect foresight steady state with the discount factor $d = 1$ and zero inflation, $\pi = 0$. Steady-state hours are given by $h = ((\theta-1)(1-\tau_w)/[\theta(1-\eta)^{\sigma}])^{1/(\sigma+\nu)}$ and the steady-state value of the nominal interest rate is: $R = 1/\beta - 1$. At time-0, agents in the model economy realize that the preference shock d_t follows a two-state Markov chain with states $(d^L, 1)$, initial condition $d_1 = d^L$, and transition probabilities $P(d_{t+1} = d^L | d_t = d^L) = p < 1$ and $P(d_{t+1} = 1 | d_t = 1) = 1$. We assume that d_{t+1} is known in period t and that the government policy $(\tau_{w,t}, \eta_t)$ depends only on d_{t+1} and is independent of time t. The economy continues in state L until a new shock to the discount factor shifter arrives at which point d reverts to 1 and the economy returns to its perfect foresight steady-state and remains there in all subsequent periods. The preference shock d^L is taken large enough to cause a binding zero lower bound when $d_{t+1} = d^L$.

Following the previous literature we consider an equilibrium in which allocations and prices take on one of two distinct values: one value obtains when the nominal rate is zero and the other applies when the nominal rate is positive an at its steady-state. We will use the superscript L to denote the former value and no subscript to indicate the latter value.

Under these assumptions the equilibrium in state L (more specifically (h^L, π^L)) is characterized by the following nonlinear AS and AD equations.

$$0 = \theta \frac{(1-\kappa^L-\eta^L)^{\sigma}(h^L)^{\sigma+\nu}}{(1-\tau_w^L)} + (1-\theta) - \gamma\pi^L(1+\pi^L) + p\beta d^L \gamma \pi^L (1+\pi^L) \tag{16}$$

$$1 = p\left(\frac{\beta d^L}{1+\pi^L}\right) + (1-p)\beta d^L \left(\frac{(1-\kappa^L-\eta^L)^{\sigma}(h^L)^{\sigma}}{(1-\eta)^{\sigma}h^{\sigma}}\right) \tag{17}$$

where $\kappa^L = \frac{\gamma}{2}(\pi^L)^2$.[5]

[5] Other equilibrium objects are recovered as $y^L = h^L$, $gdp^L = (1-\kappa^L)y^L$, $c^L = (1-\kappa^L-\eta^L)y^L$, etc. Strictly speaking, not all pairs (h^L, π^L) that solve this system are equilibria: $(1-\kappa^L-\eta^L) \geq 0$ and $R+\phi_\pi\pi^L+\phi_y\hat{y}^L - \log d^L \leq 0$ must also be satisfied.

3 A characterization of zero bound equilibria in the loglinearized economy

We start by providing a characterization of the types of time-invariant zero bound equilibria that arise when the loglinearized versions of the two previous equations. This analysis facilitates comparison with the existing literature and is also provide a useful reference point for comparison with the exact nonlinear economy.

Log-linearization of (16) and (17) about the steady-state with zero inflation described above yields:

$$0 = (\theta - 1)(\sigma + \nu)\hat{h}_L - (\theta - 1)\sigma\frac{\hat{\eta}_L}{1-\eta} + (\theta - 1)\frac{\hat{\tau}_w^L}{1-\tau_w} + (1 - p\beta)\gamma\pi^L \tag{18}$$

$$\frac{1}{\beta} - 1 = p(\hat{d}_L - \pi^L) + (1-p)(\hat{d}_L + \sigma\hat{h}_L - \sigma\frac{\hat{\eta}_L}{1-\eta}) \tag{19}$$

where $\hat{\eta}_L = \eta^L - \eta$ and $\hat{\tau}_w^L = \tau_w^L - \tau_w$. Equivalently,

$$\pi_L = \frac{(\theta-1)(\sigma+\nu)}{(1-p\beta)\gamma}\hat{h}_L - \frac{(\theta-1)\sigma}{(1-p\beta)\gamma}\frac{\hat{\eta}_L}{1-\eta} + \frac{(\theta-1)}{(1-p\beta)\gamma}\frac{\hat{\tau}_w^L}{1-\tau_w},$$

$$\pi_L = \frac{1}{p}[1 - \frac{1}{\beta} + \hat{d}_L - (1-p)\sigma\frac{\hat{\eta}^L}{1-\eta}] + \frac{1-p}{p}\sigma\hat{h}_L.$$

First, following the literature, we have loglinearized the model at a steady-state with zero inflation. As we noted above at the end of Section 2.1, this implies that the resource costs of price adjustment are absent from the loglinearized aggregate resource constraint. The second point is that equations (18)-(19) can be derived from a variety of structural models. For instance, if we were to loglinearize an economy with the same preferences and technology but with Calvo price adjustment instead, the resulting loglinearized system is identical to (18)-(19) if the value of the Calvo parameter is suitably chosen.

From the loglinearized AS and AD schedules we see that the AD and AS schedules are both upward sloping if $\beta p < 1$. We will assume this restriction is satisfied in all that follows.[6] Then it is straightforward to characterize the two types of zero bound equilibria that arise in the loglinearized economy. To make the exposition more transparent suppose for now that $\hat{\eta}^L = \hat{\tau}_w^L = 0$. The proofs to all propositions and lemmata are delegated to appendix A.

Proposition 1 Zero bound equilibrium with AD and AS upward sloping and AD steeper than AS (Type 1 equilibrium). *Suppose*

1a) $\beta(1 + \hat{d}_L) > 1$

1b) $\frac{(\theta-1)(\sigma+\nu)}{(1-p\beta)\gamma} < \sigma\frac{1-p}{p}$

Then there is a zero bound equilibrium with $(\hat{h}_L, \pi_L) < 0$*, AD and AS upward sloping, and AD steeper than AS in the loglinearized economy. This equilibrium is shown in Panel A of Figure 1.*

Recent literature on the zero bound such as Eggertsson (2011), Woodford (2011), Eggertsson and Krugman (2011) and Christiano, Eichenbaum and Rebelo (2011) has focused exclusively on this equilibrium. We next show that there is a second type of zero bound equilibrium in the loglinearized economy with both $\hat{\pi}_L < 0$ and $\hat{h}_L < 0$.

Proposition 2 Zero bound equilibrium with AD and AS upward sloping and AS steeper than AD (Type 2 equilibrium). *Suppose*

2a) $\beta(1 + \hat{d}_L) < 1$

2b) $\frac{(\theta-1)(\sigma+\nu)}{(1-p\beta)\gamma} > \sigma\frac{1-p}{p}$

If in addition $\phi_\pi > p$ *then there is a zero bound equilibrium with* $(\hat{h}_L, \pi_L) < 0$*, AD and AS upward sloping, and AS steeper than AD in the loglinearized economy. This equilibrium is shown in Panel D of Figure 1.*

[6] If instead $\beta > 1$ in the zero inflation steady-state the nominal rate $R = 1/\beta - 1$ is negative, violating the zero bound.

We will subsequently refer to equilibria of the form given in Proposition 1 as Type 1 equilibria and those given by Proposition 2 as Type 2 equilibria.

In principle, there are two other configurations of upward sloping AD and AS schedules that could arise in the state L. They are plotted in Panels B and C of Figure 1. We next show that these configurations of AD and AS do not arise in a zero bound equilibrium.

Proposition 3 Non-existence of zero bound equilibrium *Suppose 1a) holds but 1b) is not satisfied. Then there is no equilibrium in the loglinearized economy with a binding zero bound if* $\phi_\pi > p$. *This situation is shown in Panel C of Figure 1.*

Suppose instead 2a) holds but 2b) does not hold, then there is no equilibrium in the loglinearized economy, with a binding zero bound. This situation is shown in Panel B of Figure 1.

This proposition implies that as long as one considers a big/small shock that satisfies 1a)/2a), the restriction 1b)/2b) needs be imposed to have a zero bound equilibrium with low output and inflation to analyze in the loglinearized economy. This property of the loglinearized economy is rather troubling. Conditions 1b) and 2b) are restrictions on parameters. Each of these structural parameters has an economic interpretation. The possibility exists that restrictions 1b) and 2b) may rule out parameterizations that are reasonable on economic grounds. We wish to emphasize that the non-existence result hinges critically on the linearity of AD and AS curves. Below we will show that in the exact nonlinear model an equilibrium may exist under the assumptions of 1a) and 2b), or 2a) and 1b), that is, in situations in which there is no equilibrium in the loglinearized economy. Relying on Propositions 1 -3 to rule out particular configurations of parameters and/or shocks could lead the researcher to omit from consideration exactly those combinations that are most consistent with the data when viewed through the lens of the true model.

We now turn to consider uniqueness of equilibrium. Even though the AD and AS curves are linear the slope of the AD curve varies depending on whether the nominal interest rate is zero. This raises the possibility that there is a second equilibrium with a positive interest rate that is consistent with alternatively 1a) and 1b) or alternatively 2a and 2b). In order to investigate this possibility consider the loglinearized AD without imposing $R = 0$:

$$\begin{aligned}\frac{1}{\beta} - 1 &= p(\hat{d}_L + R_L - \pi^L) + (1-p)(\hat{d}_L + R_L + \sigma\hat{h}_L - \sigma\frac{\hat{\eta}_L}{1-\eta}) \\ &= (\hat{d}_L + R_L) - p\pi^L + (1-p)(\sigma\hat{h}_L - \sigma\frac{\hat{\eta}_L}{1-\eta})\end{aligned}$$

In a Markov equilibrium with a positive interest rate (if it exists), we also have $R_L = r^e + \phi_\pi \pi_L + \phi_y \hat{y}_L$ where $r^e := 1/\beta - 1 - \hat{d}_L$ and $\hat{h} = \hat{y}$. For convenience assume that $\hat{\eta}_L = 0$. Then the AD reduces to $0 = (\phi_\pi - p)\pi_L + (\phi_y + (1-p)\sigma)\hat{h}_L$, and as long as $-\frac{\phi_y + (1-p)\sigma}{\phi_\pi - p} \neq slope(AS)$ the AD and the AS has a unique intersection at $(\hat{h}_L, \pi_L) = (0, 0)$ and the implied nominal rate $R_L = r^e$. Therefore if there is a Markov equilibrium with a positive nominal rate, it is unique and $(\hat{h}_L, \pi_L, R_L) = (0, 0, r^e)$.

Consider first the Type 1 scenario. In particular, suppose that 1a) is satisfied. Under this assumption $r^e < 0$ and setting $R_L = r^e$ violates the zero bound. It follows that under the assumptions of 1a) and 1b) there is a unique time invariant Markov equilibrium, and the zero bound binds in that equilibrium.

Consider now the Type 2 scenario. If 2a) is satisfied then $r^e > 0$ and it is straightforward to show that $(\hat{h}_L, \pi_L, R_L) = (0, 0, r^e)$ satisfies the AD, the AS and the Taylor rule. Thus under the assumptions on d_L and the model parameters made in Proposition 2, there is a second time-invariant Markov equilibrium. The allocations are the same as in the steady-state allocations and the nominal rate is below its steady-state level but still positive.

This result and the configuration of parameters under which it arises is closely related to the sunspot zero bound equilibria considered by Mertens and Ravn (2010). In our Type 2 equilibrium, the d_L shock plays two roles: it affects the preference discount factor and also is a coordination device for agents signaling that the economy is in the zero bound equilibrium and not the equilibrium with a positive nominal rate. In other words, there is a change in fundamentals **and** this change is triggering a sunspot. Mertens and Ravn (2010) consider a pure sunspot that doesn't change economic fundamentals ($d_L = 1$ in our notation). In the loglinear world this amounts to assuming 2b), because otherwise there is no sunspot equilibrium. A second distinction relates to our equilibrium concept. We are restricting the equilibrium to be time invariant in state L and that restriction does not make sense in their setting with endogenous state variables.

In fact it is not difficult to show that there are also time-dependent equilibria in our setting where allocations vary in state L. To explore this possibility of time-dependent equilibria, consider the properties of the loglinearized AD and AS schedules with $R = 0$, but without imposing $(\hat{h}_t^L, \pi_t^L) = (\hat{h}_{t+1}^L, \pi_{t+1}^L)$. As noted in Eggertsson (2011) this system has a stable root when the conditions of Proposition 1 are not satisfied. Thus, there are many non-Markov (time-varying) zero bound equilibria indexed by the initial values of $(\pi_0, \hat{h}_0)$ in the first period that converge to the Markov equilibrium in Proposition 2.

3.1 Fiscal policy in the loglinearized model

The work of Christiano, Eichenbaum, and Rebelo (2011) and Eggertsson (2011) documents unorthodox policy responses of a loglinearized New Keynesian model when the zero lower bound on the nominal rate is binding. In particular, these researchers argue that the government purchases multiplier exceeds one and that hours fall if labour taxes are cut. The following proposition documents that only Type 1 equilibria have these unconventional properties.

Lemma 1 The effects of fiscal policy in the loglinear model

a) In a Type 1 equilibrium, a labor tax increase increases hours and inflation. The government purchase multiplier is above 1.

b) In a Type 2 equilibrium a labor tax increase lowers hours and inflation.

c) Suppose parameters are such that an increase in η results in an increase in government purchases then in a Type 2 equilibrium the government purchase multiplier is less than 1.[7]

[7]In the proof we provide the exact condition in terms of parameters under which an increase in η results in an increase in government purchases.

It is striking that both equilibria are associated with declines in inflation, output, hours and consumption in a zero bound equilibrium, yet have completely the opposite policy implications. Although the unconventional properties in part a) of Lemma 1 have been documented elsewhere, it is still worthwhile to provide some intuition for these results. This will help to understand why the fiscal responses are so different in a Type 2 equilibrium.

Consider first how an increase in the labor income tax translates into higher inflation for a given level of aggregate consumption and hours (h, or equivalently y). A higher labor income tax discourages households from working. At a given consumption level, an increase in the labor tax shifts the labor supply curve upward and to the left. This raises the pretax nominal wage and thus intermediate goods producers' marginal costs. In response to this, intermediate goods producers increase their relative prices. Because all firms are symmetric and aggregate output y is taken as given, their effort to increase their relative price simply ends up raising the nominal price level. Because the marginal price adjustment cost is increasing this adjustment in prices is only partial. Taking aggregate quantities as given, it follows that a labor income tax increase translates into higher inflation.

Expected inflation also goes up, because the policy changes are persistent. Given that the nominal rate is constrained by the zero lower bound, higher expected inflation lowers the real interest rate. A lower real interest rate induces people to consumption more today, and reduces future consumption (savings). This pushes up the wage further for two reasons. First, the labor supply curve shifts left further due to increased consumption. Second, to satisfy increased demand for goods, intermediate firm's labor demand curve shifts rightward. This results into further inflation, a lower real rate, higher consumption, and so on, creating a virtuous cycle.

Lemma 1a) is the end result of this virtuous cycle. Positive feedback of inflation and consumption/hours occurs, resulting in higher inflation, consumption, and hours. What is crucial though is that this virtuous cycle damps. The adjustment of inflation in the second round is smaller than the first round and the response of consumption is also smaller in the second round.

Why doesn't this same dynamic occur in a Type 2 equilibrium? The key is that under 2b) of Proposition 2 the above virtuous cycle doesn't damp; the positive feedback mechanism is amplified when iterated, and diverges. Instead, in a Type 2 equilibrium consumption must *decline* sufficiently in response to an increase in the labor tax to create deflationary price pressure. This produces a dynamic with higher real rates, and lower consumption and lower inflation. Only a this type of vicious cycle damps. Finally observe that the vicious deflationary dynamic cannot occur in a Type 1 equilibrium. This type of dynamic diverges when 1b) of Proposition 1 is satisfied.

To get a better handle on the nature of this damping effect and its relation to assumptions 1b) and 2b), it may be helpful to consider the above hypothetical adjustment process to (expected) inflation using the AD and AS schedules. An increase in τ_w shifts the AS schedule left and leaves the AD schedule unchanged (see the first panel of Figure 2).[8] In a Type 1 equilibrium at the old equilibrium $\{\hat{c}_0, \pi_0^e\}$ the (expected) inflation rate is now too low. Suppose firms take $\hat{c}_0$ as given and adjust their prices as described above. Then inflation rises to the point $\{\hat{c}_0, \pi_1^e\}$ on the new AS schedule. At this new point the real interest rate is lower, and households find consumption $\hat{c}_0$ is too low. So they adjust their consumption to satisfy the Euler equation under a lower real interest rate, and we move back to the point on the AD schedule given by $\{\hat{c}_1, \pi_1^e\}$. The size of this adjustment in consumption is given by $1/slope(AD)$. Notice next that a unit increase in $\hat{c}$ in the second step implies a slope(AS) unit increase in expected inflation in the third step. This process converges if the increase in inflation at step three is lower than in step one, or in other words when $slope(AS)/slope(AD) < 1$, which is condtion 1b). Under this assumption in each successive step the responses of (expected) inflation and consumption are damped relative to the previous step and the dynamic process converges.

Now let's consider the Type 2 equilibrium. In this equilibrium a *higher* inflation in response to an increase in the labor tax produces explosive dynamics that diverge. This can readily be seen by positing an increase in expected inflation in Figure 3. Instead, consumption and the (expected) inflation must fall. It then follows using a similar line of reasoning to above that the increase in

[8] This Figure has used the loglinearized aggregate resource constraint to express the AD and AS schedules in terms of consumption and inflation.

the tax rate generates a deflationary cycle with lower consumption, lower marginal cost, a higher real interest rate, and lower inflation. This cycle damps when $slope(AS)/slope(AD) > 1$ which is equivalent to condition 2b).

The response of hours/output in a Type 2 equilibrium depends on the type of fiscal shock. Hours unambiguously fall when the labor tax is increased. An increase in η, in contrast, may either produce an increase or reduction in hours depending on the parameterization of the model, while consumption drops regardless of the parameterization. When $\text{slope}(AD) - \text{slope}(AS)\frac{\sigma}{\sigma+\nu} > 0$, consumption drops but not so much as to offset the increase in government purchases; Hours increase and the government purchases output multiplier is positive but less than one since consumption has fallen. If instead $\text{slope}(AD) - \text{slope}(AS)\frac{\sigma}{\sigma+\nu} < 0$, consumption declines by more than the increase in government expenditure, and hours fall. The government purchase multiplier is negative if the decline in hours is moderate. However, it may actually be positive if the decline in hours is sufficiently large.[9]

3.2 How the responses of output and inflation vary with model parameters.

Previous research by Christiano, Eichenbaum, and Rebelo (2011) and Werning (2011) finds that increasing the degree of price flexibility has a counter-intuitive property when the nominal interest rate is zero. Increasing the degree of price flexibility increases the magnitude of the declines in output and inflation to a shock in d of a given size. We now show that Type 1 equilibria have this property but that Type 2 equilibria have the opposite property.

Lemma 2 Effects of more price flexibility on output and inflation responses

a) In a Type 1 equilibrium, an increase in price flexibility (lower γ) magnifies the declines in output and inflation

b) In a Type 2 equilibrium, an increase in price flexibility (lower γ) reduces the declines in output and inflation

[9] η is related to g by $\hat{g} = \hat{\eta}/\eta + \hat{h}$. If $\hat{h}$ falls by enough g also falls and the multiplier is positive.

From this Lemma we see that the result by Christiano, Eichenbaum, and Rebelo (2011) and Werning (2011) relies on the equilibrium being one in which the AD schedule is steeper than the AS schedule. If instead the equilibrium has AS steeper than AD the response of hours and inflation fall as prices become more flexible.

An important observation here is that if we let $slope(AS) \rightarrow slope(AD)$ keeping $slope(AD)$, p and β unchanged (which is possible), then both $\frac{d\hat{h}_L}{dp}$ and $\frac{d\pi_L}{dp}$ go to infinity in absolute value - numerators converge to some finite numbers and denominators go to zero. What's more interesting is that if we do that, $\frac{d\hat{h}_L}{dp}/\hat{h}_L$ and $\frac{d\pi_L}{dp}/\pi_L$ also go to infinity. Thus if we let $slope(AS) \rightarrow slope(AD)$ keeping $slope(AD)$, p, β, **and** $(\hat{h}_L, \pi_L)$ unchanged (by changing $\hat{d}_L$ and e.g. γ), then an equilibrium is unchanged but its response to a small change in p can be really wild when the AD and the AS are almost parallel.

Lemma 3 considers what happens as we increase p. A higher p implies a longer expected duration of the low interest rate state.

Lemma 3 The effect of a larger p on the magnitude of the output and inflation responses

a) In a Type 1 equilibrium, an increase in p magnifies the declines in output and inflation

b) In a Type 2 equilibrium, an increase in p reduces the declines in output and inflation

3.3 Discussion

To summarize the loglinearized economy has two types of equilibria. These equilibria arise under distinction configurations of parameters and shocks. The local properties of the equilibria are very different. In a Type 1 equilibrium an increase in price flexibility increases volatility as in De Long and Summers (1986). In a Type 2 equilibrium price volatility falls as prices become more flexible. Fiscal policy also has very different properties in the two equilibria. An increase in government purchases is highly stimulative in the Type 1 equilibrium. In a Type 2 equilibrium, however, the response of output to an increase in government purchases is muted or negative. A Type 1 equilibrium exhibits the paradox of toil, but theres is no paradox of toil in a Type 2 equilibrium. Much of the recent literature has focused on Type 1 equilibria. Two notable exceptions are Mertens and Ravn (2010) and Bullard (2010).

4 Some unpleasant properties of loglinearized equilibria

The methodology that we just illustrated for analyzing the zero bound using a loglinearized equilibrium is very convenient. It is easy to characterize the equilibrium and one can use standard techniques to solve and estimate richer versions of the model.

Unfortunately, this approach has seven unpleasant properties that we now proceed to document.

Unpleasant property I: The conditions that deliver a Type 1 and Type 2 equilibrium in the loglinearized model are invalid in the true economy. For example, we will show that a Type 1 equilibrium arises in the nonlinear model under the assumptions of Proposition 2.

Unpleasant property II: The classification into two types of zero bound equilibria is incorrect. There are at least four distinct slope configurations that arise when the zero lower bound binds in the true economy. These equilibria have different dynamic and comparative static properties.

Unpleasant property III: The loglinearized equilibrium conditions imply no equilibrium exists in some situations where there is a zero bound equilibrium. In some other situations where these conditions say there is no equilibrium there is in fact a zero bound time invariant equilibrium in the nonlinear economy.

Unpleasant property IV: Section 3 documented that if there is an equilibrium in the loglinear economy, then it is unique. Whether that unique equilibrium is of Type 1 or Type 2 depended on the size of the shock and the configuration of parameters. We will show that in the true nonlinear economy that there can be multiple zero bound equilibria. For instance, we provide examples where Type 1 and Type 2 equilibria exist for the same set of shocks and parameters.

Unpleasant property V: The conditions that deliver the paradox of toil or a government output multiplier are very different: They depend not only on parameters, but also on an equilibrium inflation rate. In particular, the paradox of toil is an unusual result that only applies when the size of the shocks are very small under standard parameterizations of the model.

Unpleasant property VI: If there exists a zero bound equilibrium in the loglinearized economy, then hours are always below their steady-state value. Hours and even inflation can be above their steady state level in a zero bound equilibrium in the nonlinear model.

Unpleasant property VII: Estimates of parameters using the loglinearized equilibrium conditions exhibit large biases and may identify an incorrect equilibrium in the true model or discard a parameterization of the model that is empirically relevant in the true model.

4.1 Resource costs of price adjustment and marginal cost

In the results that follow we will establish that abstracting from the resource costs of price adjustment can have a first order impact on the properties of the model in a liquidity trap.

To isolate the role of the resource costs of price adjustment it will be helpful to consider the following *misspecified nonlinear economy.* In this economy κ is set to its steady-state value of zero and the AD and AS schedules are given by:

$$0 = \theta \frac{(1-\eta^L)^\sigma (h^L)^{\sigma+\nu}}{(1-\tau_w^L)} + (1-\theta) - \gamma\pi^L(1+\pi^L) + p\beta d^L \gamma \pi^L (1+\pi^L) \tag{20}$$

$$1 = p\left(\frac{\beta d^L}{1+\pi^L}\right) + (1-p)\beta d^L\left(\frac{(1-\eta^L)^\sigma (h^L)^\sigma}{(1-\eta)^\sigma h^\sigma}\right) \tag{21}$$

This restriction on κ^L is of interest because the loglinearized equilibrium around a steady-state with zero inflation has this same property. Since the steady-state resource costs are zero, the value of κ^L does not appear in the loglinearized system.

To illustrate why abstracting from the resource costs might matter consider the following perfect foresight version of our model. Suppose that d can take on one of two values $\{1, d_L\}$. Assume further that $d_L > 1$ and that d_L is sufficiently large such that $R = 0$ when $d = d_L$. To keep the discussion as simple as possible suppose that the economy starts off initially in period 0 in a liquidity trap with $d = d_L$ and $R = 0$. It may have been in this situation for a number of periods. Suppose also that $d = dL$ between today and tomorrow but that in all subsequent periods $d = 1$ and the economy is in a steady-state with allocations c, h, π.[10] Under these assumptions the AD and AS schedule for this economy in the current period are given by:

$$\begin{aligned}
[(1-\eta_1-\kappa)h_1]^{-\sigma} &= c_1^{-\sigma} = \beta d_L c^{-\sigma}, \\
\pi_1(1+\pi_1) &= \frac{1}{\gamma}[\frac{\theta c_1^\sigma h_1^\nu}{1-\tau_{w,1}} + 1 - \theta].
\end{aligned}$$

The equilibrium conditions for the *misspecified* economy in which the resource costs (κ) are set to zero in the aggregate resource constraint are:

$$[(1-\kappa)h_1]^{-\sigma} = c_1^{-\sigma} = \beta d_L c^{-\sigma}, \tag{22}$$

$$\pi_1(1+\pi_1) = \frac{1}{\gamma}[\frac{\theta c_1^\sigma h_1^\nu}{1-\tau_{w,1}} + 1 - \theta]. \tag{23}$$

To illustrate the distinction between the true model and the misspecified model consider Figures 4-6. Figure 4 consists of three curves in the c and h space in a steady-state with $d = 1$. The iso-

[10]This economy has no endogenous state variables so it jumps instantly to the steady-state.

marginal utility of consumption curve, the aggregate resource constraint conditional on $\kappa = 0$ and the household intratemporal first order condition $c^\sigma h^\nu = (1 - \tau_w)mc$ conditional on a steady-state markup of 1. The steady-state equilibrium occurs at point S.S. The economy is in this steady-state in all periods from period 2 on.

Consider the *misspecified* economy first in period 1. The nominal interest rate is zero and $d = d_L$. Figure 5 reports the three schedules for the *misspecified* economy in the low state. Households are patient, $d_L > d$, and it follows from (22) that the marginal utility of consumption is above its steady-state level, $u_c > \underline{u_c}$. The intratemporal FONC is also shifted to the left. With consumption and work effort down the inflation rate is also below its steady-state level $\pi_L < \pi$ and it follows from the definition of marginal cost that the markup exceeds its steady-state value. The result is that both hours and consumption are located at point L^M.

Consider next what happens when the resource costs of price adjustment are recognized in Figure 6. There are two effects. First, the slope of the aggregate resource constraint declines. Less of gross production is available for consumption. Second, the intratemporal FONC schedule is shifted right. The equilibrium for the true economy in the state L is located at point L^T. From this we see that recognizing the resource costs of price adjustment dampens the response of marginal cost and thereby mitigates the declines in hours and inflation. Thus the marginal utility of consumption in period 0 is lower (i.e. consumption is higher) than that in the misspecified model. Note that this is isomorphic to an exogenous increase in the government spending share η in the misspecified model. It has been documented elsewhere that the government spending multiplier is much bigger than one in New Keynesian models when the zero lower bound is binding (e.g. Braun and Waki (2010) Christiano, Eichenbaum, and Rebelo (2011), Eggertsson (2011), Woodford (2011)). In this sense the resource costs κ act as an automatic stabilizer that reduce the variation in marginal cost and the inflation. This role is similar to the rule played by a monetary authority who seeks to stabilize the price level in a situation where the nominal rate is positive.

We now turn to analyze the nonlinear stochastic model.

4.2 Analytical results

In the loglinearized economy, both the AD and the AS curve are upward-sloping under the weak regularity condition that $\beta < 1$. An attractive feature of our setup is that we can also derive analytical conditions on the parameters and shocks that deliver upward sloping AD and AS schedules in the exact nonlinear economy.

Proposition 4 establishes a condition under which the AD schedule is downward sloping in the nonlinear economy.

Proposition 4 *For π^L such that $1-\kappa^L-\eta^L > 0$, the AD schedule is downward sloping at (h_L, π_L) if and only if*

$$(1 - \kappa^L - \eta)p\beta d^L + \sigma(\kappa^L)'(1 + \pi^L)^2 \left(1 - \frac{p\beta d^L}{1 + \pi^L}\right) < 0. \tag{24}$$

It is upward-sloping (vertical) if and only if the left hand side is positive (zero).

From equation (19) we know that the AD schedule is upward sloping in the loglinearized economy for values when $0 < p < 1$ and it is vertical when $p = 0$. It is clear by inspection that this restriction is very different from the restriction in equation (24). This an example of *unpleasant property I.*

The following lemma indicates that the resource costs of price adjustment are responsible for this difference in the two specifications.

Lemma 4 *In both the misspecified nonlinear economy and the loglinearized economy the AD schedule is is vertical if $p = 0$ and upward sloping if $p > 0$.*

Perhaps the single most important result in Section 3 was that the AD schedule is upward sloping. This property ruled out, for instance, the possibility that the AD and AS schedules had their conventional shapes in the loglinearized economy. Observe though that when $p = 0$ (4) implies that the AD schedule is downward sloping in the true economy. By a standard continuity argument it is also downward sloping for $p > 0$ and sufficiently small. This illustrates that the loglinearized equilibrium conditions give an incorrect classification of the types of zero bound equilibria that pertain in the true nonlinear economy. This is an example of our *unpleasant property II.*

Historical episodes with zero nominal interest rates are rather long. Japan experienced zero interest rates between 2009 and 2006. And in the U.S. the Federal Funds rate has been about zero since 2008. This suggests that we would like to consider p that are much larger than zero. It is hard to ascertain the slope of the AD schedule when p is large using (24). In addition to p, the size of the shock d^L and the configuration of other parameters matter. We will use numerical methods to ascertain the slope of the AD schedule when p is larger below.

Our results raise the possibility that the AD and AS schedules could have their conventional slopes when the nominal interest rate is zero. In order to confirm this possibility we need to ascertain that the slope of the AS schedule may indeed be positive in the true economy.

This is accomplished in the following proposition.

Proposition 5 *The AS schedule is upward sloping at* (h^L, π^L) *if and only if*

$$[\gamma\pi^L(1+\pi^L)(1-p\beta d^L)+(\theta-1)]\sigma(\kappa^L)'+\gamma(1+2\pi^L)(1-\kappa^L-\eta)(1-p\beta d^L)>0. \tag{25}$$

It is downward-sloping (vertical) if and only if the left hand side is negative (zero).

If follows that one situation in which the AD and AS schedules have their conventional slopes is when p is small and (25) is satisfied.

More generally, the AS schedule in the true model is more likely to be upward sloping when $p\beta d^L < 1$. This is the precise condition that delivers an upward sloping AS schedule in the misspecified economy that omits the resource costs of price adjustment:

Lemma 5 *If* $1 > p\beta d^L$ *then the misspecified AS nonlinear schedule is upward sloping for* $\pi_L > -\frac{1}{2}$.

In the true nonlinear economy the first term in brackets in (25) is also negative when $p\beta d^L < 1$. This acts to reduce the overall size of the first term in the sum when it is negative and also opens the door to the possibility of the first term being positive.

Recall from Section 3 that the slope of the AS schedule was upward sloping in the loglinearized economy if $\beta < 1$. It is clear then that *unpleasant property I* applies to the AS schedule as well. Our results for the AS schedule are particularly interesting because they establish that there are other important differences between the nonlinear economy and the loglinearized economy besides just the resource costs of price adjustment. Even when these resource costs are set to zero the conditions for an upward sloping AS schedule differ between the nonlinear economy and the loglinearized economy.

A final point that we wish to mention pertains to the shape of the AD and AS schedules. In the loglinearized economy the schedules were linear. In the true model the AD and AS schedules are well-defined functions when viewed as mappings from π_L to h_L. However, they may be correspondences when viewed as mappings from h_L to π_L. In the proof to Proposition 4 we show that the condition on the slope of the AD schedule can be expressed as a second order polynomial in the inflation rate. And in Proposition 2 the condition on the slope of the AS schedule can be expressed as a third order polynomial in the inflation rate. These differences between the true and loglinearized economy raise the possibility that the number of equilibria could differ across the two economies. We will see that this turns out to be the case.[11]

4.3 Numerical results

The analytical results point out some potentially important differences between the properties of the true economy and the loglinearized economy. We now turn to illustrate that the issues raised also arise in empirically relevant settings.

A distinct advantage of our model is that we can compute the exact stochastic rational expectations equilibrium for a given parameterization up to the accuracy of the computer. There is no need to use perturbation or projection methods to approximate the solution.

[11] In what follows we choose to place h_L on the x-axis and π_L on y-axis in our plots of the AD and AS schedules. This choice follows conventions in the literature and makes it easier for the reader to follow our discussion. But, the reader should keep in mind that we are plotting correspondences.

We start by considering a parameterization of the model that facilitates comparison with other recent research on the zero bound. Using the parameterization reported in Table 1 our model has the identical loglinearized equilibrium conditions and thus produces the same impulse responses of hours and inflation to a shock in d as the specifications considered in Eggertsson (2011) and Woodford (2011).[12] The parameterization is designed to reproduce facts from the great depression.

Although the loglinear dynamics of our model are identical to the models of Eggertsson (2011) and Woodford (2011), we wish to emphasize that there are some differences in the underlying nonlinear economies they consider and the nonlinear economy we consider here. Both Eggertsson (2011) and Woodford (2011) assume Calvo pricing. Woodford (2011) assumes a homogeneous labor market as we do but Eggertsson (2011) assumes instead that each type of intermediate good is produced with a distinct type of labor. Eggertsson (2011) also assumes that the tax on labor is a payroll tax. We do not believe that these differences are essential. Still, it is important that the reader keep in mind that when we refer to results for the "true" model it is for the model described above in Section 2.

Column 1 in Table 2 reports results for the loglinearized economy. The size of the shock to d^L and the parameterization of the model satisfy conditions 1a) and 1b) of Proposition 1. It follows from that proposition that both the AS and AD schedules are upward sloping and the AD schedule cuts the AS schedule from below. Output declines by 30 percent and the inflation rate falls by 10 percent. Column 2 reports the slopes of the nonlinear misspecified AD and AS schedules that arise when one imposes $\kappa^L = (\kappa^L)' = 0$. We already know from Lemma 4 that the misspecified nonlinear model delivers an upward sloping AD schedule. This parameterization also satisfies the conditions of Lemma 5 so the AS schedule is also upward sloping. Most importantly the AD is steeper than the AS schedule.

Observe that some (small) approximation errors can be seen when comparing the loglinear with the misspecified nonlinear specifications. The AS and AD schedules are both flatter in the

[12]The parameter that governs, γ, the adjustment cost of prices is set according to the formula $\gamma = \frac{((1+(\zeta-1)\theta)(\theta-1)\alpha)}{((1-\alpha)(1-\beta\alpha))}$. This formula is derived by comparing the coefficient on output in the AS schedule of Eggertsson (2011) with the same coefficient in the loglinear version of our model.

misspecified nonlinear model and this results in a larger output response and smaller inflation response as compared to the loglinearized economy. However, the overall magnitude of these errors is small and the qualitative properties of the results in the first two columns of Table 2 are very similar.

The properties of the true nonlinear economy reported in column 3 of Table 2 illustrate that *unpleasant property I and II* also arise in empirically relevant settings. According to Proposition 1 this is a Type 1 equilibrium. Yet, the AD and the AS schedules are both downward sloping! Moreover, this configuration of slopes cannot be supported as an equilibrium in the loglinearized economy.

A final noteworthy feature of Table 2 is that the true model produces much smaller responses of output and inflation. From the perspective of the true model the shock to d is much too small to reproduce the Great Depression responses of output and inflation.

One way to bring the model into better accord with the targets of a 30 percent decline in output and the 10 percent decline in inflation is to increase the size of the shock to d. Table 3 reports how the properties of the true nonlinear economy change as the size of the shock to d is varied.[13]

Before going into the details of the results reported in Table 3 it is helpful to summarize some of the properties of the loglinearized economy that we derived in Section 3. All values of d reported in this table satisfy $\beta d^L > 1$. In addition the parameterization also satisfies condition 1b) in Proposition 1. So according to the loglinearized equilibrium conditions each set of results reported in Table 3 should exhibit an upward sloping AD schedule, an upward sloping AS schedule and an AD schedule that is steeper than the AS schedule. We also know from the results in Section 3 that under these conditions there is a unique zero bound equilibrium for the loglinearized economy.

Table 3 has two panels. This is because for some configurations of parameters there are two time-invariant zero bound equilibria in the true nonlinear model.

Consider the results reported in the upper panel of Table 3 under the heading *First Equilibrium.*

[13]The value of d^L reported in Table 3 is the annualized value of d^L expressed as a net percentage: $((d^L)^4 - 1) \times 100$.

Inspection of Table 3 reveals extensive violations of Proposition 1. Results that are consistent with Proposition 1 only obtain for small shocks to d that range from 1.65 percent to 1.25 percent. If the shock size is reduced further there is no zero bound equilibrium. If the shock size is increased any further the AD and/or AS schedules do not have property that $slope(AD) > slope(AS) > 0$. It is clear from this that the conditions that deliver a Type 1 equilibrium in the loglinearized model are invalid in the true economy which is *unpleasant property I*.

Two of the configurations of AS and AD that arise in Table 3 are not possible in the loglinearized economy. This is an example of *unpleasant property II*.

What slope configurations arise? For $3.98 \geq d^L > 2.02$ percent both the AD and AS schedules have their conventional slopes. The AD schedule is downward sloping and the AS schedule is upward sloping. If $d^L > 3.98$ percent, both schedules are downward sloping. Within that range, hours and output both fall if $7.31 \geq d^L > 3.98$ percent. In contrast, if $d > 7.83$ percent, hours increase while GDP falls.

The distinction between the response of GDP and hours arises because GDP is net of the resource costs of price adjustment. For values of $d > 7.83$ percent the resource costs are so large that the sign of the response of these two variables is different. We will say more about this distinction below.

Consider next the magnitude of the responses of output and inflation. If attention is limited to choices of d that are consistent with Proposition 1 the responses of output and inflation are very small. The maximum decline in GDP consistent with this configuration of the AD and AS schedules is 2.25 percent and the corresponding annualized rate of deflation is -0.69 percent. This is far away from the Great Depression targets of a 30 percent decline in output and a 10 percent decline in inflation that were used to calibrate the loglinearized economy.

If we hold fixed the parameterization of the model, it takes an annualized 12.55 percent increase in d to reproduce a 30 percent decline in output in the true model. This parameterization though understates the 10 percent inflation target by about half. Moreover, with this choice of d^L both the AD and AS schedules are downward sloping. These results suggest that it takes a very different combination of shocks and costs of price adjustment to reproduce the Great Depression declines in output and inflation using the true model and that the dynamic and comparative state properties of the resulting equilibrium could be very different. Or put in another way estimating the parameters of the model using the loglinearized equilibrium conditions could induce some large biases and identify an incorrect equilibrium. This is *unpleasant property VII*.

Table 3 also contains an example of *unpleasant property VI*. In the loglinear economy hours always fall when d is increased. Notice though that when the shocks to d are sufficiently large hours actually increase in the upper panel of Table 3.

Another important distinction between the true model and the loglinearized model pertains to uniqueness of equilibrium. All of the results reported in Table 3 satisfy conditions 1a) and 1b) in Proposition 1. It follows that there is a unique Type 1 equilibrium in the loglinearized model. The results reported under the heading *Second Equilibrium* indicate that the true model does not share this property. For large shocks to $d^L \geq 7.31$ percent there is a second zero bound equilibrium in the nonlinear model. This second equilibrium is example unpleasant property IV: equilibrium is not always unique in the true economy.

The second equilibrium in the nonlinear model has the property that the AD and AS schedules are both upward sloping, although, the AD schedule always cuts that AS schedule from above. This is the fourth type of equilibrium we encounter in the nonlinear model. This is also an example of *unpleasant property III*. We are encountering an equilibrium with $(slope(AS) > slope(AD) > 0)$ using a combination of parameters and shocks that implies according to Proposition 1 that $(slope(AD) > slope(AS) > 0$ should obtain. This distinction in the slopes is important because a tax cut only has a contractionary effect on labor input if the AD schedule cuts the AS schedule from below. Other properties of this equilibrium are unconventional. Hours and inflation both lie above their steady-state levels. But, GDP is below its steady-state level.

So far we have limited attention to the parameterization of the model reported in Table 1 That parameterization however cannot simultaneously reproduce the output and inflation targets of a 30% GDP decline and a 10% decline in the inflation rate using the "true" model. We now consider what happens if the "true" model is reparameterized to reproduce both targets. We adjust the size of the shock to d and the size of γ. Table 4 reports results for the recalibrated model. It also illustrates how the equilibrium changes as the size of the shock to d is varied.

All of the results reported in Table 4 satisfy condition 2b) of Proposition 2. For shocks to $d^L > 1.21$ condition 1a) of Proposition 1 is satisfied. For smaller shocks to d^L condition 2a) of Proposition 1 is satisfied instead.

We find that a shock to d that increases the preference discount rate by 8.57 percent in conjunction with a value of γ that implies a Calvo parameter of 0.636 successfully reproduces both targets. For this parameterization of the model there is a unique equilibrium in the nonlinear model and the AD and AS schedules are both downward sloping with the AS schedule steeper than the AD schedule. This outcome is particularly revealing. The configuration of parameters and shocks satisfies 2b) and 1a) a configuration under which no equilibrium exists in the log-linearized economy. The values of d^L and the Calvo Parameter that fit the Great Depression facts are very different from the values to fit the same facts using the log-linearized equilibrium conditions. Finally, according to Proposition 3, this type of equilibrium does not even exist in the loglinear model! It is clear from this that using the loglinearized equilibrium conditions to estimate/calibrate the model could lead one very far astray in this setting.

Another interesting property of this nonlinear equilibrium concerns the large difference between the hours response and the GDP response. GDP is down by 30 percent but hours are 21 percent above their steady-state level. Any difference between the response of GDP and hours is attributable to the response of the resource costs and it can thus be seen that the resource costs of price adjustment are playing a central role in determining the dynamics of the model.

Table 4 also reports results for smaller shocks to d. One important distinction between the results in Tables 4 and 3 relates to the range of values of d^L that deliver a unique equilibrium. For this parameterization of the model there are two equilibria not only when d^L is large but also when d^L is small. When d^L is large the two equilibria are qualitatively similar to before. The first equilibrium is associated with large declines in hours, inflation and GDP whereas hours and inflation are both above their steady-state values in the second equilibrium.

The two equilibria for small d^L have very different properties. Hours and inflation are low in each of the two equilibria. The first equilibrium is qualitatively similar to the first equilibrium reported in Table 3. The AD and AS schedules take on three distinct configurations depending on the value of d^L. The second equilibrium though satisfies $slope(AS) > slope(AD) > 0$ and is thus qualitatively similar to the Type 2 equilibrium.

In order to understand how two equilibria can arise for small d^L we plot the case of a shock to d of 0.80 percent in Figure 7. Observe that both schedules are convex. For purposes of comparison we also report the loglinearized equilibrium. It is closer to the second equilibrium and exhibits the same slope configuration. In terms of magnitudes the inflation response in the loglinearized model is -1.5 percent. This compares with an inflation rate of -1.08 percent in the second equilibrium. The hours response in the loglinearized model is -0.97 percent. This compares with an hours response of 0.97 percent and a GDP response of -1.19 percent in the nonlinear model.

The U.S. economy has changed considerably since the Great Depression and it is interesting to understand the properties of the model if one sets the size of the shock to d and γ to reproduce events from a more recent event. Christiano, Eichenbaum, and Rebelo (2011) parameterize their model to reproduce declines in inflation and output that are consistent with outcomes during the recent financial crisis in the U.S. They target a one percent decline in the inflation rate and a seven percent decline in output. When we recalibrate the "correct" model to reproduce these outcomes by altering γ and d while holding fixed the other parameters, the resulting value of the Calvo parameter is 0.836 and the value of $d = 3.6$ percent. We find that equilibrium is unique and that the sign of the slope of the AD schedule is negative and the sign of the slope of the AS schedule is positive.

4.4 Fiscal policy in the nonlinear model

4.4.1 A labor tax cut

We now turn to document the part of *unpleasant property V* that relates to the response of hours to a labor tax cut. We have encountered four distinct configurations of the AD and AS schedules in the nonlinear model. Informal graphical analysis would suggest that only in the case where $slope(AD) > slope(AS) > 0$ would a tax cut produce a decline in hours. The following results formalize this intuition.

Proposition 6 Response of hours and inflation to a change in the labor tax[14]

a) Assume that both the AS and the AD schedule are upward sloping and the AD schedule is steeper. Then a labor tax cut lowers hours and inflation.

b) Assume that both the AS and the AD schedule are upward sloping and the AS schedule is steeper. Then a labor tax cut increases hours and inflation.

c) Assume that both the AS and AD schedule is downward sloping and the AS schedule is steeper or assume that the AS schedule is upward sloping and the AD schedule is downward sloping. Then a labor tax cut increases hours and lowers inflation.

The following Lemma summarizes the differences between the nonlinear economy on the one hand and the loglinear and the misspecified model on the other.

Lemma 6 Response of hours to a tax cut when p is small

1. *When $p = 0$ a tax cut increases labor input in the true economy. However, the loglinear and nonlinear economies that abstract from the resource costs of price adjustment imply instead that labor input does not change.*

[14] In this proposition and what follows we limit attention to the four distinct equilibrium configurations of the AS and AD schedules we encountered in the previous section.

2. *For $p > 0$ but sufficiently small, a tax cut increases labor input in the true economy. The loglinear and nonlinear economies that abstract from the resource costs of price adjustment incorrectly imply that labor input should fall.*

We now illustrate using numerical methods that these results are also relevant when p is large. Panel a) of Table 5 reports responses to a labor tax cut for the parametrization given in Table 2. Column 1 reports the responses for the loglinearized model. The equilibrium is of Type 1 and it follows from our analysis in Section 3 that hours and the inflation rate fall. As in Eggertsson (2011) hours fall by 1 percent in response to a 1 percentage point drop in the labor tax rate. Column 2 of Table 5 reports the labor tax multiplier for the misspecified nonlinear model. That model shares the property that hours fall in response to a tax cut. Observe that the labor tax multiplier is smaller. It increased from -1 percent using the loglinearized solution to -0.44 percent. The results for the nonlinear model reported in Column 3 are an example of *unpleasant property V* . Proposition 6 implies that hours increase in response to a labor tax cut because both schedules are downward sloping as documented in Table 2. We can see that the magnitude of the increase is quite substantial and hours rise by 0.56 percent.

Proposition 6 can also be used to ascertain the hours and inflation response to tax cuts for the other simulations we reported above. Inspection of the slopes in Table 3 reveals that a labor tax cut raises labor in the "true" model for all shocks except for $d^L \leq 1.65$. It is worthing noting that $d^L = 1.25$ is the smallest sized shock that is consistent with a zero nominal interest rate.

Consider next the results reported in Table 4. Recall that the third column from the left is calibrated to reproduce the Great Depression. For that parameterization of the model Proposition 6 implies that labor input increases in response to a tax cut. More generally hours increase in response to a tax cut with exception of the right most equilibrium in the upper panel.

Hours also increase in response to a tax cut when the model is parameterized to reproduce output and inflation responses from the recent financial crisis because that calibration has the property that the AD is downward sloping and the AS is upward sloping.

Taken together these results constitute our claim that the result that hours fall on a tax cut in the nonlinear model is an unusual result. Our results imply instead that labor input generally increases in response to a labor tax cut. In particular, hours increase regardless whether we use the Eggertsson (2011) parameterization, recalibrate our true model to reproduce the Great Depression or to the recent financial crisis. For a broad range of shocks to d hours increase too. Labor input only falls for small shocks that lie in a very small neighborhood of the point where the nominal interest rate falls to zero.

Table 5 also reports results for the response of GDP. When the resource costs of price adjustment are ignored there is no distinction between hours and GDP. Recall though that in the true model GDP and hours are related by (12). Interestingly, the results in Column 3 show that while hours increase, the response of GDP is zero. For smaller shocks, GDP falls, while GDP increases for lager shocks. To see why this can happen observe that the GDP labor tax multiplier can be expressed as:

$$\frac{\Delta \ln(gdp^L)}{\Delta \tau_w^L} = \frac{\Delta \ln(1-\kappa^L)}{\Delta \tau_w^L} + \frac{\Delta \ln(h^L)}{\Delta \tau_w^L} \tag{26}$$

This decomposition shows that GDP can fall with a labor tax cut if the savings in resource costs associated with a higher price level are sufficiently large.[15] The final row of panel a) in Table 5 reports the value of $\frac{\Delta \ln(1-\kappa^L)}{\Delta \tau_w^L}$. These savings are quite substantial and exactly crowd out the positive response of gross output (hours).

4.4.2 An increase in government purchases

Christiano, Eichenbaum, and Rebelo (2011) show that the government purchases multiplier is much larger than one when the nominal interest rate is zero. They derive their result using a log-linearized economy and limit attention to configurations of shocks and parameters that satisfy

[15]This decomposition also has a second order term but it is very small for small changes in τ_w^L.

Proposition 1. We now turn to analyze the properties of increases in government purchases in the nonlinear economy.

It is difficult to provide an analytical characterization of the size of the government purchase multiplier in the nonlinear model. So instead we briefly document its properties using numerical solutions. Panel b) in Table 5 reports the government purchase multipliers for hours and GDP using the parameterization of the model reported in Table 1.[16] As documented in the first column, the hours multiplier in the loglinear economy equals 1.55 and it is 1.41 in the misspecified nonlinear economy. Recall that both of these models abstract from the resource costs of price adjustment. Once these costs are recognized in the third column, the hours multiplier is 0.4 while the GDP multiplier equals 1.19. As in the case of a tax shock, the difference can be attributed to the change in the resource costs of price adjustment, which is sizable as shown in the last row of panel b) in Table 5.[17]

More generally, for a broad range of parameterizations we have considered the response of hours to an increase in government purchases is less than one. It is not unusual for the hours response to be negative when e.g. $slope(AS) > slope(AD) > 0$. The GDP response though is typically greater than one. This arises when the inflation rate increases. In this situation the savings in resource costs associated with an increase in the inflation rate leave more output available for consumption and this acts to magnify the response of GDP.

4.5 How the responses of hours and inflation vary with model parameters

We have also investigated how the properties of the nonlinear model change as we reduce the costs of price adjustment and reduce the expected duration of the low state. The properties of the nonlinear model in response to more price flexibility are qualitatively consistent with the loglinear model in the following sense. More price flexibility is stabilizing only if condition 2b) of Proposition

[16]The government spending multiplier for X is defined as $\frac{dX}{dg}$. Note that Eggertsson (2011) e.g. defines the multiplier as $\frac{d\hat{X}}{d\hat{g}}$ which explains the difference to the results reported in his paper.

[17]The change in the resource costs does not exactly fill the gap between the GDP and hours multiplier, but it equals the difference between the GDP and hours response to a government spending shock.

2 holds and both schedules are upward sloping with the AS being more steeper.[18] This is a rather interesting case because among other things it implies that a zero bound equilibrium can arise even when prices are fully flexible. For other configurations of the model parameters and shocks as the extent of price flexibility is increased eventually a threshold is reached beyond which there is no zero bound equilibrium. In many cases hours is above its steadystate when this occurs.

Altering p has somewhat more interesting effects and these effects vary depending on the size of d^L and the value of γ. A longer expected duration of the low state (larger p) magnifies the declines in hours and inflation if both schedules are upward sloping and the AD is steeper. If the AS is steeper instead and the AD is upward sloping, increasing the duration of the zero interest rate episode reduces the decline in hours and inflation.

Interestingly, if we condition on large shocks to dL and if the expected duration of zero interest rates is $p = 0.8$ or less then the AD and AS schedules have orthodox slopes. In our simulations we conditioned on $p = 0.9030$ These experiments suggest that with a somewhat smaller value of p events like the Great Depression would be associated with orthodox configurations of the AD and AS schedules in the true model.

5 Conclusion

A large body of recent research has analyzed the zero bound by taking a short cut. That short cut is to loglinearize all equilibrium conditions except for the monetary policy rule around a steady-state with a stable price level. This paper has illustrated that this common practice has a range of unpleasant properties. We have shown that the dynamics of the nonlinear economy and loglinearized economy are often entirely different when evaluated using the same parameterization and shocks. Our results suggest that one should be very cautious about relying on results from log-linearized economies when analyzing the effects of fiscal policy or estimating parameters of economies that are facing zero interest rates.

[18]The results in this section are established using both numerical and analytical methods. The analytical results are established using the same techniques as for the results on fiscal policy.

An advantage of our setup is that the results for the true nonlinear economy are exact. There is no need to discretize the state-space or use global numerical methods to approximate the "true" equilibrium. In addition, it is very convenient to be able to express the properties of our model in terms of aggregate demand and aggregate supply relations. One question though is whether our results are robust to the form of price adjustment costs. It is not possible to provide exact solutions to a stochastic model like ours using Calvo pricing. The duration of the episode of zero interest rates is exogenous in our model. Under Calvo price setting relative price dispersion is an endogenous state variable and it follows that the duration of zero interest rates is endogenous. So one has no alternative but to use numerical methods to approximate the solution. We are not sure what will happen here but perfect foresight simulations we have performed suggest that similar results to the ones documented here also arise if one considers e.g. Calvo price adjustment and model has a homogenous labour market as in the Rotemberg model analyzed here. In particular, we have found a similar reversal of the paradox of toil. And we have also found multiple equilibria. Conducting a more complete analysis under Calvo price setting is the subject of our current research.

6 Apendix A: Proofs

Proof of Proposition 1 To derive the first restriction observe that the AS schedule passes through $(\hat{h}_L, \pi_L) = (0,0)$ and that the intercept of the AD schedule determines the relative position of two curves at $\hat{h}_L = 0$. At $\hat{h}_L = 0$, the AD schedule is above AS if $1 - \frac{1}{\beta} + \hat{d}_L > 0$. This final restriction is equivalent to condition 1a) in the proposition.

Condition 1b) states that the AD is steeper than the AS. It follows from condition 1a) and the linearity of the two schedules that $\hat{h}_L < 0$ and $\pi_L < 0$ at their intersection. Given that $(\hat{h}_L, \pi_L) < 0$, the linear part of the Taylor rule in equation (10) prescribes $r^e + \phi_\pi \pi_L + \phi_y \hat{y}_L < r^e := 1/\beta - 1 - \hat{d}_L < 0$, which implies that the nominal interest rate is zero. □

Proof of Proposition 2 See the proof for Proposition 1 above. The only difference is that (ϕ_π, ϕ_y) needs be sufficiently large to have $r^e + \phi_\pi \pi_L + \phi_y \hat{y}_L < 0$, since $r^e := 1/\beta - 1 - \hat{d}_L$ is positive under 2a). Since $slope(AD)$ is positive, $\pi_L < icept(AD) = -r^e/p$. Thus

$$r^e + \phi_\pi \pi_L + \phi_y \hat{y}_L < r^e - phi_\pi r^e/p = (1 - \phi_\pi/p) r^e.$$

If $\phi_\pi > p$, then the rightmost term is negative under 2a). □

Proof of Proposition 3 Suppose 1a) holds but 1b) is not satisfied. Then the AD is no steeper than the AS, and the intercept at $\hat{h}_L = 0$ is strictly higher for the AD. When the AD and the AS are parallel, then there is no intersection and thus no equilibrium with $R = 0$. When the AS is strictly steeper than the AD, then their intersection satisfies $(\hat{h}_L, \pi_L) > 0$. Since the AD is upward sloping, $(\hat{h}_L, \pi_L) > (0, icept(AD)) = (0, \frac{1}{p}[1 - 1/\beta + \hat{d}_L])$. Thus,

$$r^e + \phi_\pi \pi_L + \phi_y \hat{y}_L > r^e + \frac{\phi_\pi}{p}[1 - 1/\beta + \hat{d}_L] = \frac{\phi_\pi - p}{p}[1 - 1/\beta + \hat{d}_L] > 0,$$

where the last inequality follows from 1a) and $\phi_\pi > p$. Thus the zero bound is not binding at this intersection. Taken together, there is no equilibrium with a binding zero bound when 1a) holds but 1b) doesn't. The same argument goes through for the case where 2a) holds but 2b) doesn't. One difference is that because r^e is positive in this case, the zero bound doesn't bind at any intersection on the positive orthant. Hence we don't need the assumption $\phi_\pi > p$. □

Proof of Lemma 2 We first solve analytically for the intersection $(\hat{h}_L, \pi_L)$ in terms of parameters, and then differentiate them with respect to γ. Assuming that $\hat{\eta}^L = \hat{\tau}^w_L = 0$ and that $slope(AS) \neq slope(AD)$, we have

$$\hat{h}_L = \frac{1}{slope(AS) - slope(AD)} \frac{1}{p}[1 - \frac{1}{\beta} + \hat{d}_L] \tag{27}$$

$$\pi_L = \frac{slope(AS)}{slope(AS) - slope(AD)} \frac{1}{p}[1 - \frac{1}{\beta} + \hat{d}_L] \tag{28}$$

Therefore,

$$\frac{d\hat{h}_L}{d\gamma} = \frac{-\frac{dslope(AS)}{d\gamma}}{(slope(AS) - slope(AD))^2}[1 - \frac{1}{\beta} + \hat{d}_L]/p \tag{29}$$

Then observe that $[1 - \frac{1}{\beta} + \hat{d}_L] > 0$ is equivalent to $\beta(1 + d_L) > 1$. But the latter condition equals assumption $1a)$ in Proposition 1. The result for inflation and for part b) of the proof can be derived in a similar way.

Proof of Lemma 3 We first solve analytically for the intersection $(\hat{h}_L, \pi_L)$ in terms of parameters, and then differentiate them with respect to p. Assuming that $\hat{\eta}^L = \hat{\tau}^w_L = 0$ and that $slope(AS) \neq slope(AD)$, we have

$$\hat{h}_L = \frac{1}{slope(AS) - slope(AD)} \frac{1}{p}[1 - \frac{1}{\beta} + \hat{d}_L] \tag{30}$$

$$\pi_L = \frac{slope(AS)}{slope(AS) - slope(AD)} \frac{1}{p}[1 - \frac{1}{\beta} + \hat{d}_L] \tag{31}$$

We also have

$$\frac{dslope(AS)}{dp} = slope(AS)\frac{\beta}{1-p\beta} \tag{32}$$
$$\frac{dslope(AD)}{dp} = slope(AD)\frac{-1}{p(1-p)} \tag{33}$$

Thus

$$\begin{aligned}
\frac{d\hat{h}_L/dp}{[1-\frac{1}{\beta}+\hat{d}_L]} &= \frac{[\frac{-1}{p^2}\{slope(AS)-slope(AD)\}-\frac{1}{p}\{\frac{dslope(AS)}{dp}-\frac{dslope(AD)}{dp}\}]}{[slope(AS)-slope(AD)]^2} \\
&= \frac{\frac{1}{p^2}[(-1-\frac{p\beta}{1-p\beta})slope(AS)+(1-\frac{1}{1-p})slope(AD)\}]}{[slope(AS)-slope(AD)]^2} \\
&= \frac{\frac{1}{p^2}[\frac{-1}{1-p\beta}slope(AS)-\frac{p}{1-p}slope(AD)\}]}{[slope(AS)-slope(AD)]^2} < 0
\end{aligned}$$

Thus in a Type-1 equilibrium where $[1-\frac{1}{\beta}+\hat{d}_L]$ is positive, $\frac{d\hat{h}_L}{dp} < 0$, while in a Type-2 equilibrium $\frac{d\hat{h}_L}{dp} > 0$.

$$\begin{aligned}
\frac{d\pi_L/dp}{[1-\frac{1}{\beta}+\hat{d}_L]} &= \frac{[(\frac{-slope(AS)}{p^2}+\frac{dslope(AS)}{dp}\frac{1}{p})\{slope(AS)-slope(AD)\}-\frac{slope(AS)}{p}\{\frac{dslope(AS)}{dp}-\frac{dslope(AD)}{dp}\}]}{[slope(AS)-slope(AD)]^2} \\
&= \frac{\frac{slope(AS)}{p^2}[\frac{2p\beta-1}{1-p\beta}\{slope(AS)-slope(AD)\}-p\{\frac{dslope(AS)}{dp}-\frac{dslope(AD)}{dp}\}]}{[slope(AS)-slope(AD)]^2} \\
&= \frac{\frac{slope(AS)}{p^2}[-slope(AS)+(-\frac{2p\beta-1}{1-p\beta}-\frac{1}{1-p})slope(AD)]}{[slope(AS)-slope(AD)]^2} \\
&= \frac{\frac{slope(AS)}{p^2}[-slope(AS)+\frac{p(2p\beta-1-\beta)}{(1-p\beta)(1-p)}slope(AD)]}{[slope(AS)-slope(AD)]^2} < 0
\end{aligned}$$

The last inequality follows from $2p\beta-1-\beta = (p\beta-1)+(p\beta-\beta) < 0$. Thus, again, in a Type-1 equilibrium where $[1-\frac{1}{\beta}+\hat{d}_L]$ is positive, $\frac{d\pi_L}{dp} < 0$, while in a Type-2 equilibrium $\frac{d\pi_L}{dp} > 0$.

Proof of Lemma 1 We first solve for $(\hat{h}_L, \pi_L)$, allowing non-zero $\hat{\eta}_L$ and τ_w^L. AS and AD are:

$$\pi_L = \frac{(\theta-1)(\sigma+\nu)}{(1-p\beta)\gamma}\hat{h}_L - \frac{(\theta-1)\sigma}{(1-p\beta)\gamma}\frac{\hat{\eta}_L}{1-\eta} + \frac{(\theta-1)}{(1-p\beta)\gamma}\frac{\hat{\tau}_w^L}{1-\tau_w}$$

$$\pi_L = \frac{1}{p}[1 - \frac{1}{\beta} + \hat{d}_L - (1-p)\sigma\frac{\hat{\eta}^L}{1-\eta}] + \frac{1-p}{p}\sigma\hat{h}_L$$

Let

$$\begin{aligned}
slope(AS) &= \frac{(\theta-1)(\sigma+\nu)}{(1-p\beta)\gamma} \\
slope(AD) &= \frac{1-p}{p}\sigma \\
icept(AS) &= -\frac{(\theta-1)\sigma}{(1-p\beta)\gamma}\frac{\hat{\eta}_L}{1-\eta} + \frac{(\theta-1)}{(1-p\beta)\gamma}\frac{\hat{\tau}_w^L}{1-\tau_w} \\
icept(AD) &= \frac{1}{p}[1 - \frac{1}{\beta} + \hat{d}_L - (1-p)\sigma\frac{\hat{\eta}^L}{1-\eta}].
\end{aligned}$$

Then

$$\begin{aligned}
\hat{h}_L &= \frac{icept(AD) - icept(AS)}{slope(AS) - slope(AD)} \\
\pi_L &= icept(AS) + slope(AS)\hat{h}_L \\
(&= icept(AD) + slope(AD)\hat{h}_L)
\end{aligned}$$

We give two equivalent expression for π_L, for it is sometimes convenient to switch between them.

Differentiating by a generic parameter δ (e.g. $\delta = \gamma$, $\delta = \hat{\eta}_L$, etc.),

$$\begin{aligned}
\frac{d\hat{h}_L}{d\delta} &= \frac{(slope(AS) - slope(AD))(\frac{dicept(AD)}{d\delta} - \frac{dicept(AS)}{d\delta}) - (icept(AD) - icept(AS))(\frac{dslope(AS)}{d\delta} - \frac{dslope(AD)}{d\delta})}{(slope(AS) - slope(AD))^2} \\
\frac{d\pi_L}{d\delta} &= \frac{dicept(AS)}{d\delta} + \frac{dslope(AS)}{d\delta}\hat{h}_L + slope(AS)\frac{\hat{h}_L}{d\delta} \\
(&= \frac{dicept(AD)}{d\delta} + \frac{dslope(AD)}{d\delta}\hat{h}_L + slope(AD)\frac{\hat{h}_L}{d\delta})
\end{aligned}$$

Consider a paradox of toil: $\delta = \hat{\tau}_w^L$. In this case only $\frac{dicept(AS)}{d\delta} = \frac{(\theta-1)}{(1-p\beta)\gamma} > 0$ is non-zero, and thus

$$\frac{d\hat{h}_L}{d\hat{\tau}_w^L} = \frac{-\frac{dicept(AS)}{d\hat{\tau}_w^L}}{slope(AS) - slope(AD)}$$
$$\frac{d\pi_L}{d\hat{\tau}_w^L} = slope(AD)\frac{\hat{h}_L}{d\hat{\tau}_w^L},$$

where we used the second expression for $\frac{d\pi_L}{d\delta}$. When $slope(AD) > 0$ (i.e. $p < 1$) then π_L and $\hat{h}_L$ move in the same direction. The sign of $\frac{d\hat{h}_L}{d\hat{\tau}_w^L}$ is the same as that of $slope(AD) - slope(AS)$, since the numerator is negative. Thus under condition $1b$) we have the paradox of toil, i.e. $\frac{d\hat{h}_L}{d\hat{\tau}_w^L} > 0$ and $\frac{d\pi_L}{d\hat{\tau}_w^L} > 0$ (an increase (a decrease) in labor income tax rate increases (decreases) equilibrium hours and inflation). Under $2b$), however, we have $\frac{d\hat{h}_L}{d\hat{\tau}_w^L} < 0$ and $\frac{d\pi_L}{d\hat{\tau}_w^L} < 0$ and there is no paradox.

Consider a change in government expenditure share: $\delta = \hat{\eta}^L$. Slopes are unchanged with respect to this change, but both intercepts change: $\frac{dicept(AS)}{d\eta} = -\frac{(\theta-1)\sigma}{(1-p\beta)\gamma}\frac{1}{1-\eta} = -slope(AS)\frac{\sigma}{\sigma+\nu}\frac{1}{1-\eta}$ and $icept(AD) = -\frac{1-p}{p}\frac{\sigma}{1-\eta} = -slope(AD)\frac{1}{1-\eta}$.

$$\frac{d\hat{h}_L}{d\hat{\eta}^L} = \frac{1}{1-\eta}\frac{(-slope(AD) + slope(AS)\frac{\sigma}{\sigma+\nu})}{(slope(AS) - slope(AD))}$$
$$\frac{d\pi_L}{d\hat{\eta}^L} = -slope(AD)\frac{1}{1-\eta} + slope(AD)\frac{\hat{h}_L}{d\hat{\eta}^L}$$

Under assumption $1b$), $\frac{d\hat{h}_L}{d\hat{\eta}^L}$ is positive, since both the denominator and the numerator on the RHS are negative. Moreover,

$$\frac{d\hat{h}_L}{d\hat{\eta}^L} = \frac{1}{1-\eta}\frac{(slope(AD) - slope(AS)\frac{\sigma}{\sigma+\nu})}{(slope(AD) - slope(AS))}$$
$$= \frac{1}{1-\eta}[1 + \frac{slope(AS)\frac{\nu}{\sigma+\nu}}{(slope(AD) - slope(AS))}] > \frac{1}{1-\eta}$$

and

$$\begin{aligned}\frac{d\pi_L}{d\hat{\eta}^L} &= -slope(AD)\frac{1}{1-\eta} + slope(AD)\frac{\hat{h}_L}{d\hat{\eta}^L} \\ &= slope(AD)\frac{1}{1-\eta}[-1 + \frac{\hat{h}_L}{d\hat{\eta}^L}(1-\eta)] > 0.\end{aligned}$$

Thus assumption 1b) implies that both $\frac{d\hat{h}_L}{d\hat{\eta}^L}$ and $\frac{d\pi_L}{d\hat{\eta}^L}$ are positive, with $\frac{d\hat{h}_L}{d\hat{\eta}^L}$ is greater than $1/(1-\eta)$.

Under assumption 2b), there are two cases: (1) $slope(AD) - slope(AS)\frac{\sigma}{\sigma+\nu} > 0$ and (2) $slope(AD) - slope(AS)\frac{\sigma}{\sigma+\nu} < 0$.

Consider the case (1). In this case $\frac{d\hat{h}_L}{d\hat{\eta}^L}$ is negative. $\frac{d\pi_L}{d\hat{\eta}^L}$ is negative too.

Consider the case (2). In this case, although $\frac{d\hat{h}_L}{d\hat{\eta}^L}$ is positive,

$$\begin{aligned}\frac{d\hat{h}_L}{d\hat{\eta}^L} &= \frac{1}{1-\eta}\frac{(slope(AS)\frac{\sigma}{\sigma+\nu} - slope(AD))}{(slope(AS) - slope(AD))} \\ &= \frac{1}{1-\eta}[1 - \frac{slope(AS)\frac{\nu}{\sigma+\nu}}{(slope(AS) - slope(AD))}] < \frac{1}{1-\eta}\end{aligned}$$

and

$$\begin{aligned}\frac{d\pi_L}{d\hat{\eta}^L} &= -slope(AD)\frac{1}{1-\eta} + slope(AD)\frac{\hat{h}_L}{d\hat{\eta}^L} \\ &= slope(AD)\frac{1}{1-\eta}[-1 + \frac{\hat{h}_L}{d\hat{\eta}^L}(1-\eta)] < 0.\end{aligned}$$

Hence $\frac{d\pi_L}{d\hat{\eta}^L}$ is negative, and $\frac{d\hat{h}_L}{d\hat{\eta}^L}$ is smaller than $1/(1-\eta)$.

Another interesting difference is the response of consumption. Since $c = (1-\eta-\kappa)h$, we have $\hat{c}_L = -\frac{\hat{\eta}^L}{1-\eta} + \hat{h}_L$ and thus

$$\frac{d\hat{c}_L}{d\eta_L} = -\frac{1}{1-\eta} + \frac{d\hat{h}_L}{d\eta^L}.$$

The sign of the consumption response to $\hat{\eta}_L$ is the sign of the RHS, which is determined by whether $\frac{d\hat{h}_L}{d\eta^L}$ is larger or smaller than $\frac{1}{1-\eta}$. Under 1b) this is larger, so the consumption response is positive. (Observe that in this case all consumption, government expenditure level, and hours increase.) Under 2b), in the case (1) the second term on the RHS is nagative, so the consumption response is negative. (Observe that the consumption level decreases more than hours do, so the level of government expenditure should be increasing.) In the case (2) the second term is positive but less than $1/(1-\eta)$, making the RHS negative. (Since hours and η increase, government expenditure level increases too. Consumption drops in response to this.) Thus under 2b), the consumption response to η shock is negative.

Therefore, if we measure the government expenditure multiplier on hours $d\hat{h}_L/d\hat{g}_L$ then it is greater than one under 1b), negative in case (1) under 2b), and positive but less than 1 in case (2). This is because whether $d\hat{h}_L/d\hat{g}_L$ is greater than one or not hinges on the sign of the consumption response in the loglinear economy with $\kappa = 0$.

In sum, under 2b), the paradox of toil disappears, a positive government expenditure shock reduces inflation, and the size of the government expenditure multiplier is less than 1.

Proof of Proposition 4 First observe that the slope of the aggregate demand schedule (17) is given by

$$\frac{D\pi^L}{Dh^L} = -\frac{AD_{h^L}}{AD_{\pi^L}} \equiv -\frac{\partial AD(h^L,\pi^L)/\partial h^L}{\partial AD(h^L,\pi^L)/\partial \pi^L} = \frac{\frac{\sigma(1-p)(1-\kappa^L-\eta^L)^\sigma (h^L)^{\sigma-1}}{(1-\eta)^\sigma h^\sigma}}{\frac{p}{(1+\pi^L)^2} + \frac{(1-p)\sigma(h^L)^\sigma(1-\kappa^L-\eta^L)^{\sigma-1}(\kappa^L)'}{(1-\eta)^\sigma h^\sigma}} \tag{34}$$

Consider the numerator. It is unambiguously positive. Next consider the denominator and observe that the AD schedule is downward sloping at (h^L, π^L) iff

$$(1-p)\beta d^L \frac{\sigma(\kappa^L)'(1-\kappa^L-\eta)^{\sigma-1}(h^L)^\sigma}{(1-\eta)^\sigma h^\sigma} + p\frac{\beta d^L}{(1+\pi^L)^2} < 0,$$

or equivalently,

$$(1-p)\beta d^L \frac{(1-\kappa^L-\eta)^\sigma (h^L)^\sigma}{(1-\eta)^\sigma h^\sigma}\frac{\sigma(\kappa^L)'}{(1-\kappa^L-\eta)} + p\frac{\beta d^L}{(1+\pi^L)^2} < 0.$$

Since (π^L, h^L) is on AD,

$$(1-p)\beta d^L \frac{(1-\kappa^L-\eta)^\sigma (h^L)^\sigma}{(1-\eta)^\sigma h^\sigma} = 1 - p\frac{\beta d^L}{1+\pi^L}.$$

Using this,

$$\begin{array}{rc} & (1-p)\beta d^L \frac{(1-\kappa^L-\eta)^\sigma (h^L)^\sigma}{(1-\eta)^\sigma h^\sigma}\frac{\sigma(\kappa^L)'}{(1-\kappa^L-\eta)} + p\frac{\beta d^L}{(1+\pi^L)^2} < 0 \\ = & (1-p\frac{\beta d^L}{1+\pi^L})\frac{\sigma(\kappa^L)'}{(1-\kappa^L-\eta)} + p\frac{\beta d^L}{(1+\pi^L)^2} < 0 \\ = & \frac{p\beta d^L}{(1+\pi^L)^2}\frac{1}{(1-\kappa^L-\eta)}[(1-\kappa^L-\eta) + \frac{\sigma}{p\beta d^L}(\kappa^L)'(1+\pi^L)^2\left(1-\frac{p\beta d^L}{1+\pi^L}\right)] < 0 \end{array}$$

For π^L such that $1-\kappa^L-\eta>0$, the last term is less than zero iff

$$(1-\kappa^L-\eta)p\beta d^L + \sigma(\kappa^L)'(1+\pi^L)^2\left(1-\frac{p\beta d^L}{1+\pi^L}\right) < 0. \tag{35}$$

Suppose that the AD schedule is misspecified, $(\kappa^L = (\kappa^L)' = 0)$, then the left hand side of (35) simplifies to $1-\eta$, which is unambiguously positive. This is not the case for the true AD schedule. Instead (35) can be expressed as the following quadratic function of π^L

$$[\sigma - \frac{p\beta d^L}{2}]\gamma(\pi^L)^2/(1+\pi_L) + \sigma\gamma(1-p\beta d^L)\pi^L + p\beta d^L(1-\eta).$$

Proof of Proposition 5 Observe that the slope of the AS schedule is given by:

$$\frac{D\pi^L}{Dh^L} = -\frac{AS_{h^L}}{AS_{\pi^L}} \equiv -\frac{\partial AS(h^L,\pi^L)/\partial h^L}{\partial AS(h^L,\pi^L)/\partial \pi^L} \tag{36}$$

$$= \frac{\theta(\sigma+\nu)(1-\kappa^L-\eta)^\sigma (h^L)^{\sigma+\nu-1}}{\theta\sigma(1-\kappa^L-\eta)^{\sigma-1}(\kappa^L)'(h^L)^{\sigma+\nu} + (1-\tau_w)\gamma(1-p\beta d^L)(1+2\pi^L)} \tag{37}$$

Since the numerator is positive, the AS schedule is upward sloping at (h^L, π^L) iff

$$\theta\sigma(1-\kappa^L-\eta)^{\sigma-1}(\kappa^L)'(h^L)^{\sigma+\nu} + (1-\tau_w)\gamma(1-p\beta d^L)(1+2\pi^L) > 0 \tag{38}$$

or equivalently,

$$\frac{\theta(1-\kappa^L-\eta)^\sigma(h^L)^{\sigma+\nu}}{1-\tau_{w,L}}\frac{\sigma(\kappa^L)'}{1-\kappa^L-\eta}+\gamma(1-p\beta d^L)(1+2\pi^L)>0.$$

Since (π^L, h^L) is on AS,

$$\frac{\theta(1-\kappa^L-\eta)^\sigma(h^L)^{\sigma+\nu}}{1-\tau_{w,L}}=(1-p\beta d^L)\gamma\pi^L(1+\pi^L)+(\theta-1).$$

Eliminating h^L, we get

$$\begin{aligned}&\tfrac{\theta(1-\kappa^L-\eta)^\sigma(h^L)^{\sigma+\nu}}{1-\tau_{w,L}}\tfrac{\sigma(\kappa^L)'}{1-\kappa^L-\eta}+\gamma(1-p\beta d^L)(1+2\pi^L)\\ =\quad&[(1-p\beta d^L)\gamma\pi^L(1+\pi^L)+(\theta-1)]\tfrac{\sigma(\kappa^L)'}{1-\kappa^L-\eta}+\gamma(1-p\beta d^L)(1+2\pi^L).\end{aligned}$$

For π^L such that $1-\kappa^L-\eta>0$, this is greater than zero iff

$$[\gamma\pi^L(1+\pi^L)(1-p\beta d^L)+(\theta-1)]\sigma(\kappa^L)'+\gamma(1+2\pi^L)(1-\kappa^L-\eta)(1-p\beta d^L)>0. \tag{39}$$

Consider first the misspecified AS schedule, $(\kappa^L=(\kappa^L)'=0)$. Inspection of (39) indicates that the AS schedule is upward sloping when $(1-p\beta d^L)>0$ and $\pi^L<-1/2$. Finally observe that the left hand side of (39) can be expressed as a cubic polynomial in π^L:

$$(\sigma\gamma-1)(\pi^L)^3+\gamma(\sigma-\frac{1}{2})(\pi^L)^2+\left[\frac{\sigma(\theta-1)(1+\tau_s)}{(1-p\beta d^L)}+2(1-\eta)\right]\pi^L+(1-\eta).$$

Proof of Lemma 6 To derive the responses start by applying the chain rule to (16) and (17) to get

$$AS_{h^L}Dh^L+AS_{\pi^L}D\pi^L+AS_{\tau_w^L}D\tau_w^L=0 \tag{40}$$

$$AD_{h^L}Dh^L+AD_{\pi^L}D\pi^L=0 \tag{41}$$

We then solve the total differential of the AD schedule for $D\pi^L$ and substitute it into the total differential of the AS schedule to get:

$$\left(AS_{h^L} - AS_{\pi^L}\frac{AD_{h^L}}{AD_{\pi^L}}\right)Dh^L + AS_{\tau_w^L}D\tau_w^L = 0 \tag{42}$$

or

$$\frac{Dh^L}{D\tau_w^L} = -\frac{AS_{\tau_w^L}}{AS_{h^L} - AS_{\pi^L}\frac{AD_{h^L}}{AD_{\pi^L}}} \tag{43}$$

Proceeding in a similar yields the following expression for the inflation response:

$$\frac{D\pi^L}{D\tau_w^L} = -\frac{AS_{\pi^L} - AS_{h^L}\frac{AD_{\pi^L}}{AD_{h^L}}}{AS_{\tau_w^L}} \tag{44}$$

where the various derivatives are as follows

$$AS_{h^L} = \frac{\theta(\sigma+\nu)(1-\kappa^L-\eta)^\sigma(h^L)^{\sigma+\nu-1}}{(1-\tau_w^L)} \tag{45}$$

$$AS_{\pi^L} = -\frac{\theta\sigma(1-\kappa^L-\eta)^{\sigma-1}(\kappa^L)'(h^L)^{\sigma+\nu}}{(1-\tau_w^L)} + \gamma(p\beta d^L - 1)(1+2\pi^L) \tag{46}$$

$$AS_{\tau_w^L} = \frac{\theta(1-\kappa^L-\eta)^\sigma(h^L)^{\sigma+\nu}}{(1-\tau_w^L)^2} \tag{47}$$

$$AD_{\pi^L} = -\frac{p\beta d^L}{(1+\pi^L)^2} + \frac{(p-1)\beta d^L\sigma(h^L)^\sigma(1-\kappa^L-\eta)^{\sigma-1}(\kappa^L)'}{(1-\eta)^\sigma h^\sigma} \tag{48}$$

$$AD_{h^L} = \frac{\sigma(1-p)\beta d^L(1-\kappa^L-\eta)^\sigma(h^L)^{\sigma-1}}{(1-\eta)^\sigma h^\sigma} \tag{49}$$

Then note that

$$\frac{Dh^L}{D\tau_w^L} = -\frac{AS_{\tau_w^L}}{AS_{h^L} - AS_{\pi^L}\frac{AD_{h^L}}{AD_{\pi^L}}} \tag{50}$$

$$= \frac{\frac{-AS_{\tau_w^L}}{AS_{\pi^L}}}{\frac{AS_{h^L}}{AS_{\pi^L}} - \frac{AD_{h^L}}{AD_{\pi^L}}} \tag{51}$$

$$= \frac{\frac{-AS_{\tau_w^L}}{AS_{\pi^L}}}{-slope(AS) + slope(AD)} \tag{52}$$

Similarly we have

$$\frac{D\pi^L}{D\tau_w^L} = -\frac{1/slope(AS) - 1/slope(AD)}{\frac{AS_{\tau_w^L}}{AS_h^L}} \tag{53}$$

The response of hours in each of the cases can be derived using 50 and the response of inflation can be derived using 53

References

Adam, K., and R. Billi (2006): "Optimal Monetary Policy under Commitment with a Zero Lower Bound on Nominal Interest Rates," *Journal of Money, Credit and Banking*, 38(7), 1877–1905. 1

Aldrich, E., and H. Kung (2009): "Computational Methods for Production-Based Asset Pricing Models with Recursive Utility," mimeo. 1

Bodenstein, M., C. J. Erceg, and L. Guerrieri (2009): "The effects of foreign shocks when interest rates are at zero," mimeo. 1

Braun, R. A., and L. M. Körber (2011): "New Keynesian Dynamics in a Low Interest Rate Environment," *Journal of Economic Dynamics and Control*, 35(12), 2213–2227, Federal Reserve Bank of Atlanta Working Paper. 1

Braun, R. A., and Y. Waki (2006): "Monetary Policy During Japan'S Lost Decade," *The Japanese Economic Review*, 57(2), 324–344. 1

——— (2010): "On the Size of the Fiscal Multiplier When the Zero Bound on Nominal Interest Rates is Binding," mimeo. 1, 4.1

Bullard, J. (2010): "Seven Faces of "The Peril"," *Federal Reserve Bank of St. Louis Review*, 92(5), 339–352. 3.3

Calvo, G. A. (1983): "Staggered prices in a utility-maximizing framework," *Journal of Monetary Economics*, 12(3), 383–398. 1

Christiano, L. (2011): "Comment on "What Fiscal Policy is Effective at Zero Interest Rates?"," in *NBER Macroeconomics Annual 2010, Volume 25*, NBER Chapters, pp. 113–124. National Bureau of Economic Research, Inc.

Christiano, L., M. Eichenbaum, and S. Rebelo (2011): "When is the Government Spending Multiplier Large?," *Journal of Political Economy*, 119(1), 78–121. 1, 3.1, 3.2, 3.2, 4.1, 4.3, 4.4.2

Coenen, G., A. Orphanides, and V. Wieland (2004): "Price Stability and Monetary Policy Effectiveness when Nominal Interest Rates are Bounded at Zero," *Advances in Macroeconomics*, 4(1), 1187–1187. 1

De Long, J. B., and L. H. Summers (1986): "Is Increased Price Flexibility Stabilizing?," *American Economic Review*, 76(5), 1031–44. 3.3

Del Negro, M., G. Eggertsson, A. Ferrero, and N. Kiyotaki (2010): "The Great Escape? A Quantitative Evaluation of the Fed's Non-Standard Policies," mimeo. 1

Den Haan, W., and P. Rendahl (2009): "Solving the Incomplete Markets Model with Aggregate Uncerctainty using Explicit Aggregation," mimeo. 1

Eggertsson, G. (2011): "What Fiscal Policy is Effective at Zero Interest Rates?," in *NBER Macroconomics Annual 2010, Volume 25*, NBER Chapters, pp. 59–112. National Bureau of Economic Research, Inc. 1, 2, 2.2, 4, 2.3, 3.1, 4.1, 4.3, 12, 4.4.1, 16

Eggertsson, G., and P. Krugman (2010): "Debt, Deleveraging and the Liquidity Trap," mimeo. 1, 2.2

Eggertsson, G., and M. Woodford (2003): "The Zero Interest Rate Bound and Optimal Monetary Policy," *Brookings Panel on Economic Activity*. 2.3

Erceg, C., and J. Lindé (2010): "Is There a Fiscal Free Lunch in a Liquidity Trap?," Board of Governors Board of Governors International Finance Discussion Papers No. 1003. 1

Jung, T., Y. Teranishi, and T. Watanabe (2005): "Optimal Monetary Policy at the Zero-Interest-Rate Bound," *Journal of Money, Credit, and Banking*, 37(5), 813–835. 1

Mertens, K., and M. O. Ravn (2010): "Fiscal Policy in an Expectations Driven Liquidity Trap," CEPR Discussion Papers 7931, C.E.P.R. Discussion Papers. 1, 1, 3, 3.3

Nakov, A. (2008): "Optimal and Simple Monetary Policy Rules with Zero Floor on the Nominal Interest Rates," *International Journal of Central Banking*, 4(2), 73–127. 1

Rotemberg, J. (1996): "Prices, Output and Hours: An Empirical Analysis based on a Sticky Price Model,," *Journal of Monetary Economics*, 37, 503–533. 1, 2

Werning, I. (2011): "Managing a Liquidity Trap: Monetary and Fiscal Policy," mimeo. 3.2, 3.2

Wolman, A. (2005): "Real Implications of the Zero Bound on Nominal Interest Rates," *Journal of Money, Credit and Banking*, 37(2), 273–296. 1

Woodford, M. (2011): "Simple Analytics of the Government Expenditure Multiplier," *American Economic Journal: Macroeconomics*, 3(1), 1–35. 1, 4.1, 4.3

Table 1: Parameterization

Symbol	Value	Description
β	0.997	Discount factor
σ	1.1599	Consumption curvature
ν	1.5692	Leisure curvature
p	.9030	Probability of a low state in the next period
θ	12.7721	Elasticity of substitution of intermediate goods
α	0.7747	Calvo Parameter
γ	3742.9	Implied price adjustment cost parameter
τ_w	0.2	Labour tax rate
d^L	1.015	Preference discount factor shock
η	0.2	Government purchase share of output
ϕ_π	1.5	Inflation coefficient on the Taylor rule
ϕ_y	0.125	Output coefficient on the Taylor rule
d_L	6.14%	Annualized shock to discount rate in the low state

Table 2

Properties of low (zero interest rate) state using Eggertsson (2011) parameterization of the model

Solution Procedure	log-linearized	nonlinear (κ^L=0)	nonlinear
Percentage change in hours	-29.92%	-36.71%	-1.98%
Inflation rate	-9.92%	-9.13%	-3.24%
Slope of AS schedule	0.12	0.06	-0.16
Slope of AD schedule	0.17	0.11	-0.03

Table 3

Zero interest rate equilibria for alternative sized shocks to preferences using Calvo parameter of 0.7747.

First Equilibrium												
Annualized increase in d (in %)	12.55	10.38	8.24	7.83	7.31	6.98	6.12	4.06	3.98	2.02	1.65	1.25
Percentage change in hours	9.16	4.31	0.59	0.00	-0.68	-1.09	-1.98	-3.18	-3.19	-2.34	-1.69	-0.25
Annualized inflation rate (in %)	-5.18	-4.62	-3.99	-3.85	-3.67	-3.56	-3.24	-2.34	-2.30	-1.05	-0.69	-0.09
Percentage change in GDP	-30.37	-25.65	-20.66	-19.67	-18.39	-17.56	-15.39	-9.91	-9.68	-3.64	-2.25	-0.26
Slope of AS schedule	-0.03	-0.04	-0.07	-0.08	-0.10	-0.11	-0.16	-5.15	20.85	0.16	0.12	0.09
Slope of AD schedule	-0.01	-0.01	-0.02	-0.02	-0.02	-0.02	-0.03	-0.05	-0.05	-0.26	8.39	0.15
Second Equilibrium												
Annualized increase in d (in %)	12.55	10.38	8.24	7.83	7.31	6.98	6.12	4.06	3.98	2.02	1.65	1.25
Percentage change in hours	26.90	23.62	20.20	19.51	18.63	NA	NA	NA	NA	NA	NA	NA
Annualized inflation rate (in %)	4.92	4.54	4.10	4.01	3.88	NA	NA	NA	NA	NA	NA	NA
Percentage change in GDP	-5.04	-2.95	-0.98	-0.62	-0.17	NA	NA	NA	NA	NA	NA	NA
Slope of AS schedule	0.02	0.02	0.02	0.03	0.03	NA	NA	NA	NA	NA	NA	NA
Slope of AD schedule	0.01	0.01	0.01	0.01	0.01	NA	NA	NA	NA	NA	NA	NA

Table 4

Zero interest rate equilibria for alternative sized shocks to preferences using Calvo parameter of 0.636.

First Equilibrium									
Annualized increase in d (in %)	12.55	10.38	8.57	4.06	2.02	1.21	0.80	0.68	0.68
Percentage change in hours	36.69	27.70	20.91	6.68	1.55	-0.17	-0.91	-1.07	-1.07
Annualized inflation rate (in %)	-11.41	-10.71	-10.00	-7.48	-5.54	-4.28	-3.18	-2.29	-2.22
Percentage change in GDP	-37.73	-33.87	-30.00	-18.02	-11.01	-7.41	-4.82	-3.06	-2.93
Slope of AS schedule	-0.01	-0.02	-0.03	-0.07	-0.15	-0.28	-0.70	14.49	5.44
Slope of AD schedule	-0.01	-0.01	-0.01	-0.03	-0.07	-0.12	-0.27	-3.90	115.77
Second Equilibrium									
Annualized increase in d (in %)	12.55	10.38	8.57	4.06	2.02	1.21	0.80	0.68	0.68
Percentage change in hours	15.22	12.57	NA	NA	NA	-0.01	-0.76	-1.05	-1.06
Annualized inflation rate (in %)	7.15	6.36	NA	NA	NA	-0.01	-1.08	-1.97	-2.05
Percentage change in GDP	-4.51	-2.82	NA	NA	NA	-0.01	-1.19	-2.52	-2.64
Slope of AS schedule	0.05	0.06	NA	NA	NA	0.29	0.52	1.77	2.22
Slope of AD schedule	0.02	0.02	NA	NA	NA	0.13	0.25	1.13	1.60

Table 5

a) A labor tax cut

Solution Procedure	log-linearized	nonlinear (κ^L=0)	nonlinear
Response of hours	-1.02	-0.44	0.56
Response of GDP	-1.02	-0.44	0.00
$\Delta\ln(1-\kappa^L)/\Delta\tau_w$	0	0	-0.56

b) An increase in government spending

Solution Procedure	log-linearized	nonlinear (κ^L=0)	nonlinear
Government spending hours Multiplier	1.55	1.41	0.40
Government spending GDP Multiplier	1.55	1.41	1.19
$\Delta\ln(1-\kappa^L)/\Delta g$	0	0	0.98

Figure 1: Possible configurations of aggregate demand and aggregate supply schedules when the nominal rate is zero for the loglinear economy. Panels A and D are equilibria. However, Panels C and B are not.

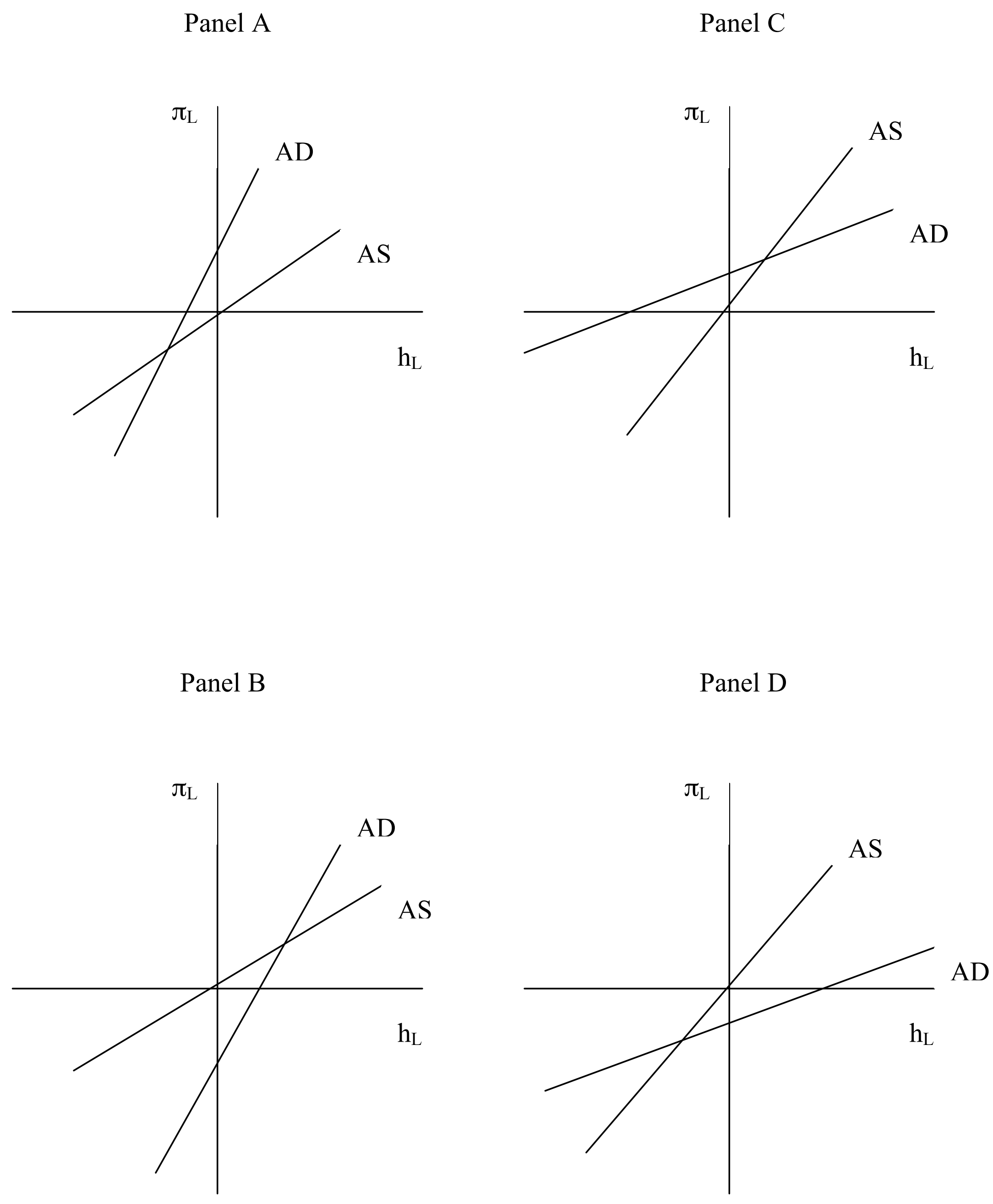

Figure 2: Dynamic adjustments of consumption in expected inflation in to a higher labor tax in a Type 1 equilibrium.

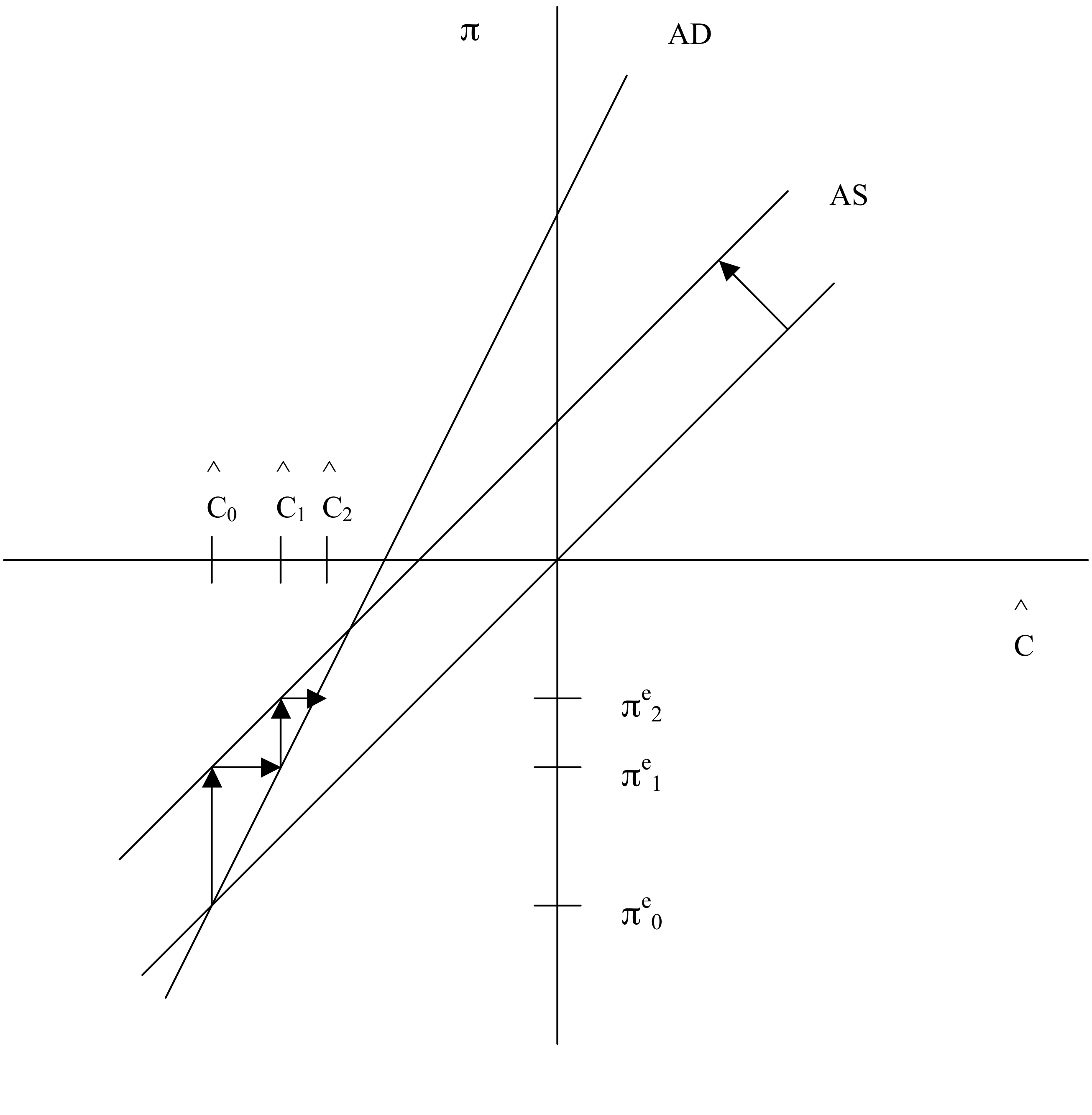

Figure 3: Dynamic adjustments of consumption in expected inflation in to a higher labor tax in a Type 2 equilibrium.

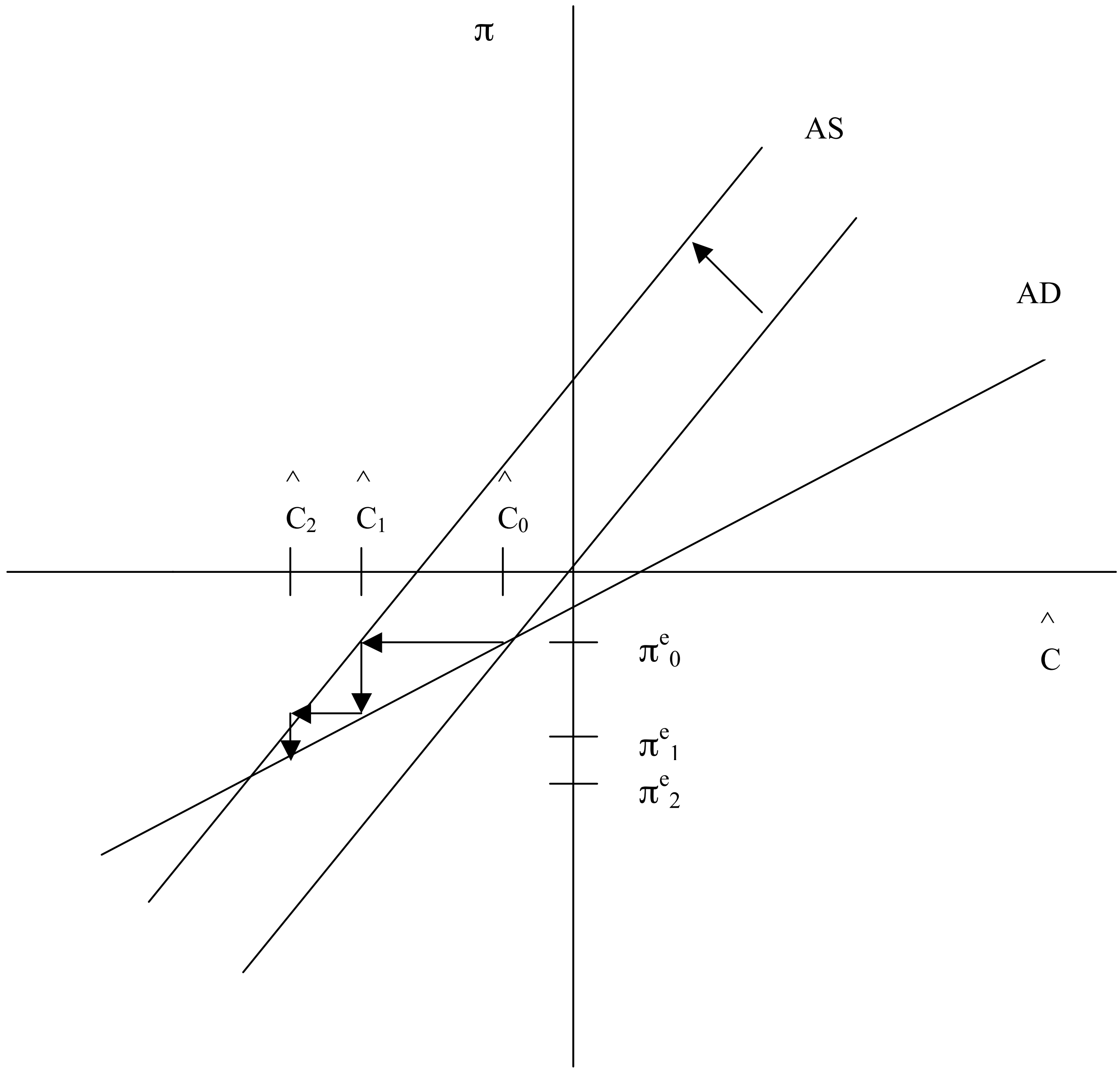

Figure 4: Model in the steady-state

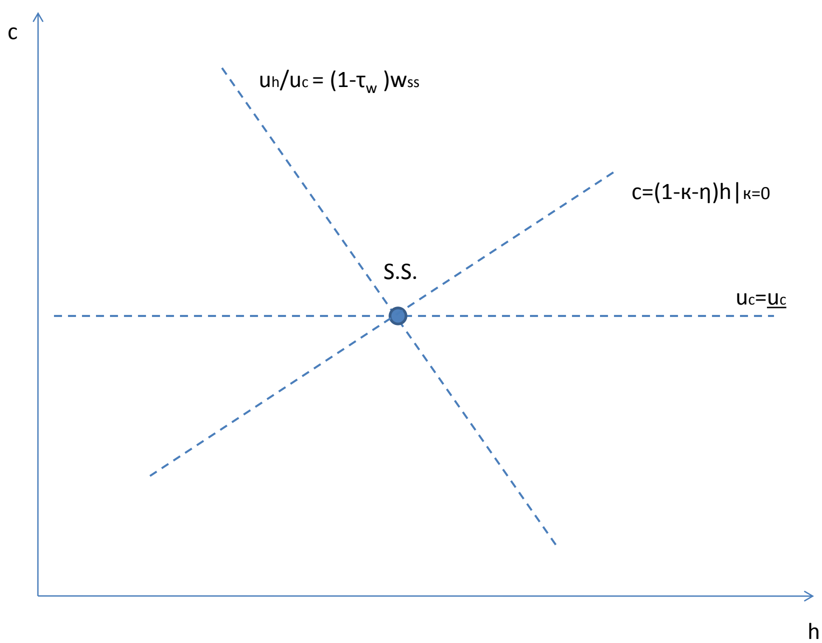

Figure 5: Misspecified Model ($\kappa = 0$) in state L

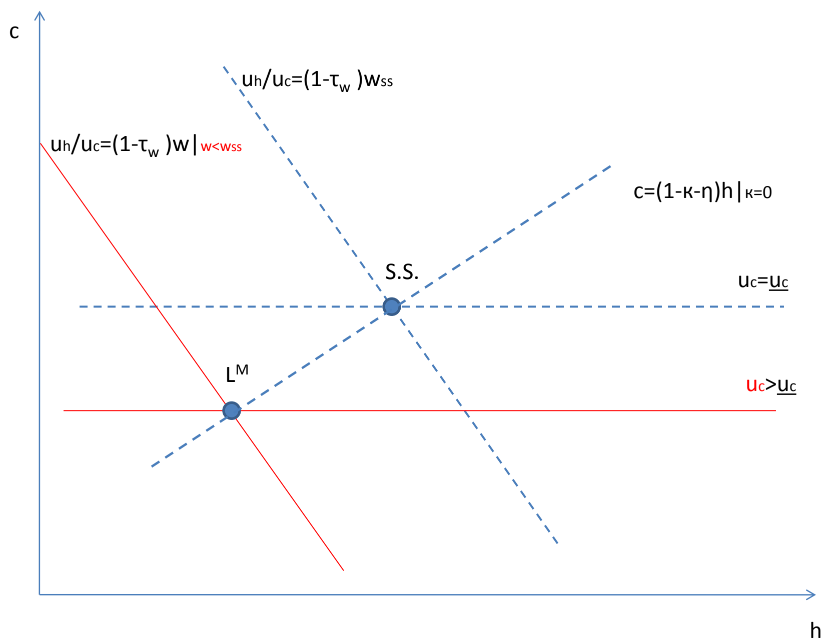

Figure 6: True Model ($\kappa > 0$) in state L

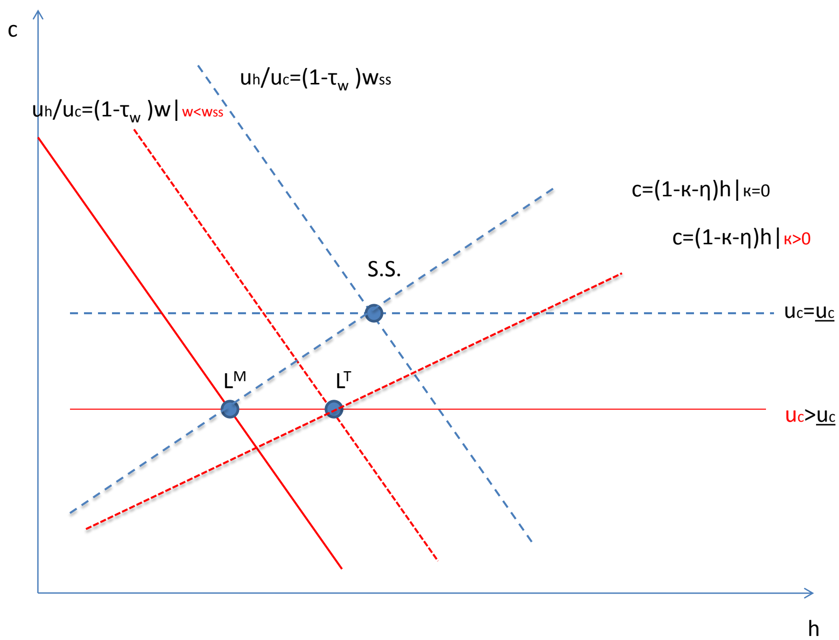

Figure 7: AD and AS schedules from a parameterized version of the model with $\gamma = 0.636$, and a value of d of 0.80.