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Liquidity Backstops and Dynamic Debt Runs

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Abstract: Liquidity backstops have important implications for financial stability. In this paper, we provide a microfoundation for the important role of liquidity backstops in mitigating runs (or, conversely, the role of the lack of liquidity backstops in exacerbating runs) based on a dynamic model of debt runs. We focus on the municipal bond markets for variable rate demand obligations (VRDOs) and auction rate securities (ARS). The different experiences in these markets during the recent financial crisis of 2007–09 provide a natural experiment to identify the value of a liquidity backstop in mitigating runs. Through structural estimation of the model, we show that the value of a liquidity backstop is about 14.5 basis points per annum. The results in this paper shed light on one central difference between shadow banks and traditional banks in terms of their differential access to public liquidity backstops.

JEL classification: G10, G20, G21

Key words: liquidity backstop, debt run

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1 Introduction

An important form of bank lending is credit lines and other liquidity commitments, which serve as liquidity backstops to businesses. The recent financial crisis of 2007-09 demonstrated that liquidity backstops have important implications for financial stability. For example, concerns about provision of liquidity backstops by banks contributed to runs on asset-backed commercial paper programs in the summer 2007 (Covitz, Liang, and Suarez (2013)). These runs together with those in several other markets (e.g., markets for repo, money market funds) tightened credit going to firms and households and inflicted widespread damage to the US and global economy. In this paper we provide a microfoundation for the important role of liquidity backstops in mitigating runs (or, conversely, the role of the lack of liquidity backstops in exacerbating runs) based on a dynamic model of debt runs. We focus on the municipal bond markets for variable rate demand obligations (VRDOs) and auction rate securities (ARS). As we will describe shortly, these markets provide an ideal laboratory to identify and quantify the value of a liquidity backstop in mitigating runs.

VRDOs and ARS are essentially short-term municipal bonds with floating interest rates that are reset on a periodic basis (typically weekly). Despite many similarities, they differ in terms of the nature of liquidity support. VRDOs are typically structured with liquidity backstop facilities committed by liquidity providers (usually large banks) who act as "buyers of last resort." By contrast, there are no explicit liquidity backstops in the ARS market: in the event that there are insufficient bids at an auction and the auction agent does not intervene to provide liquidity, the auction will fail and selling creditors will be stuck with their holdings until the next successful auction. Section 2 provides more detail about these markets.

The key difference in the nature of liquidity support in these two markets leads to dramatically different experiences during the crisis. Prior to the crisis, market participants viewed both ARS and VRDOs as almost identical securities. From some anecdotal evidence, such view partly stems from the flawed perception that auction agents would always provide liquidity to prevent ARS auctions from failing. Consistent with the treatment of ARS and VRDOs as almost identical securities, the average ARS and VRDO interest rates, shown in Figure 1, were very close before 2007.

[**Insert Figure 1 About Here**]

However, at the onset of the crisis, several major banks serving as auction agents were forced to cut back on lending including uncommitted lending in the ARS market and decided not to intervene in early 2008. Consequently, auctions in the ARS market started to fail and at the peak in mid-February 2008 "about two-thirds of auctions have failed per day."[1] The wave of auction failures shattered the misperception and triggered a run on ARS. The run is evident in the spike of the ARS rate around 6.6% in the second half of February and March in 2008 as shown in Figure 1. By contrast, explicit liquidity backstops in the VRDO market helped to stabilize its interest rate around 2%. Later that year, in September 2008, the Lehman bankruptcy cast into doubt the strength of explicit liquidity backstops. Investors worried whether banking institutions with explicit liquidity commitment would be able to meet their obligations and consequently runs on both VRDO and ARS occurred. As shown in Figure 1, the average VRDO and ARS rates jumped in unison to around 8% on September 24, 2008.

These run episodes suggest that whether or not an explicit liquidity backstop exists can lead to dramatically different dynamics in otherwise almost identical markets. This is the basis for our identification of the value of a liquidity backstop. Moreover, the run on ARS in early 2008 demonstrates its "dynamic" nature; that is, once auctions started to fail, fear of future auction failures propelled more investors to run on ARS.

Motivated by the above observations, we develop a continuous time model of dynamic debt runs in the markets for ARS and VRDO based on He and Xiong (2012, HX hereafter). Our model captures several key characteristics of ARS and VRDOs: the floating interest

[1] See "Florida Schools, California Convert Auction-Rate Debt," Bloomberg L.P., February 22, 2008.

rate, the pre-specified interest rate cap or maximum interest rate, and, more importantly, the committed versus uncommitted liquidity provision. In the model there is a unique threshold equilibrium in which creditors' decision to run or not depends on whether the fundamental falls below a certain endogenous threshold (referred to as the "rollover threshold"). Runs may arise in the model due to the "maturity-mismatch" problem — short-term securities (ARS or VRDOs in our case) are issued to finance an illiquid long-term project. As argued in HX, a run by future creditors may force the project to be prematurely liquidated, resulting in a loss from premature liquidation that is borne by earlier creditors who decide to roll over the debt. Ex ante, such a negative externality prompts creditors to run more often, giving rise to so-called *dynamic* debt runs.

One key departure of our model from HX is that we formally model and distinguish between committed and uncommitted liquidity provision. The former is referred to as liquidity backstops. As one main result, we show theoretically that the lack of a liquidity backstop can exacerbate runs because of a new type of externalities arising from a creditor's running decision. The intuition is as follows. Absent liquidity backstops, liquidity may evaporate unexpectedly, especially at times when it is needed most. In the context of the ARS market, a run may trigger an auction to fail and the failure is likely to occur when creditors need to liquidate their bond holdings. Put differently, the decision to run imposes negative externalities on future maturing creditors who may find themselves unable to sell and suffer losses amid auction failures. The resulting negative externalities make a run more likely to happen, ex ante. Theoretically, we prove that the rollover threshold is higher in the ARS market than that in the VRDO market because of the lack of a liquidity backstop. This explains why the ARS market was more susceptible to runs than the VRDO market during the crisis.

In another main result, we conceptualize and evaluate the value of a liquidity backstop based on structural estimation of the model. As shown in Figure 1, following the eruption of auction failures, the ARS rate started to diverge from the VRDO rate since November 2007, as ARS creditors began to take into account the previously ignored possibility of auction failures. This structural shift helps us to identify the value of a liquidity backstop in a spirit similar to the "difference-in-difference" approach. In the model, we assume that the probability of auction failure is considered to be zero in the pre-crisis period so that the running decision in both ARS and VRDO markets is the same, characterized by a common rollover threshold. We further assume that in the crisis and post-crisis periods, ARS creditors correctly recognize the lack of a liquidity backstop in the ARS market. Once a positive probability of auction failure is taken into consideration, ARS creditors face a higher rollover threshold than that in the VRDO market, implying that creditors in the ARS market are more likely to run.

The value of a liquidity backstop can now be conceptualized in the following thought experiment. If an ARS issuer acquires the same liquidity backstop facility as in the VRDO market, the rollover threshold in the ARS market will be reduced to the same threshold in the VRDO market. Alternatively, without acquiring the VRDO liquidity facility, the ARS issuer could obtain the same VRDO threshold by permanently increasing the ARS rate by a large enough amount at each point of time. A risk neutral ARS issuer would be indifferent between these two methods, as long as the fee for acquiring the liquidity backstop is the same as the constant ARS rate increase. Therefore, the value of a liquidity backstop can be determined as the hypothetical permanent increase in the ARS rate that equalizes the rollover thresholds in both markets.

Lastly, to quantify a liquidity backstop's value we structurally estimate the model using the Quasi-Maximum Likelihood (QML) method based on the following identification assumptions. One identification assumption is that issuers of ARS and VRDOs have similar fundamentals. In fact, many municipalities issue both types of securities. We first infer the unknown fundamental process based on the historical and model-implied VRDO rates. Then we apply the inferred fundamental process to the ARS market to compute the likelihood function. Another identification assumption is that there is a structural change in the beliefs of ARS investors at the onset of the crisis. Specifically, we assume that ARS investors have started to recognize the lack of a liquidity backstop and the possibility of auction failures beginning November 2007. As a result, the ARS rollover threshold has since then jumped to a higher level than the VRDO threshold, and the resulting ARS interest rate is determined by taking into account the higher rollover threshold as well as possible losses from auction failures. Based on these identification assumptions and using the historical and model-implied ARS rates, we can estimate several key parameters by maximizing the likelihood function. The estimation results imply that auctions are expected to fail within a month with a unconditional probability of 0.3%. We are also able to estimate the value of a liquidity backstop to be about 14.5 basis points per annum.

Our paper contributes to the theoretical debt-run literature that examines the determinants of runs.[2] Our model is built upon He and Xiong (2012), which extends the literature on static bank-run models (Diamond and Dybvig (1983); Rochet and Vives (2004); Goldstein and Pauzner (2005), etc.). The He-Xiong model highlights the dynamic coordination problem and one main finding is that fear of future rollover risk could motivate each creditor to run ahead of others. In a closely related paper, Schroth, Suarez, and Taylor (2014) extend and apply the He-Xiong model to the ABCP market. The authors show that an endogenous "dilution risk", arising from higher yields demanded by maturing creditors, increases the likelihood of runs. Cheng and Milbradt (2012) develop a dynamic debt run model to study the optimal maturity structure of debt to best trade off incentive provision against inefficient rollover freezes. Our different focus on the effect of liquidity backstops on the likelihood of runs distinguishes our paper from these papers.

This paper is also related to the literature on the role of banks as liquidity providers. Kashyap, Rajan, and Stein (2002) provide a convincing argument that banks have a natural

[2]For empirical studies, please see Carey, Correa, and Kotter (2009) and Covitz, Liang, and Suarez (2012) for the run on ABCP, Gorton and Metrick (2012) for the run on repo, Kacperczyk and Schnabl (2013), Wermers (2012) for the run on money market mutual funds, and Shin (2009) on the run on Northern Rock, and Han and Li (2009) and McConnell and Saretto (2010) for the run on ARS.

advantage of acting as liquidity providers to provide liquidity on demand. The advantage stems from a synergy between deposit-taking and loan commitments to the extent that both types of activities require banks to hold large balances of liquid assets. The synergy exists as long as both activities are not too highly correlated, which holds up very well during normal times or several recent episodes of market stress.[3] However, as Acharya and Mora (2015) argue, during the banking crisis of 2007-08, the role of banks as liquidity providers was itself in crisis as both sides of their balance sheet were hit simultaneously. Nagel (2012) shows empirical evidence for the reluctance of market makers to absorb inventories during times of crises. In this paper, we further investigate the destabilizing effects when banks as liquidity providers cut back on uncommitted lending (e.g., the wave of auction failures in the ARS market). By contrasting the run episodes in the VRDO and ARS markets, our paper is able to shed new light on how valuable the role of banks is in providing backup liquidity.

The remainder of this paper is structured as follows. In Section 2, we provide an overview of the VRDO and ARS markets and the turmoil in these markets during the financial crisis. Section 3 presents the model. Section 4 characterizes the equilibrium. In Section 5 we discuss key model implications, including the externalities imposed on future creditors by the running decisions of current creditors. In Section 6, we present our estimation procedure and results. Section 7 concludes. Most proofs are in the appendix at the end of this paper. A companion internet appendix provides omitted proofs and additional derivations.

2 Overview of the Markets for VRDOs and ARS

In this section, we first provide a description of VRDOs and ARS, and an overview of these markets, followed by a narrative on the disruptions in these markets in 2008 during the recent financial crisis.

[3] See Gatev and Strahan (2006) and Gatev, Schuermann, and Strahan (2009) for evidence of a negative correlation between deposit withdrawals and commitment draw-downs in the commercial paper market.

2.1 Background

In this subsection we provide some background information on VRDOs and ARS.

Auction Rate Securities. ARS are long-term municipal bonds with interest rates that are periodically reset through a Dutch auction process at short-term intervals, usually 7, 28 or 35 days. Following a successful auction, buyers purchase the bonds at par and receive the market clearing interest rate until the next interest reset date. ARS have nominally long-term maturities that usually range from 20 to 30 years. Nonetheless, the interest rate reset mechanism provides creditors with frequent opportunities to sell their holdings through auctions, and thus makes ARS priced and traded as short-term instruments.

At each auction, the auction agent accepts bids from market participants. Existing bond holders can submit one of three types of orders: a "hold at market" order if they wish to maintain their positions regardless of the market-clearing rate; a "sell at market" (market sell) order if they wish to sell regardless of the market-clearing rate; a "hold at rate" (limit sell) order if they commit to sell their positions under the condition that the market-clearing rate is equal to or lower than the specified rate. Potential buyers can submit a limit buy order to buy the bond if the bid is less than or equal to the market-clearing rate. The auction agent then receives all the bids and can submit his/her own order.

The market-clearing interest rate is bounded from above by a pre-specified maximum interest rate, often shortened to "max rate" in Wall Street parlance.[4] The max rates are either fixed, or floating and usually tied to a reference rate (e.g., LIBOR). Fixed max rates are specified for all ARS, in a wide range of 9% to 25%. For ARS that also have floating max rates, the binding max rate is the minimum of the two.[5]

An auction fails when there are not sufficient bids to clear the market at a rate less than the max rate. In the case of auction failure, the max rate is imposed, however, importantly, creditors are stuck with the bonds until the next successful auction. Until the ARS market

[4] Throughout this paper we use the terms "maximum interest rate" and "max rate" interchangeably.

[5] Please see McConnell and Saretto (2010) for some examples of how floating max rates are set.

froze in mid-Feburary 2008, auction failures were extremely rare — there were only 13 failed auctions between 1984 and 2006.[6] However, as described shortly in the next subsection, after the financial crisis broke out, a tidal wave of auction failures hit the market.

Variable Rate Demand Obligations. VRDOs are very similar to ARS; they are also long-term floating-rate bonds with periodic interest rate resets. Unlike ARS, interest rates of VRDOs are reset periodically through "remarketing agents" so that the securities can be sold at par.

The key distinguishing characteristic of VRDOs is the existence of an explicit liquidity facility/backstop. VRDO creditors have a "tender" or "put" option which allows them to put the bonds at par value (plus any accrued interest) to the remarketing agent who then try to resell (remarket) the tendered bonds to new investors. To make the tender option feasible, VRDOs are usually structured with a liquidity facility provided by a third-party "liquidity provider." The liquidity provider, usually a large bank, acts as a buyer of last resort; it provides liquidity support by buying the bonds if the remarketing agent is unable to remarket them. In this case, the bonds become the so-called "bank bonds" showing up on the liquidity provider's balance sheet.

The liquidity facility is in the form of a *direct Letter of Credit* (LOC) under which the liquidity provider acts as the first source of payment of principal and interest, or a *standby* LOC under which the issuer is the first source of liquidity and the liquidity provider acts as a back-up, or a *Standby Bond Purchase Agreement* (SBPA) under which the VRDO instrument is insured by an investment-grade insurer and in the case of unsuccessful remarketing, the liquidity provider is obligated to buy the tendered bonds as long as the insurer maintains its investment grade rating. Regardless of which structure is used, the liquidity provider is the ultimate source of liquidity. As a result, the VRDO instrument carries the short-term rating of its liquidity provider.

[6] "Prolonged disruption of the auction rate market could have negative impact on some ratings," Special Report, Moody's Investors Service, February 20, 2008.

VRDOs are also typically sold with credit enhancement, which takes the form of a municipal bond insurance policy provided by some monoline bond insurers. The credit enhancement protects creditors and the liquidity provider from long-term credit risk. Therefore, the VRDO instrument carries the long-term rating of its insurer, typically triple A.

2.2 The VRDO and ARS Markets and the Crisis in 2008

The VRDO and ARS markets are significant components of the \$3.7 trillion municipal bond market, with sizes of about \$200 billion and \$500 billion in 2008 at their peak time, respectively. The markets were an attractive financing venue for municipal issuers because they allow for the issuance of long-term obligations using short-term interest rates that are typically lower than long-term interest rates. For investors, these securities were also attractive because they offered better returns than traditional money market investments. Both markets have existed since 1980s and had functioned well until the financial crisis broke out in 2007. In the aftermath of the financial crisis, the ARS market collapsed afterwards and there have been no new ARS issuances since 2008. Meanwhile, new issuance of VRDOs surged in 2008 as many existing ARS were converted into VRDOs. Figure 2 below plots the annual amount of issuance in both markets since 1988, calculated using SDC platinum.

[Insert Figure 2 About Here]

The ARS market encountered significant problems in early 2008. Since mid-2007, the disruption in the subprime mortgage market spread to the monoline insurance market where several major municipal bond insurers (e.g., Ambac and MBIA) were downgraded because of their exposure to subprime mortgage debt. These downgrades resulted in increased selling pressure in ARS. On the other hand, the subprime mortgage meltdown also significantly strained balance sheets of auction agents (e.g., Citibank, Goldman Sachs, Lehman Brothers, UBS, Royal Bank of Canada and JP Morgan) to the extent that they decided not to intervene and let the auctions fail in mid-February 2008. Reportedly, about 60% to 80% of auctions failed in the second half of February in 2008.[7] The wave of auction failures drove up the ARS rate to as high as 6.6% around mid-February 2008 as shown in Figure 1. The sheer volume of failed auctions and fear of future auction failures propelled more investors to run on ARS.

The run on ARS highlighted the implicitness of the liquidity provision in the ARS market: although in less tumultuous times prior to 2007, auction agents had almost always stepped in to buy some of these securities to help keep the market functioning, they had no contractual obligations to do so. During the financial crisis, major auction agents indeed chose to no longer be "buyers of last resort." By contrast, the VRDO market was not affected as much in early 2008 due to the explicit structure of its liquidity facility.

However, later in 2008 the VRDO (as well as ARS) market experienced a run as a result of the bankruptcy of Lehman Brothers declared on September 15, 2008 and the subsequent panic in the market of money market mutual funds (e.g., runs on the Reserve Primary Fund that "broke the buck", and other money market mutual funds). Investors worried about whether banking institutions that explicitly provided liquidity facility would be able to meet their obligations. The run on VRDO is evident in the spike of 7.96% of the average VRDO rate on September 24, 2008, as shown in Figure 1.

The runs on ARS and VRDO in 2008 allow us to distinguish the differential effects of explicit and implicit liquidity provisions on the running decision of investors. In particular, we build a dynamic-debt-run model of the VRDO and ARS markets to illustrate why the ARS market became more susceptible to runs in early 2008 once investors started to recognize the implicitness of the liquidity provision. Furthermore, we also structurally estimate the model to assess the value of providing an explicit liquidity facility as in the case of VRDOs.

[7] See, "Breakdown of auction rate securities markets," congressional testimony by Leslie Norwood, September 18, 2008.

3 The Model

We develop a model of dynamic debt runs for the markets for VRDOs and ARS, based on He and Xiong (2012). The model contains several common features shared by VRDOs and ARS: a floating short-term interest rate and a pre-specified maximum interest rate. Moreover, the model captures a unique feature of VRDOs that ARS do not have: a liquidity backstop (facility), structured as imperfect credit lines, to support the tender option of VRDO investors. We use "floating-rate (municipal) bonds" or simply "bonds" to refer to both VRDOs and ARS when describing the setting that applies to both.

3.1 Asset

At time 0, a government-related entity (referred to as a municipality, hereafter) issues floating-rate municipal bonds (i.e., VRDOs or ARS) to borrow \$1 to finance a long-horizon project that generates cash flow at a constant rate r. At a random arrival time τ_ϕ according to a Poisson process with intensity $\phi > 0$, the project is terminated with a final payoff. The final payoff is the realization of a geometric Brownian motion process y_t at time τ_ϕ,

$$dy_t = y_t \left(\mu dt + \sigma dZ_t\right), \tag{1}$$

where $\{Z_t\}$ is a standard Brownian motion. The project's fundamental value under a discount rate ρ is determined as follows:

$$F\left(y_t\right) = E_t \left[\int_t^{\tau_\phi} e^{-\rho(s-t)} r ds + e^{-\rho\left(\tau_\phi - t\right)} y_{\tau_\phi}\right] = \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y_t. \tag{2}$$

Due to tax exemption, the discount rate ρ equals the after-tax risk-free rate, that is,

$$\rho = r_f \left(1 - \tau\right), \tag{3}$$

where r_f denotes the taxable Treasury yield and τ the marginal tax rate. The discount rate ρ is identical for all creditors.

3.2 VRDO/ARS Financing

The long-term project is financed by the issuance of VRDOs or ARS whose maturity coincides with the termination of the project. That is, the issued VRDOs or ARS have a long-term nominal maturity, equal to $1/\phi$ or the expected time until the project is terminated. Despite the long-term nominal maturity, VRDOs and ARS have been considered as short-term securities in practice, because of periodical (typically, weekly) remarketing or auctions through which creditors can sell their holdings (see Section 2). To capture the short-term nature of VRDO/ARS financing in a tractable way, we assume that there is a continuum of risk neutral creditors with measure 1, and each creditor decides to sell his bond holdings at a random time τ_δ which arrives following a Poisson process with intensity $\delta >> \phi > 0$. This assumption shares the spirit of the Calvo (1983) staggered-pricing model: at each time interval $[t, t + dt]$, a fixed fraction δdt of creditors arrive to make their rollover decision. For example, creditors may experience idiosyncratic private liquidity shocks such that their rollover decision making is uniformly spread out across time.

A coordination problem between current and future creditors arises in the model. This is because current creditors face a so-called rollover risk where they may suffer losses if future creditors choose not to roll over their debt. As a result, each creditor's rollover decision depends on what he anticipates the action of future creditors to be. This dynamic nature of creditors' rollover decisions makes this model distinct from the static bank-run models (Diamond and Dybvig (1983)).

Another prominent feature associated with financing via VRDO or ARS is the floating interest rate. In the next section, we will discuss in detail how the interest rate is determined.

3.3 Runs, Liquidity Backstops, and Auction Failures

A run occurs if creditors decide not to roll over their debt. In the model, creditors may refuse to roll over their debt for fear of future distressed liquidations, but also for fear of auction failures in the ARS market that lacks a liquidity backstop, both of which can be triggered by a run by future creditors. The latter fear of auction failures due to the lack of a liquidity backstop is an innovative feature of our model. As we will show shortly, a run induces very different dynamics in these markets, because liquidity backstops exist in one market (i.e., VRDO) but not in the other one (i.e., ARS).

VRDOs are structured with an explicit liquidity backstop (i.e., a liquidity facility in the form of a letter of credit or stand-by purchase agreement) committed by a liquidity provider. Upon a run when the remarketing agent cannot find enough buyers, the liquidity provider is contractually obligated to provide liquidity and buy the bonds. However, the liquidity facility may not be perfectly reliable, even though it is explicitly committed. For example, the liquidity provider may become so severely financially distressed (e.g., Lehman Brothers) that it may fail to honor its liquidity commitment. To model the extent of unreliability of the liquidity facility, we assume that with probability $\theta\delta dt$, the committed liquidity support may fail, and, once it fails, the asset will be forced into premature liquidation, sold at a fraction α of its fundamental value (e.g., fire sale). That is, the liquidation value is

$$L(y_t) = \alpha F(y_t) = \frac{\alpha r}{\rho + \phi} + \frac{\alpha \phi}{\rho + \phi - \mu} y_t \equiv L + l y_t. \tag{4}$$

If the liquidation value is not enough to pay off all the creditors, a bankruptcy occurs. Therefore, a run in the future will expose creditors to possible bankruptcy losses. In anticipation of the bankruptcy losses, creditors may refuse to roll over the debt earlier on.

By contrast, ARS are not structured with a liquidity backstop. As a result, ARS creditors face an additional risk of auction failures. Without a liquidity commitment, the auction agent can choose whether or not to participate in an auction. Upon a run when there are insufficient buyers, the auction agent has no contractual obligations to act as the residual bidder and the auction would fail if the agent decided not to step in. To capture this layer of uncertainty due to the lack of a liquidity backstop in the ARS market, we assume that upon a run, with probability $\kappa\delta dt$, the auction agent will not step in to intervene in the market and the auctions will fail; with probability $1-\kappa\delta dt$, the auction agent will intervene to keep the auctions functioning. For tractability, we further assume that once an auction fails, all the following auctions continue to fail. In the event of successful auctions, the market-clearing interest rate r_t prevails and premature liquidation occurs with probability $\theta\delta dt$. In the autarkic event of failed auctions, the max rate $\overline{r}$ is imposed and premature liquidation occurs with probability $\overline{\theta}\delta dt$.

3.4 Timeline

Figures 3A and 3B summarize the sequence of events in the model of VRDOs and ARS, respectively. All participants observe the fundamental y_t and the max rate $\overline{r}$. At the beginning, the (remarketing or auction) agent announces and commits to an interest rate formula $R(y_t; y_*)$ which may depend on a certain (endogenously determined) parameter y_* which we will discuss shortly in the following section.

In the case of VRDOs (shown in Figure 3A), at each time t, a fraction δdt of creditors decide whether to roll over their debt or to run. They base their decision on the observation of the fundamental y_t and the interest rate r_t reset by the remarketing agent. If they decide to roll over, the game continues to the next instant. If they decide to run, the liquidity facility is drawn upon to purchase the tendered bonds, but the facility may fail with probability $\theta\delta dt$. If it fails, the game ends and the project is liquidated to pay off all the creditors. If it succeeds, the game continues to the next instant.

[Insert Figure 3 About Here]

The case of ARS, shown in Figure 3B, has a similar timeline as VRDOs, except that when the creditors decide to run, with probability $\kappa\delta dt$ the auction agent may decide not to intervene and then the auctions would continue to fail until the project fails eventually. This additional layer of uncertainty, highlighted by the flowchart within the dashed circle in Figure 3B, captures the central distinction between VRDOs and ARS in terms of the existence of a liquidity backstop.

3.5 Parameter Restrictions

We impose a few parameter restrictions for the model to be meaningful. We keep the same parameter restrictions as in HX:

$$\begin{aligned} \mu &< \rho + \phi, & (5) \\ \phi &< \theta\delta\left(1 - L - l\right) + \kappa\delta\left(1 - U\left(1\right)\right), & (6) \\ \alpha &< \left[\frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu}\right]^{-1}. & (7) \end{aligned}$$

The first one of the above three restrictions imposes an upper bound on the growth rate of the fundamental to ensure the fundamental value is finite. The second restriction ensures that the parameter θ is sufficiently high so that bankruptcy becomes likely when some creditors choose to run. $U\left(\cdot\right)$ denotes the value function in the case of continued auction failures, which we derive shortly in the next section. The third restriction stipulates a sufficiently low premature recovery rate so that $L + l < 1$.

In addition, we impose the following restriction for the additional parameters in our model, namely, $\overline{r}$, Δ, and κ:

$$\overline{r} > \rho + \Delta + \kappa\delta\left(1 - U\left(\frac{1-L}{l}\right)\right), \tag{8}$$

$$0 \leq \Delta < \frac{\eta_1\left(\rho + \phi\right)\left(\rho + \phi + \theta\delta\left(1 - L\right) - \overline{r}\right)}{\gamma_2\left(\rho + \phi + \delta\left(1 + \theta + \kappa\right)\right)}, \tag{9}$$

where η_1 and γ_2 as well as other constants are defined in Appendix A. The restriction (8) ensures that the max rate is sufficiently high for the model to be meaningful. The restriction (9) rules out the degenerate case where the liquidity component Δ is too large and it is thus always profitable to hold the bonds, implying that the equilibrium threshold y_* is zero. Note that $\rho + \phi + \theta\delta(1 - L)$ is an endogenous upper bound on the max rate. Furthermore, this restriction also implies $U(0) < 1$. Lastly, we also impose some parameter restrictions to ensure monotonicity of the value function. To simplify exposition, we assume $\overline{\theta} = 1 + \theta$ throughout the paper.

4 Equilibrium

We now turn to the characterization of monotone equilibriums in which creditors choose to roll over if and only if the fundamental is above a threshold. In this section, we first analyze an individual creditor's problem of optimal threshold choice. Then we study how the interest rate should be set in a monotone equilibrium. Lastly, we derive a unique symmetric monotone equilibrium in closed form in which the optimal threshold, denoted by y_*, is unique for all creditors and the equilibrium interest rate is set in a way that the debt is priced at par whenever $y_t \geq y_*$. We also discuss an extension of the model which is needed when we conceptualize and estimate the value of a liquidity backstop.

4.1 Value Functions

We derive the optimal rollover threshold y_* by solving an individual creditor's optimal rollover problem. Consider an individual creditor who is making his rollover decision. Suppose all the other creditors choose a rollover threshold y_* and the (remarketing or auction) agent resets the interest rate $r_t = R(y_t; y_*)$ based on the same threshold y_*. Denote by $V(y_t; y_*)$ the creditor's value function when auctions have been successful, and by $U(y_t)$ the value function when auctions have failed.

First, we determine the value function $V(y;y_*)$ when auctions have been successful. For each unit of debt, each creditor receives a stream of interest payments $R(y_t;y_*)$ until the earliest of the following four events occur. The first event occurs at stopping time τ_ϕ when the asset matures and the creditor gets a final payoff of $\min\left(1, y_{\tau_\phi}\right)$. The second event occurs at stopping time τ_δ when the creditor gets the opportunity to decide whether to roll over the debt. Whether or not the creditor decides to roll over depends on whether or not the continuation value $V(y_{\tau_\delta};y_*)$ exceeds the one-dollar par value. The third and fourth events occur when the fundamental falls below other creditors' rollover threshold y_*: upon a run by other creditors, with probability $1_{\{y_t \le y_*\}}\kappa\delta dt$, the third event occurs at the stopping time τ_κ when auctions fail and the creditor will be stuck with the debt valued at $U(y_{\tau_\kappa})$; with probability $1_{\{y_t \le y_*\}}\theta\delta dt$, the fourth event occurs at the stopping time τ_θ when the project is forced to premature liquidation with payoff $\min(1, L + ly_{\tau_\theta})$. The stopping time $\tau \equiv \min\{\tau_\phi, \tau_\delta, \tau_\kappa, \tau_\theta\}$ is the minimum of these four stopping times, representing the earliest time when any of these four events occur. Due to risk neutrality, the value function $V(y;y_*)$ is given by

$$
\begin{aligned}
V(y_t;y_*) &= E_t\left[\int_t^\tau e^{-\rho(s-t)}R(y_s;y_*)\,ds + e^{-\rho(\tau-t)}\min(1,y_\tau)\,1_{\{\tau=\tau_\phi\}}\right. \\
&\quad + e^{-\rho(\tau-t)}\min(1, L+ly_\tau)\,1_{\{\tau=\tau_\theta\}} + e^{-\rho(\tau-t)}U(y_t)\,1_{\{\tau=\tau_\kappa\}} \\
&\quad \left. + e^{-\rho(\tau-t)}\max_{\textit{rollover or run}}\left(V(y_\tau;y_*), 1\right)1_{\{\tau=\tau_\delta\}}\right].
\end{aligned} \tag{10}
$$

The Hamilton-Jacobi-Bellman (HJB) equation is given below:

$$
\begin{aligned}
\rho V(y_t;y_*) &= \mu y_t V_y(y_t;y_*) + \frac{\sigma^2}{2}y_t^2 V_{yy}(y_t;y_*) + R(y_t;y_*) \\
&\quad + \phi\left[\min(1,y_t) - V(y_t;y_*)\right] \\
&\quad + \theta\delta 1_{\{y_t \le y_*\}}\left[\min(1, L+ly_t) - V(y_t;y_*)\right] \\
&\quad + \kappa\delta 1_{\{y_t \le y_*\}}\left[U(y_t) - V(y_t;y_*)\right] \\
&\quad + \delta \max_{\textit{rollover or run}}\left(0, 1 - V(y_t;y_*)\right).
\end{aligned} \tag{11}
$$

It shows that the creditor's required return on the left hand side, $\rho V(y_t; y_*)$, is equal to the expected increase in his continuation value as summarized by the terms on the right hand side. The creditor will choose to roll over the debt if and only if $V(y_t; y_*) \geq 1$. If we denote as $y'_* = \inf\{y_t : V(y_t; y_*) \geq 1\}$ the minimum fundamental value at which the continuation value is no less than 1, then y'_* is the creditor's optimal rollover threshold since $V(y'_*; y_*) = 1$ and $V(y; y_*) < 1$ for $y < y'_*$. In the symmetric equilibrium we consider below, each creditor's optimal threshold choice y'_* must coincide with other creditors' threshold y_*. Thus the optimality condition is

$$V(y_*; y_*) = 1.$$

Similarly, we can determine the value function $U(y_t)$ when auctions have continued to fail. Under the assumption that the auctions, once failed, would continue to fail, creditors' rollover decision becomes irrelevant and thus the value function $U(y_t)$ does not depend on their rollover threshold y_*. In this autarkic scenario, the max rate $\overline{r}$ is imposed until the asset matures at the stopping time τ_ϕ or the project is prematurely liquidated at the stopping time $\tau_{\overline{\theta}}$. As a result, the value function $U(y_t)$ is given by

$$\begin{aligned} U(y_t) &= E_t\left[\int_t^{\tau_\phi \wedge \tau_{\overline{\theta}}} e^{-\rho(s-t)}\overline{r}ds + e^{-\rho\left(\tau_\phi - t\right)} \min\left(1, y_{\tau_\phi}\right) 1_{\left\{\tau_\phi \leq \tau_{\overline{\theta}}\right\}}\right. \\ &\quad \left. + e^{-\rho\left(\tau_{\overline{\theta}} - t\right)} \min\left(1, L + l y_{\tau_{\overline{\theta}}}\right) 1_{\left\{\tau_\phi > \tau_{\overline{\theta}}\right\}}\right]. \end{aligned} \tag{12}$$

In Lemma 1, we derive the value function in closed form as below, and prove that it is strictly monotonically increasing:

$$U(y_t) = \begin{cases} \overline{K}_1 + \overline{K}_2 y_t + U_1 y_t^{\eta_3}, & \text{if } y \in (0, 1] \\ \overline{K}_3 + \overline{K}_4 y_t + U_2 y_t^{-\gamma_3} + U_3 y_t^{\eta_3}, & \text{if } y \in (1, \frac{1-L}{l}] \\ \overline{K}_5 + U_4 y_t^{-\gamma_3}, & \text{if } y \in (\frac{1-L}{l}, \infty) \end{cases}, \tag{13}$$

where the coefficients $\overline{K}_1$, etc. are defined in Appendix A or in the proof of Lemma 1.

Lemma 1 $U(y)$ *is strictly monotonically increasing.*

To determine the value function $V(y; y_*)$, we need to spell out how the floating interest rate $r_t = R(y_t; y_*)$ is determined first, to which we will turn next.

4.2 Floating Interest Rate

We now consider how the interest rate r_t is reset at each point in time. We first derive the unconstrained interest rate in the benchmark case where there is no interest rate cap, and then determine the constrained interest rate once an interest rate cap is imposed.

In the absence of a maximum interest rate, for any fixed threshold $y_* \geq 0$, the interest rate is unbounded and can be set arbitrarily high to ensure that VRDOs or ARS are priced at par. Based on the HJB equation (11), the value function is always equal to one under the following unconstrained interest rate $R^u(y_t; y_*)$

$$R^u(y_t; y_*) = \rho + \phi(1 - y_t)^+ + 1_{\{y_t \leq y_*\}} \left[\theta\delta(1 - [L + ly_t])^+ + \kappa\delta(1 - U(y_t))\right] \tag{14}$$

where $(x)^+$ denotes x if $x > 0$, or zero otherwise. The unconstrained interest rate schedule takes a different shape when y_* is in a different range: $y_* \leq 1$, $1 < y_* \leq \frac{1-L}{l}$, and $y_* > \frac{1-L}{l}$, as shown in Panels A, B, and C of Figure 4, respectively.

[Insert Figure 4 About Here]

The unconstrained interest rate in Eq. (14) can be decomposed into three components: a risk-free component ρ, a component related to losses at maturity $\phi(1 - y_t)^+$, and the last component associated with possible credit losses. Intuitively, the unconstrained interest rate decreases with the fundamental y_t. That is, creditors are generally paid by a higher interest rate when the fundamental deteriorates. As we will show shortly, such countercyclicality of the interest rate tends to alleviate runs. Furthermore, the unconstrained interest rate jumps to a higher value when the threshold y_* is reached from above. The upward jumps occur in these cases because of the possible losses incurred due to premature liquidation $(1 - [L + ly_t])^+$ and auction failures $(1 - U(y_t))$. However, in the absence of a maximum rate, the interest rate can freely adjust to guarantee the value of debt is always equal to one. As a result, in equilibrium, creditors are indifferent between rolling over and running.

In the presence of an interest rate cap $\overline{r}$, creditors will be under-compensated in bad states if $\overline{r}$ is imposed instead of the higher unconstrained interest rate. Therefore, if the interest rate is given as the lower value between $\overline{r}$ and $R^u(y_t; y_*)$, then creditors' continuation value is strictly less than 1 and thus they always prefer to run, i.e., $y_* = \infty$. To avoid such a degenerate case and to keep tractability, throughout the rest of the paper, we add a new component $\Delta \geq 0$ to the unconstrained interest rate, which we refer to as a "*liquidity premium*", and then impose the following interest rate schedule:

$$R(y_t; y_*) = \min\left(R^u(y_t; y_*) + \Delta, \overline{r}\right). \tag{15}$$

Depending on the values of Δ and $\overline{r}$, as shown in Figure 5, there are totally eight different cases (Case A, $\cdots$, Case H) where the constrained interest rate schedule takes a different functional form.

[Insert Figure 5 About Here]

The component Δ introduces a trade-off for creditors. On one hand, when the fundamental falls below the rollover threshold, creditors are undercompensated by the max rate, which is usually lower than what the market-required rate would have been had creditors decided to roll over the debt. On the other hand, when the fundamental remains above the threshold, creditors are overcompensated by an amount equal to Δ. Therefore, however small Δ is, creditors receive the benefit of overcompensation during tranquil times and trade off the benefit against the loss due to undercompensation during future run scenarios. As we prove in the next subsection, the trade-off guarantees uniqueness of the symmetric equilibrium where the threshold y_* is uniquely pinned down at which creditors are just indifferent between rolling over and running. Another possible interpretation of the component Δ is that it can also be an extra compensation demanded by the (auction or remarketing) agent for possible inventory risk or holding costs. Lastly and importantly, it is also related to the concept of the value of a liquidity backstop we introduce shortly in the next section. For the above reasons, we will refer to Δ as a "*liquidity premium.*"

4.3 Unique Threshold Equilibrium

We focus on symmetric monotone equilibria where all creditors in equilibrium will choose the same threshold y_* and the agent resets the interest rate based on y_*. The threshold y_* is defined as the minimum value at which $V(y_t; y_*) \geq 1$, i.e., $y_* = \min\{y_t : V(y_t; y_*) \geq 1\}$. When y_t falls below the threshold y_*, due to monotonicity of the value function, the decision to run is strictly preferable since $V(y; y_*) < 1$ for $y < y_*$. Theorem 1 below proves the existence of a unique symmetric monotone equilibrium.

Theorem 1 *There exists a unique symmetric monotone equilibrium in which the rollover threshold y_* is uniquely determined — each maturing creditor chooses to roll over his debt if $y_t > y_*$, and to run otherwise.*

Proof. See Appendix C. ■

5 Model Implications

In this section, we explore the main implications of our model. The model has two key ingredients: a liquidity backstop and a floating interest rate. The rest of this section is devoted to understanding how these features affect equilibrium outcomes, in particular, the likelihood of runs. We first examine the role of a floating interest rate by focusing on the model of VRDOs where κ is set to zero to reflect the existence of an explicit liquidity backstop. We then turn to the model of ARS to examine the role of (lack of) a liquidity backstop where $\kappa > 0$ is positive to reflect the possibility of auction failures. Lastly, we formally define and quantify the value of a liquidity backstop based on structural estimation of the model in the next section.

5.1 Implications of Floating Interest Rate: The VRDO Model

To single out the role of a floating interest rate, we start with a special case where κ is set to be zero — the liquidity support, albeit imperfect, is explicitly committed such that the liquidity provider has contractual obligations to honor its liquidity commitment. In this special case where $\kappa = 0$, the model reduces to the one of VRDOs.

The floating interest payment $\{r_t\}$ in this paper, a key departure from HX, affects the creditors' rollover decision in a profound way. For example, as the fundamental deteriorates, the interest rate increases in a manner so as to compensate creditors for credit losses. However, the magnitude of overall interest payments to creditors depends on two factors: the maximum interest rate $\overline{r}$ and the liquidity premium Δ.

We show in Proposition 1 below that the equilibrium rollover threshold y_* decreases with the maximum interest rate $\overline{r}$ or the liquidity premium Δ. The result that y_* decreases with the liquidity premium is very important when we define and measure the value of a liquidity backstop later. The intuition is straightforward: a higher maximum interest rate $\overline{r}$ or a higher liquidity premium Δ allows the interest rate to increase further in a severely adverse environment; therefore, it increases the expected interest income for creditors and they will roll over more frequently. In the extreme case where the maximum interest rate $\overline{r}$ is sufficiently high, then the rollover threshold is zero (i.e., $y_* = 0$), that is, the likelihood of runs is zero.

Proposition 1 *The equilibrium rollover threshold y_* decreases with the maximum interest rate $\overline{r}$ or the liquidity premium Δ. In particular, when Δ goes to 0, the equilibrium rollover threshold y_* tends to infinity.*

Proof. See Appendix C. ■

5.2 Implications of Liquidity Backstop: The ARS Model

Next, we examine how the lack of a liquidity backstop in the ARS market affects equilibrium outcomes. To examine the role of (lack of) a liquidity backstop in isolation, we assume away the floating interest rate. In particular, we assume that the interest rate is always fixed at $\overline{r}$, the max rate, regardless of auction success or failure. This simplified model is very similar to the one in HX, except that there is an additional risk of auction failures.

In Proposition 2 below, we prove that when the max rate is low enough, increasing κ from zero to a positive value makes creditors more likely to run. Intuitively, a low enough max rate leads to a very low continuation value $U(y)$ in the event of failed auctions and thus, ex ante, creditors choose to run more often.

Proposition 2 *If $\overline{r}$ is sufficiently low, the equilibrium rollover threshold y_* increases as the arrival intensity of auction failures κ increases from zero; that is,* $\frac{dy_*}{d\kappa}\Big|_{\kappa=0} > 0$.

Proof. See Appendix C. ■

Proposition 2 illustrates how the lack of a liquidity backstop may exacerbate runs, which provides an explanation for the turmoil in the ARS market in early 2008 when investors started to factor in the possibility of auctions failures. As we show below, the destabilizing effect of the lack of liquidity backstops results from a new type of externality. The running decision of current creditors accelerates the issuer's default probability and may also trigger auction failures. Therefore, their decision to run affects payoffs of future creditors. Table 1 summarizes the current and future creditors' payoffs in different scenarios depending on whether the current creditors run or not.

[Insert Table 1 About Here]

From Table 1, we can see that the current creditors will choose to run if and only if $1 \cdot (1 - \kappa\delta dt - \theta\delta dt) + L(y) \cdot \theta\delta dt + U(y) \cdot \kappa\delta dt > V(y) \cdot 1$, or $V(y) < 1$ after ignoring higher order terms. Furthermore, because of the lack of a committed liquidity facility, a run on ARS may lead to auction failure when the auction agent stops providing liquidity, which imposes an additional implicit cost on future maturing creditors. Specifically, a run by the current creditors reduce the future creditors' value function by

$$\begin{aligned} & V(y) - [V(y) \cdot (1 - \kappa\delta dt - \theta\delta dt) + L(y) \cdot \theta\delta dt + U(y) \cdot \kappa\delta dt] \\ = \; & \underbrace{[V(y) - L(y)]\theta\delta dt}_{\text{cost due to default loss}} + \underbrace{[V(y) - U(y)]\,\kappa\delta dt}_{\text{cost due to auction failure}}. \end{aligned}$$

Besides the implicit cost of default loss as studied in HX, a run in our model also induces an additional cost in the event of auction failure. This additional externality, absent in the VRDO market, makes the ARS market more susceptible to runs: in anticipation of possible auction failures and the associated losses as a result of runs by future creditors, the current creditors have less incentive to roll over their debt.

5.3 The Value of a Liquidity Backstop

We are now in position to define the value of a liquidity backstop. From our earlier discussion (Section 2), prior to the crisis, ARS were considered to have the same explicit liquidity backstops as VRDOs. Put differently, before 2007 investors perceived the probability of auction failures to be negligible, i.e., $\kappa = 0$ for both ARS and VRDOs. However, the wave of auction failures in 2008 during the crisis revealed the implicit nature of liquidity support in the ARS market, and investors started to realize that $\kappa > 0$. As proved in Proposition 2, an increase in κ increases the rollover threshold y_*. Our estimation results reported in the following section confirms that after the crisis broke out, the estimated rollover threshold $y_*^{ARS}(\Delta, \kappa)$ in the ARS market is substantially higher than $y_*^{VRDO}(\Delta, 0)$ in the VRDO market. Note that we explicitly express the rollover threshold as a function of the arguments

Δ and κ (in the case of VRDOs, κ is always zero).

To define the value of a liquidity backstop, let us consider the following thought experiment. On one hand, an ARS issuer can pay a certain fee per annum to purchase a liquidity backstop from a liquidity provider, and effectively reduce the rollover threshold y_*^{ARS} to the same level as $y_*^{VRDO}(\Delta, 0)$. On the other hand, the ARS issuer can raise the level of interest rate by a constant amount $\Gamma > 0$, and thus effectively increases Δ to $\Delta + \Gamma$. According to Proposition 1, a higher liquidity premium $\Delta + \Gamma$ induces a lower rollover threshold. From the perspective of the (risk neutral) issuer, the two methods are equivalent as long as the fee to purchase a liquidity backstop is the same as Γ. Therefore, the increment in the ARS rate Γ measures the value of a liquidity backstop, satisfying

$$y_*^{ARS}(\Delta + \Gamma, \kappa) = y_*^{VRDO}(\Delta, 0). \tag{16}$$

In the next section, we use the historical data to estimate Γ based on the estimated thresholds $y_*^{ARS}(\Delta, \kappa)$ and $y_*^{VRDO}(\Delta, 0)$.

6 Estimation

The markets for VRDOs and ARS provide an ideal laboratory for us to identify the value of a liquidity backstop. The identification scheme hinges on the structural change in the belief of ARS investors following the wave of auction failures in mid-February 2008. We first describe the data and our estimation methodology, and then report estimation results.

6.1 Data

The weekly data of 1-week tax-exempt VRDO and ARS rates are obtained directly from the Securities Industry and Financial Markets Association (SIFMA) website.[8] The historical

[8]The website's URL is http://archives.sifma.org/swapdata.html.

data for the VRDO rate is available for the period from May 22, 1991 to October 24, 2012, while the ARS historical rate is only available for a shorter period from May 31, 2006 to December 30, 2009. We also obtain the 1-week Treasury repo rate from Bloomberg for the same period as the VRDO sample period.

For the purpose of calibration, we obtain information about characteristics of VRDOs or ARS (e.g., the max rate) from the Municipal Securities Rulemaking Board (MSRB)'s SHORT database from its inception date of April 1, 2009 through November 8, 2012. The SHORT database has been built from the Short-term Obligation Rate Transparency (SHORT) System and the Real-Time Transaction Reporting System (RTRS), which the MSRB launched in early 2009 to collect and disseminate interest rates and important descriptive information about these ARS and VRDOs. The SHORT database provides a centralized source of information about municipal ARS and VRDOs that was previously unavailable. Starting from May 2011, MSRB rules require VRDO remarketing agents to report to the MSRB the aggregate amount of par value of bonds held by investors or remarketing agents. There are 20,547 distinct VRDOs in the SHORT database during our sample period. We focus on the VRDOs with weekly interest resets, which accounts for 90.7% of the whole sample (i.e., 18,630). The SHORT database does not contain maturity information. Therefore, we merge it with the Mergent Municipal Bond database to collect information on maturities.

6.2 Calibration

There are eleven primitive parameters in the model: $\bar{r}, r, \phi, \rho, \delta, \alpha, \mu, \sigma, \Delta, \theta, \kappa$. Using the SHORT database and the Mergent Municipal Bond database, we first calibrate the parameters $\bar{r}, r, \rho, \phi, \delta, \alpha$. We then use the quasi-maximum likelihood (QML) method to estimate the remaining parameters.

The contractual maximum interest rate $\bar{r}$ is calibrated to be 12% using the SHORT database. Among the 18,630 VRDOs with weekly interest resets in the SHORT database, 53.42% of them have the max rate of 12%, 26.13% of them have the max rate of 10%, and

10.37% of them have the max rate of 15%. The weighted average of these three rates is 11.76%. Therefore, we set $\bar{r}$ as 12%. The cash flow rate from the project r is set to be equal to the average VRDO interest rate, or $r = 2.39\%$. That is, the municipality issuers have balanced budgets.

The average debt maturity of our merged VRDO sample from the SHORT and Mergent databases is 25.2 years (and the median is 25.96 years). We therefore set $1/\phi$, the expected asset maturity, to 25 based on the assumption that the average maturity coincides with the average asset maturity; that is, $\phi = 0.04$. The tax-adjusted risk-free rate ρ is set to the average value of the tax-adjusted repo rate, or $\rho = 0.0195$, during the sample period between 1991 and 2012 using a tax rate of 40% following Longstaff (2011).

The parameter δ represents the arrival intensity of creditors who make the running decision. In the model, once a run occurs, the proportion of creditors who decide not to roll over the debt is $\delta\Delta t$, where we set $\Delta t = 7/365.25$ to reflect the weekly frequency of the interest rate reset for the constituent VRDOs/ARS in the SIFMA indexes. In reality, VRDO/ARS creditors come to the remarketing or auction agent to buy or sell the securities on the interest rate reset dates. A run is considered to occur if a significant number of creditors decide to not roll over the debt. As a result, we set $\delta = 12$, meaning that on average creditors make the running decision on a monthly basis, and upon a run, about $\delta\Delta t = 23\%$ of the securities are not rolled over. Furthermore, we set the recovery rate $\alpha = 50\%$.[9] The calibration results are reported in Table 2 Panel A.

[Insert Table 2 About Here]

[9] The recovery rate of municipal bonds is not readily available given municipal bankruptcy is rare. See, for instance, Coval and Stafford (2007) for the estimates of the recovery rate for stocks; and Andrade and Kaplan (1998), Hennessy and Whited (2007) , Ellul, Jotikashira, and Lundblad (2010), for the estimates of the recovery rate for corporate bonds.

6.3 Estimation Methodology

We use the Quasi-Maximum Likelihood (QML) method to structurally estimate the model. The key equation in the structural estimation is the following equation:

$$rx_t^{ARS} = RX^{ARS}\left(y_t; y_*\left(\Theta_t\right)\right) + w_t, \text{ where } w_t \sim N\left(0, \nu^2\right), \tag{17}$$

where rx_t^{ARS} is the ARS interest rate in excess of the repo rate at time t, and $RX^{ARS}\left(y_t; y_*\left(\Theta_t\right)\right) \equiv R\left(y_t; y_*\left(\Theta_t\right)\right) - \rho$ is the model-implied excess interest rate for the ARS market (see Equation (15) for the expression of $R^{ARS}\left(y_t; y_*\left(\Theta_t\right)\right)$), and w_t denotes the pricing error that is assumed to follow a normal distribution with mean zero and standard deviation v. Note that the risk-free rate ρ is assumed to be constant in the model for tractability. We thus work directly with the excess rates in our structural estimation.

Note that the model-implied ARS excess rate depends on the equilibrium rollover threshold $y_*\left(\Theta_t\right)$, which in turn depends on the vector of the parameters to estimate $\Theta_t = (\mu, \sigma, \Delta, \theta, \kappa_t, v)$, as well as the other parameters calibrated. The subscript t reflects one important identification assumption that there is a structural change in the beliefs of ARS investors. Denote by τ the date of structural change. The probability of auction failures is considered to be zero before time τ, and becomes positive and equal to $\kappa\delta dt > 0$ at time τ and onwards once investors realize that auction agents have no contractual obligations to provide liquidity (i.e., no explicit liquidity backstops). That is,

$$\kappa_t = \begin{cases} 0, t \in [T_1, \tau) \\ \kappa > 0, t \in [\tau, T_2] \end{cases}, \tag{18}$$

where T_1 and T_2 denotes the start and end time periods of the ARS sample period, respectively. As we will describe shortly, we choose the date of structural change τ as November 14, 2007 when the VRDO and ARS rates start to diverge, and the ARS sample period is between May 31, 2006 and December 30, 2009.

The model-implied ARS excess rate also depends on the fundamental value y_t, which, however, is unobservable in the data. Based on the other identification assumption that both VRDO and ARS markets have the same fundamental process, we can infer y_t from the VRDO market. As mentioned before, this identification assumption is realistic since municipality issuers of VRDOs or ARS are very similar (and in fact the same in many cases). To infer y_t, we assume that there are no measurement errors in the VRDO market

$$\begin{aligned} rx_t^{VRDO} &= RX^{VRDO}\left(y_t; y_*^{VRDO}(\Theta_0)\right) \\ &= \phi(1-y_t)^+ + 1_{\{y_t \leq y_*^{VRDO}(\Theta_0)\}}\theta\delta(1-[L+ly_t])^+ . \end{aligned}$$

Note that the explicit arrangement of liquidity backstops in the VRDO market implies $\kappa = 0$. Therefore, the VRDO rollover threshold $y_*^{VRDO}(\Theta_0)$ remains unchanged throughout the sample period, and $\Theta_0 = (\mu, \sigma, \Delta, \theta)$ denotes the set of constant parameters to estimate. Also note that the model-implied VRDO excess rate is always non-negative while the VRDO excess rate in the data are sometimes (but very infrequently) negative. In this case, we set $y_t = \max\left(1, y_*^{VRDO}(\Theta_0)\right)$. We denote by $\widehat{y}_t^{VRDO}$ the value of y_t inferred from the VRDO data.

The above method of assuming zero measurement error to extract latent factors in one market segment and then applying the extracted factors to the other market segment assuming non-zero measurement errors is widely used in the literature of affine term-structure models (ATSMs). In this literature, it is very common to assume that the Treasury yield curve is driven mainly by a finite number of latent factors (e.g., level, slope, and curvature, etc.). The usual way to estimate such ATSMs is to extract the latent factors by assuming zero measurement errors for the same number of Treasury securities, and then to estimate the model using the extracted factors together with the rest of the yield curve. Our method is similar, but has a major difference: when we extract the unobservable fundamental process $\left\{\widehat{y}_t^{VRDO}\right\}$ and apply it to the ARS market, our structural estimation takes into consideration the structural change in November 2007 when the probability of auction failures is (correctly) perceived to be positive.

Lastly, we can estimate the parameters $\Theta = (\mu, \sigma, \Delta, \theta, \kappa, v)$ from the quasi-maximum likelihood (QML) method. That is, the QML estimator is the maximizer of log-likelihood function $\ln L\left(\Theta; rx_{T_1:T_2}^{ARS}, \widehat{y}_{T_1:T_2}^{VRDO}\right)$:

$$\widehat{\Theta} = \arg\max_{\Theta} \ln \mathcal{L}\left(\Theta; rx_{T_1:T_2}^{ARS}, \widehat{y}_{T_1:T_2}^{VRDO}\right),$$

where $T_1 : T_2$ denotes the sequence of time periods $\{T_1, T_1 + 1, \cdots, T_2\}$. The log-likelihood function is constructed as follows:

$$\begin{aligned} & \ln \mathcal{L}\left(\Theta; rx_{T_1:T_2}^{ARS}, \widehat{y}_{T_1:T_2}^{VRDO}\right) \\ = \ & \sum\nolimits_{t=T_1}^{(\tau-1)} \ln f^{(0)}\left(rx_t^{ARS} \left| \widehat{y}_t^{VRDO}, \Theta\right.\right) + \sum\nolimits_{t=\tau}^{T_2} \ln f^{(1)}\left(rx_t^{ARS} \left| \widehat{y}_t^{VRDO}, \Theta\right.\right) \end{aligned}$$

where

$$\begin{aligned} f^{(0)}\left(rx_t^{ARS} \left| \widehat{y}_t^{VRDO}, \Theta\right.\right) &= \frac{1}{\sqrt{2\pi v}} \exp\left\{-\frac{\left(rx_t^{ARS} - RX^{ARS}\left(\widehat{y}_t^{VRDO}; y_*^{VRDO}(\Theta_0)\right)\right)}{2v^2}\right\}, \\ f^{(1)}\left(rx_t^{ARS} \left| \widehat{y}_t^{VRDO}, \Theta\right.\right) &= \frac{1}{\sqrt{2\pi v}} \exp\left\{-\frac{\left(rx_t^{ARS} - RX^{ARS}\left(\widehat{y}_t^{VRDO}; y_*^{ARS}(\Theta)\right)\right)}{2v^2}\right\}. \end{aligned}$$

As discussed above, in the subperiod $[T_1, \tau)$, the probability of auction failures is assumed to be zero by ARS investors. Therefore, in this subperiod, the rollover threshold is the same as in the VRDO market, i.e., $y_*^{VRDO}(\Theta_0)$, which is used in the density function $f^{(0)}(\cdot)$. However, in the subperiod $[\tau, T_2]$ following the structural change, the ARS rollover threshold $y_*^{ARS}(\Theta)$ jumps to a higher level as a result of positive probability of auction failures (or $\kappa > 0$). The density function $f^{(1)}(\cdot)$ captures the structural change by using the higher threshold $y_*^{ARS}(\Theta)$.

6.4 Estimation Results

We apply the above estimation methodology to the SIFMA historical interest rate indexes for the VRDO and ARS markets. The VRDO sample period ranges between May 22, 1991 and October 24, 2012, while the ARS sample period ranges between May 31, 2006 and December 30, 2009 when the SIFMA stopped producing the ARS index. Recall that from Figure 1 the ARS rate had largely moved in lockstep with the VRDO rate until November 14, 2007, and has diverged since then. Based on this observation, we set κ to zero for the pre-crisis period when using the ARS data, but allow for a positive κ between November 14, 2007 and December 30, 2009 as a reflection the structural change in investors' beliefs. For the VRDO data, we restrict κ to zero for the entire sample period. In addition, we set $\mu = \sigma^2/2$ so that the (log) fundamental process has zero expected growth rate.

We back out the fundamental process using the VRDO historical rate, which is plotted by solid line in Figure 6 Panel A below. In plotting the figure, we focus on the period between May 2006 and December 2009 when both ARS and VRDO data are available. In Panel A, we also plot the VRDO rollover threshold (dashed line) and the ARS rollover threshold (dash-pointed line). The latter is plotted only after November 2007 when the structural change took place. The estimation results confirm that once a positive probability of auction failures is taken into account, ARS investors face a higher threshold and are more likely to run. Moreover, in February and March 2008, it is only the higher ARS rollover threshold that is crossed, not the VRDO threshold. This is consistent with the differential crisis experiences in these markets in early 2008 when there was a run in the ARS market, but not in the VRDO market. Moreover, in late 2008 following the Lehman's bankruptcy, both rollover thresholds were crossed, indicating runs in both markets. This is consistent with the market commentary that Lehman's bankruptcy put in doubt the ability of liquidity providers to honor their commitments.

In Panels B and C of Figure 6, we also plot the actual and model-implied excess interest rates in both markets. Because the actual VRDO excess interest rate is used to exactly fit the model-implied one in order to back out the fundamental process, these two series coincide with each other as shown in Panel B of Figure 6. We then estimate the model using QML to best fit the actual ARS excess interest rate. Panel C of Figure 6 shows a reasonably good fit between the model and the data. In particular, consistent with the data our model is able to generate spikes in the ARS excess interest rate in both run episodes in 2008.

[**Insert Figures 6 and 7 About Here**]

The structural estimation also allows us to compute the likelihood of a run to occur within the following week. Figure 7 plots such run probabilities for both VRDOs (Panel A) and ARS (Panel B). From Panel A, the model implies a 50% chance of a VRDO run within a week following the Lehman's bankruptcy. The run probability of 50% is very likely to understate the true likelihood of runs. This is due to the truncation in the fundamental process when we infer it from the historical VRDO rate. Recall that when the rollover threshold is hit for the first time, the interest rate jumps (see Panel A of Figure 4). Observing a spike in the VRDO rate indicates that the fundamental hits and may fall below the rollover threshold. In our estimation, we truncate the fundamental in this case at the threshold level. This leads to the artificial probability value of 50%. The run probability estimates for the ARS market are not subject to this problem. As a result, we can see from Panel B of Figure 7 that the run probability increases to about 80% during the first ARS run and to about 100% during the second ARS run in 2008. Furthermore, except these run episodes, the run probabilities are close to zero. In summary, our model is able to reproduce the differential crisis experiences for both markets.

We now turn to the quantitative estimation results, reported in Panels B-D in Table 2. We report the estimated parameter values in Panel B of Table 2. The default intensity θ is estimated to be 0.0111 so that the average time from a run to eventual bankruptcy is equal to $1/(\theta\delta) = 7.5$ years, which is roughly in line with the bankruptcy experience of Jefferson

County, AL (Woodley (2012)). Furthermore, the parameter κ is estimated to be 0.003, under which the fraction of auctions that have failed within the 14-week window between November 14, 2007 and February 20, 2008 is about 1%.[10] Based on the formula $L = \frac{\alpha r}{\rho+\phi}$ and $l = \frac{\alpha\phi}{\rho+\phi-\mu}$ and the estimated parameter values, the parameters governing the recovery rate of the asset in the worst case scenario are calibrated as: $(L, l) = (20.2\%, 55.8\%)$. The estimated value of Δ equal to 1 basis points accounts for about 0.4% of the average VRDO interest rate (i.e., 2.4%), or 0.3% of the average ARS interest rate (i.e., 3.02%) in the data.

Standard errors, reported in parentheses, are constructed from Monte Carlo simulations. Each simulation begins by randomly generating 119 weekly data of the underlying fundamental process. We then reestimate the key parameters with maximum likelihood using these data. We repeat this procedure 1000 times to construct the standard errors. The standard errors should be treated with caution. They are correct assuming the model is specified accurately. However, because of focus on the run episodes during the crisis, our data sample contain some extreme observations (e.g., spikes in the historical interest rates in the markets considered), which are likely to be less informative about the data-generating process than the Gaussian model implies.

The equilibrium thresholds are reported in Table 2 Panel C. First, the estimation results confirm that with a positive probability of auction failures (i.e., $\kappa > 0$), the rollover threshold for ARS investors is indeed higher than that for VRDO investors: $y_*^{ARS} > y_*^{VRDO}$. This is consistent with the economic intuition discussed in Section 5.2 that the fear of getting stuck when future auctions fail propels ARS creditors more likely to run, ex ante, relative to VRDO creditors. The higher rollover threshold for ARS creditors reflects the lack of a liquidity backstop in the ARS market.

We also study the effect of the floating interest rate on the run behavior. Recall that in

[10] By definition, following a run, a fraction $(\kappa\delta dt)$ of auctions will fail in the first week, or $(1-\kappa\delta dt)$ of auctions will surive the first week. Similarly, among the ARS whose auctions succeeded in the first week, a fraction of them, $(1-\kappa\delta dt)^2$, will continue to surve in the second week, $\cdots$. The cumulative fraction of auctions that have failed with N weeks equals $1-(1-\kappa\delta dt)^N$. Plugging in $\kappa = 0.003$, $\delta = 12$, $dt = 7/365$ leads to a failure rate of 1% in a 14-week window.

our structural estimation, we set the cash flow r to the average VRDO interest rate. That is, on average, the municipality issuer has a balanced budget. To compare with the HX model which has a fixed interest rate, we consider a fixed interest rate equal to the constant cash flow rate r. The equilibrium rollover threshold in this case is labeled as y_*^{HX}, which turns out to be higher than the rollover thresholds in either VRDO or ARS markets, according to the estimation results in Panel C of Table 2. This result suggest that floating interest rates tend to mitigate runs in both markets. This is an intuitive result because the floating interest rate moves inversely with the fundamental. When the issuer's fundamental deteriorates, the interest rate increases to compensate investors for the higher default risk, and thus makes them more willing to roll over. It is worthwhile to note that this result is contrary to the finding in Schroth, Suarez, and Taylor (2012) where due to the dilution risk time-varying yields in the ABCP market tend to make investors more likely to run because only maturing creditors enjoy a higher yield. By contrast, the dilution risk does not exist in our model because the floating interest rate, once reset, applies to all creditors. As a result, the floating interest rate in our paper tends to make investors less likely to run.

Lastly, we estimate the value of a liquidity backstop (denoted by Γ) and the estimation result is reported in Table 2 Panel D. Recall that from Proposition 1, the rollover threshold y_* decreases with the liquidity/risk premium Δ. Therefore, to measure the value of a liquidity backstop Γ, we should find out how much the interest rate needs to be increased so that the ARS rollover threshold can be reduced to the same level of the VRDO rollover threshold. The required increase Γ in the interest rate is a measure of the value of a liquidity backstop. Mathematically, we express the rollover thresholds $y_*^{ARS}(\Delta, \kappa)$ and $y_*^{VRDO}(\Delta, 0)$ to denote their dependence on Δ, and define the value of a liquidity backstop $\Gamma > 0$ as the solution to the following equation:

$$y_*^{ARS}(\Delta + \Gamma, \kappa) = y_*^{VRDO}(\Delta, 0).$$

The estimated value of a liquidity backstop is about 14 basis points. In present value (i.e., $\Gamma / (\rho + \phi)$), a liquidity backstop is evaluated to be about 2.4% of the par value. Therefore, the implied value (or cost) of providing liquidity backstops for the ARS market is about $4.7 billion for the $200 billion ARS market at the peak level before its collapse.

6.5 Policy Implications

Our study has several policy implications. First, this paper sheds light on (the costs of) government bailouts, or *public* liquidity backstops, although we focus on (private) liquidity backstops provided by banks.[11] In particular, the value of a private liquidity backstop estimated in this paper can serve as a lower bound for that of a public liquidity backstop and thus can be used to estimate the costs of government bailouts. For example, the FDIC offered on October 13, 2008 a three-year government guarantee on new unsecured bank debt issues with an annualized fee equal to 75 basis points.[12]

Second, the stabilizing role of liquidity backstops studied in this paper helps us better understand the fragility in the shadow banking system. The shadow banking system provides important liquidity and credit transformation outside of the traditional banking system (Moreira and Savov (2013)), however, as pointed out in Tarullo (2012),

> "[s]shadow banking also refers to the creation of assets that are thought to be safe, short-term, and liquid, and as such, "cash equivalents" similar to insured deposits in the commercial banking system. Of course, as many financial market actors learned to their dismay, in periods of stress these assets are not the same as insured deposits."

Due to a lack of government guarantees that backstop the traditional banking system (e.g., federal deposit insurance and the central bank's lender-of-last-resort capacity), "shadow money" created by the shadow banking system is runnable. In fact, the recent financial crisis

[11] See Veronesi and Zingales (2010) for the study of the costs and benefits of government intervention during the financial crisis.

[12] More detail about the Temporary Liquidity Guarantee Programm can be found at https://www.fdic.gov/news/board/08BODtlgp.pdf.

can be considered as a modern bank run on the shadow banking system (Gorton and Metrick (2010, 2012)). The value of a liquidity backstop, the key focus of this paper, speaks to the central difference between the shadow banking system and the traditional banking system. Related to the "neglected risk" view of shadow banking in Gennaioli, Shleifer, and Vishny (2013) studied in a static setting, this paper contributes to the growing literature on shadow banking by showing that in a *dynamic* setting, recognition of "neglected risk" in conjunction with strategic complementaries among investors can further exacerbate runs.[13]

7 Concluding Remarks

In this paper, we develop a model of dynamic debt runs to provide a microfoundation for the important role of liquidity backstops in mitigating runs. We focus on the municipal bond markets for ARS and VRDOs, which provide an idea laboratory to identify and quantify the value of a liquidity backstop in terms of its run-mitigating role. As discussed in the paper, ARS were considered almost identical to VRDOs prior to the financial crisis, however, investors started to recognize the lack of a liquidity backstop in the ARS market at the onset of the crisis. The structural change in investors' beliefs drove a wedge in the experiences of these two markets during the crisis: the liquidity-backstop-lacking ARS market was more susceptible to runs and collapsed, while the liquidity-backstop-possessing VRDO market survived. Such structural change is also the key in identifying the value of a liquidity backstop. Through structural estimation, we estimate that a liquidity backstop is valued at about 14.5 basis points per annum.

Our model provides a microfoundation for the destabilizing effects of the lack of a liquidity backstop observed during the crisis. In the model, a run by future creditors imposes two types of negative externalities on earlier creditors who have decided to roll over the debt.

[13] The dynamic effects in exacerbating runs have empirical evidence from runs on money market or open-end mutual funds (see, Kacperczyk and Schnabl (2013) and Chen, Goldstein, and Jiang (2010)). Also see Pozsar, et al. (2012) for a comprehensive overview of the shadow banking system and the references therein.

The first type of externalities, studied in HX, results from losses due to premature liquidation of the project triggered by the run, while the second type, new in this paper, arises from losses due to illiquidity when the run cause liquidity to completely dry up unexpectedly (e.g., auction failures in the ARS market). The latter explains why the ARS market that lacks liquidity backstops was more susceptible to runs than the VRDO market.

Note that the microfoundation has broad applications beyond these municipal bond markets studied in the paper. For example, when the Reserve Primary Fund broke the buck in September 2008 following the Lehman's bankruptcy, the fund sponsor did not cover the losses, which made investors realize that fund sponsors have the option, but not the obligation, to support failing money market funds. Similar to the run on ARS in early 2008, the structural change in beliefs triggered a wide-spread run on other money market funds. However, focusing on the run on money market funds is not enough for us to identify the value of a liquidity backstop. Instead, we consider the markets for ARS and VRDOs that allow for such identification in a spirit similar to the "difference-in-difference" approach. Through structural estimation of the model based on the QML method, we are able to quantify the value of a liquidity backstop as about 14 basis points per annum.

Consistent with the literature on the "neglected risk" view of shadow banking, "shadow money" (e.g., ARS in this paper, or money market fund shares) can stop being liquid or safe once investors take account of tail risks that are previously neglected (see Gennaioli, Shleifer, and Vishny (2013)). We further demonstrate possible amplifying effects of neglecting tail risks in a dynamic setting in the context of the markets for VRDOs and ARS. Despite our focus on these specific markets, the key model implication is more general: the possibility of shadow money becoming illiquid in the future prompts investors to run more often, ex ante. Having a public liquidity backstop (e.g., deposit insurance) effectively mitigates runs induced by such liquidity risk. Therefore the value of a liquidity backstop studied in this paper speaks to the central difference between the shadow banking system and the traditional banking system in terms of their differential access to public liquidity backstops.

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Appendix A: Notation

We denote by $-\gamma_i$ and η_i two real roots of the quadratic equation $\frac{1}{2}\sigma^2 x(x-1) + \mu x - (\rho + \phi + \delta_i) = 0$, $i = 1, 2, 3$,

$$\begin{aligned} -\gamma_i &= -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2\left[\rho + \phi + \delta_i\right]}}{\sigma^2} < 0, \\ \eta_i &= -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2\left[\rho + \phi + \delta_i\right]}}{\sigma^2} > 0. \end{aligned}$$

where $\delta_1 = \delta(1 + \theta + \kappa)$, $\delta_2 = 0$, $\delta_3 = \overline{\theta}\delta$. The following notation is used in determining equilibrium threshold

$$\begin{array}{llll}
K_1 = \frac{\overline{r}+\delta(1+\theta L)}{\rho+\phi+\delta(1+\theta+\kappa)} & K_6 = \frac{\phi}{\rho+\phi-\mu} & \overline{K}_1 = \frac{\overline{r}+\overline{\theta}\delta L}{\rho+\phi+\overline{\theta}\delta} & \overline{K}_6 = \kappa\delta\frac{\overline{K}_1}{\rho+\phi+\delta(1+\theta+\kappa)} \\
K_2 = \frac{\phi+\theta\delta l}{\rho+\phi+\delta(1+\theta+\kappa)-\mu} & K_7 = \frac{\overline{\rho}+\phi+\delta(1+\theta+\kappa)}{\rho+\phi+\delta(1+\theta+\kappa)} & \overline{K}_2 = \frac{\phi+\overline{\theta}\delta l}{\rho+\phi+\overline{\theta}\delta-\mu} & \overline{K}_7 = \kappa\delta\frac{\overline{K}_2}{\rho+\phi+\delta(1+\theta+\kappa)-\mu} \\
K_3 = \frac{\overline{r}+\phi+\delta(1+\theta L)}{\rho+\phi+\delta(1+\theta+\kappa)} & K_8 = \frac{\overline{\rho}+\phi}{\rho+\phi} & \overline{K}_3 = \frac{\overline{r}+\phi+\overline{\theta}\delta L}{\rho+\phi+\overline{\theta}\delta} & \overline{K}_8 = \kappa\delta\frac{\overline{K}_3}{\rho+\phi+\delta(1+\theta+\kappa)} \\
K_4 = \frac{\theta\delta l}{\rho+\phi+\delta(1+\theta+\kappa)-\mu} & K_9 = \frac{\overline{r}+\phi+\delta(1+\theta)}{\rho+\phi+\delta(1+\theta+\kappa)} & \overline{K}_4 = \frac{\overline{\theta}\delta l}{\rho+\phi+\overline{\theta}\delta-\mu} & \overline{K}_9 = \kappa\delta\frac{\overline{K}_4}{\rho+\phi+\delta(1+\theta+\kappa)-\mu} \\
K_5 = \frac{\overline{r}}{\rho+\phi} & K_{10} = \frac{\overline{r}+\phi}{\rho+\phi} & \overline{K}_5 = \frac{\overline{r}+\phi+\overline{\theta}\delta}{\rho+\phi+\overline{\theta}\delta} & \overline{K}_{10} = \kappa\delta\frac{\overline{K}_5}{\rho+\phi+\delta(1+\theta+\kappa)}
\end{array}$$

Appendix B: Proofs

Proof. [Proof of Lemma 1] The HJB equation for $U(y)$ is the following

$$\rho U(y_t) = \mu y_t U_y(y_t) + \frac{\sigma^2}{2} y_t^2 U_{yy}(y_t) + \overline{r} + \phi\left[\min(1, y_t) - U(y_t)\right] + \overline{\theta}\delta\left[\min(1, L + l y_t) - U(y_t)\right].$$

Depending on the value of y, the HJB equation can be re-expressed as

$$\left(\rho + \phi + \overline{\theta}\delta\right) U - \mu y U_y - \frac{\sigma^2}{2} y^2 U_{yy} = \begin{cases} \overline{r} + \phi y + \overline{\theta}\delta\left(L + l y\right), \text{ if } y \in (0, 1]; \\ \overline{r} + \phi + \overline{\theta}\delta\left(L + l y\right), \text{ if } y \in (1, \frac{1-L}{l}]; \\ \overline{r} + \phi + \overline{\theta}\delta, \text{ if } y \in (\frac{1-L}{l}, \infty). \end{cases}$$

Therefore, the solution has the functional form in Eq. (13). We determine the unknown coefficients $U_1, \cdots, U_4$ from the value-matching and smooth-pasting conditions:

$$\begin{aligned} U_1 &= \left(\overline{K}_3 + \overline{K}_4\right) - \left(\overline{K}_1 + \overline{K}_2\right) + U_2 + U_3, \\ U_2 &= -\frac{\eta_3 \left(\overline{K}_3 - \overline{K}_1\right) + (\eta_3 - 1)\left(\overline{K}_4 - \overline{K}_2\right)}{\eta_3 + \gamma_3}, \\ U_3 &= \frac{-\gamma_3 \left(\overline{K}_3 - \overline{K}_5\right) - (\gamma_3 + 1)\overline{K}_4 \left(\frac{1-L}{l}\right)}{(\eta_3 + \gamma_3)\left(\frac{1-L}{l}\right)^{\eta_3}}, \\ U_4 &= -\frac{\overline{K}_4 \left(\frac{1-L}{l}\right) - \gamma_3 U_2 \left(\frac{1-L}{l}\right)^{-\gamma_3} + \eta_3 U_3 \left(\frac{1-L}{l}\right)^{\eta_3}}{\gamma_3 \left(\frac{1-L}{l}\right)^{-\gamma_3}}. \end{aligned}$$

To prove the monotonicity of $U(y)$, we first prove that $U_i < 0$, for $i = 1, \cdots, 4$. Substituting the expressions of $\overline{K}_1, \cdots, \overline{K}_5$ into U_1, U_2, U_3, we have

$$\begin{aligned} U_1 &= \frac{\phi + \overline{\theta}\delta (1-L)\left(\frac{1-L}{l}\right)^{-\eta_3}}{(\eta_3 + \gamma_3)} \left[\frac{\gamma_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\gamma_3 + 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] < 0; \\ U_2 &= -\frac{\phi}{\eta_3 + \gamma_3} \left[\frac{\eta_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\eta_3 - 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] < 0; \\ U_3 &= \frac{\overline{\theta}\delta (1-L)}{(\eta_3 + \gamma_3)\left(\frac{1-L}{l}\right)^{\eta_3}} \left[\frac{\gamma_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\gamma_3 + 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] < 0. \end{aligned}$$

Lastly, from the above expression of U_3 and the result $U_2 < 0$, we have

$$\gamma_3 \left(\frac{1-L}{l}\right)^{-\gamma_3} U_4 = -\left[\frac{(\eta_3 - 1)\overline{\theta}\delta (1-L)}{(\eta_3 + \gamma_3)\left(\rho + \phi + \overline{\theta}\delta - \mu\right)} - \gamma_3 U_2 \left(\frac{1-L}{l}\right)^{-\gamma_3}\right] < 0.$$

Next, we prove $U(y)$ is monotonically increasing. For $y > \frac{1-L}{l}$, $U'(y) = U_4 (-\gamma_3) y^{-\gamma_3 - 1} > 0$ for since $U_4 < 0$. For $0 < y \leq 1$, because $\left(\frac{1-L}{l}\right)^{-\eta_3} < \frac{l}{1-L}$ and definitions of η_3 and γ_3, we have

$$\begin{aligned} & U'(y) = \overline{K}_2 + \eta_3 U_1 y^{\eta_3 - 1} \geq \overline{K}_2 + \eta_3 U_1 \\ = \; & \frac{\phi + \overline{\theta}\delta l}{\rho + \phi + \overline{\theta}\delta - \mu} + \eta_3 \frac{\phi + \overline{\theta}\delta (1-L)\left(\frac{1-L}{l}\right)^{-\eta_3}}{(\eta_3 + \gamma_3)} \left[\frac{\gamma_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\gamma_3 + 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] \\ > \; & \frac{\phi + \overline{\theta}\delta l}{\rho + \phi + \overline{\theta}\delta - \mu} + \eta_3 \frac{\phi + \overline{\theta}\delta l}{(\eta_3 + \gamma_3)} \left[\frac{\gamma_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\gamma_3 + 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] \\ = \; & \frac{\gamma_3 \left(\phi + \overline{\theta}\delta l\right)}{\eta_3 + \gamma_3} \left[\frac{\eta_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\eta_3 - 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] > 0. \end{aligned}$$

For $1 < y \le \frac{1-L}{l}$,

$$
\begin{aligned}
U'(y) &= \overline{K}_4 + (-\gamma_3) U_2 y^{-\gamma_3-1} + \eta_3 U_3 y^{\eta_3-1} > \overline{K}_4 - \gamma_3 U_2 + \eta_3 U_3 \left(\frac{1-L}{l}\right)^{\eta_3-1} \\
&= \frac{\overline{\theta}\delta l}{\rho+\phi+\overline{\theta}\delta-\mu} + \gamma_3 \frac{\phi}{\eta_3+\gamma_3}\left[\frac{\eta_3}{\rho+\phi+\overline{\theta}\delta} - \frac{\eta_3-1}{\rho+\phi+\overline{\theta}\delta-\mu}\right] \\
&\quad + \eta_3 \frac{\overline{\theta}\delta l}{(\eta_3+\gamma_3)}\left[\frac{\gamma_3}{\rho+\phi+\overline{\theta}\delta} - \frac{\gamma_3+1}{\rho+\phi+\overline{\theta}\delta-\mu}\right] \\
&= \frac{\gamma_3\left(\phi+\overline{\theta}\delta l\right)}{\eta_3+\gamma_3}\left[\frac{\eta_3}{\rho+\phi+\overline{\theta}\delta} - \frac{\eta_3-1}{\rho+\phi+\overline{\theta}\delta-\mu}\right] > 0.
\end{aligned}
$$

■

The following lemmas are needed. Lemma 2's proof is straightforward and thus omitted.

Lemma 2 *For η_i and γ_i, $i = 1, 2, 3$, defined in Appendix A, we can show that: (i)* $\frac{\eta_i\gamma_i}{\rho+\phi+\delta_i} = \frac{(\eta_i-1)(\gamma_i+1)}{\rho+\phi+\delta_i-\mu} = \frac{2}{\sigma^2}$*; (ii) Under the restriction* $\rho+\phi > \mu$, $\chi_i \equiv \frac{\eta_i}{\rho+\phi+\delta_i} - \frac{\eta_i-1}{\rho+\phi+\delta_i-\mu} > 0$.

Lemma 3 *Under the parameter restrictions in Eqs. (5)-(9) and the following restrictions*

$$
\begin{aligned}
(\eta_1-1)\left(K_2+\overline{K}_7\right) + \eta_3(\eta_1-\eta_3)U_1 &> 0, \qquad (19)\\
(\eta_1-1)\left(K_4+\overline{K}_9\right) + \eta_3(\eta_1-\eta_3)U_3\left((1-L)/l\right)^{\eta_3-1} &> 0, \qquad (20)
\end{aligned}
$$

the function $W(y)$ is strictly increasing.

Proof. [Proof of Lemma 3] Because there are eight different cases and in each different case the function $W(y)$ takes a different form. Below we prove the monotonicity of $W(y)$ in each of the ten cases. Define $y^C_{**} = \frac{\overline{\rho}+\phi-\overline{r}}{\phi} < 1$, and y^A_{**}, y^E_{**} as solutions to the following equations:

$$
\begin{aligned}
\overline{r} &= \overline{\rho} + \phi\left(1-y^A_{**}\right) + \theta\delta\left(1-\left[L+ly^A_{**}\right]\right) + \kappa\delta\left(1-\left[\overline{K}_1+\overline{K}_2 y^A_{**} + U_1\left(y^A_{**}\right)^{\eta_3}\right]\right), \\
\overline{r} &= \overline{\rho} + \theta\delta\left(1-\left[L+ly^E_{**}\right]\right) + \kappa\delta\left(1-\left[\overline{K}_3+\overline{K}_4 y^E_{**} + U_2\left(y^E_{**}\right)^{-\gamma_3} + U_3\left(y^E_{**}\right)^{\eta_3}\right]\right).
\end{aligned}
$$

(i) In Case A where $\overline{r} > \overline{\rho} + \phi(1-y) + \theta\delta(1-[L+ly]) + \kappa\delta(1-U(y))$, the function $W(y) = W_A(y) = \frac{\eta_1 K_7+\gamma_2 K_8}{\eta_1+\gamma_2} + \frac{\eta_1+\gamma_1}{\eta_1+\gamma_2} A_2 y^{-\gamma_1}$, for $y \in (0,1]$. To prove $W_A(y)$ is strictly increasing, we only need to prove $A_2 = \frac{\eta_1\left(K_1+\overline{K}_6-K_7\right)+(\eta_1-1)\left(K_2+\overline{K}_7\right)y^A_{**}+(\eta_1-\eta_3)U_1\left(y^A_{**}\right)^{\eta_3}}{(\eta_1+\gamma_1)\left(y^A_{**}\right)^{-\gamma_1}} < 0$. Substituting the expression of r^A_{**} into the above equation yields

$$
(\eta_1+\gamma_1)\left(y^A_{**}\right)^{-\gamma_1} A_2 = -\left(\phi+\theta\delta l+\kappa\delta\overline{K}_2\right) y^A_{**}\chi_1 + \left[\eta_1-\eta_3-\frac{\kappa\delta\eta_1}{\rho+\phi+\delta(1+\theta+\kappa)}\right] U_1\left(y^A_{**}\right)^{\eta_3}.
$$

From Condition (19) and Lemma 2, we have

$$\frac{(\eta_1 - 1)\left(\phi + \theta\delta l + \kappa\delta\overline{K}_2\right)}{\rho + \phi + \delta(1 + \theta + \kappa) - \mu} = \left(\phi + \theta\delta l + \kappa\delta\overline{K}_2\right)\gamma_1\chi_1 > -\eta_3(\eta_1 - \eta_3)U_1,$$

and, note that $\eta_1 - \gamma_1 = \eta_3 - \gamma_3$ and $\frac{\eta_1\gamma_1}{(1+\theta+\kappa)} = \frac{\eta_3\gamma_3}{\rho+\phi+\theta\delta}$,

$$\eta_3(\eta_1 - \eta_3) + \gamma_1\left(\eta_1 - \eta_3 - \frac{\kappa\delta\eta_1}{\rho + \phi + \delta(1 + \theta + \kappa)}\right) = -\eta_3\gamma_3 + \frac{(\rho + \phi + \delta(1 + \theta))\eta_1\gamma_1}{\rho + \phi + \delta(1 + \theta + \kappa)} = 0.$$

Therefore, since $0 < \left(y_{**}^A\right)^{\eta_3} < y_{**}^A < 1$, we have

$$(\eta_1 + \gamma_1)\left(y_{**}^A\right)^{-\gamma_1}A_2 < \left[\eta_3(\eta_1 - \eta_3) + \gamma_1\left(\eta_1 - \eta_3 - \frac{\kappa\delta\eta_1}{\rho + \phi + \delta(1 + \theta + \kappa)}\right)\right]U_1\frac{\left(y_{**}^A\right)^{\eta_3}}{\gamma_1} = 0.$$

(ii) In Case B where $\overline{\rho} + \phi(1 - y) \leq \overline{r} \leq \overline{\rho} + \phi(1 - y) + \theta\delta(1 - [L + ly]) + \kappa\delta(1 - U(y))$, the function $W(y) = W_B(y)$, for $y \in (0, 1]$

$$W_B(y) = \frac{\eta_1\left(K_1 + \overline{K}_6\right) + \gamma_2 K_8 + (\eta_1 - 1)\left(K_2 + \overline{K}_7\right)y}{\eta_1 + \gamma_2} + \frac{(\eta_1 - \eta_3)U_1 y^{\eta_3}}{\eta_1 + \gamma_2}.$$

Under Condition (19) and $y \leq 1$, we have

$$\begin{aligned} &(\eta_1 + \gamma_2)W_B'(y) = (\eta_1 - 1)\left(K_2 + \overline{K}_7\right) + \eta_3(\eta_1 - \eta_3)U_1 y^{\eta_3 - 1} \\ \geq\ &(\eta_1 - 1)\left(K_2 + \overline{K}_7\right) + \eta_3(\eta_1 - \eta_3)U_1 > 0. \end{aligned}$$

(iii) In Case C where $\overline{r} < \overline{\rho} + \phi(1 - y) < \overline{\rho} + \phi$, the function $W(y) = W_C(y)$, for $y \in (0, 1]$

$$\begin{aligned} W_C(y) = &\frac{\gamma_2 K_5 + \eta_1\left(K_1 + \overline{K}_6\right) + (\gamma_2 + 1)K_6 y + (\eta_1 - 1)\left(K_2 + \overline{K}_7\right)y}{(\eta_1 + \gamma_2)} \\ &+ \frac{\eta_1 - \eta_3}{\eta_1 + \gamma_2}U_1 y^{\eta_3} + \frac{\gamma_2(K_8 - K_5) - (\gamma_2 + 1)K_6 y_{**}^C}{(\eta_1 + \gamma_2)\left(y_{**}^C\right)^{\eta_2}}y^{\eta_2}. \end{aligned}$$

When $\overline{r} < \overline{\rho} + \phi$, it holds that $\gamma_2(K_8 - K_5) - (\gamma_2 + 1)K_6 y_{**}^C = (\overline{\rho} + \phi - \overline{r})\left[\frac{\gamma_2}{\rho+\phi} - \frac{\gamma_2+1}{\rho+\phi-\mu}\right] < 0$. Furthermore, since $y < y_{**}^C < 1$ and $U_1 < 0$, we have that under Condition (19),

$$\begin{aligned} (\eta_1 + \gamma_2)W_C'(y) &> \left[\begin{array}{c} (\eta_1 - 1)\left(K_2 + \overline{K}_7\right) + (\gamma_2 + 1)K_6 + \eta_3(\eta_1 - \eta_3)U_1\left(y_{**}^C\right)^{\eta_3 - 1} \\ + \left[\gamma_2(K_8 - K_5) - (\gamma_2 + 1)K_6 y_{**}^C\right]\frac{\eta_2}{y_{**}^C} \end{array}\right] \\ &= (\eta_1 - 1)\left(K_2 + \overline{K}_7\right) + \eta_3(\eta_1 - \eta_3)U_1\left(y_{**}^C\right)^{\eta_3 - 1} > (\eta_1 - 1)\left(K_2 + \overline{K}_7\right) + \eta_3(\eta_1 - \eta_3)U_1 > 0. \end{aligned}$$

(iv) In Case D where $\overline{r} \geq \overline{\rho} + \theta\delta(1-L-l) + \kappa\delta(1-U(1))$, the function $W(y) = W_D(y) = W_A(y)$, for $y \in \left(1, \frac{1-L}{l}\right)$. The proof in (i) applies here too.

(v) In Case E where $\overline{\rho} + \theta\delta(1-L-ly) + \kappa\delta(1-U(y)) < \overline{r} < \overline{\rho} + \theta\delta(1-L-l) + \kappa\delta(1-U(1))$, the function $W(y) = W_E(y) = \frac{\eta_1 K_7 + \gamma_2 K_8}{\eta_1+\gamma_2} + \frac{\eta_1+\gamma_1}{\eta_1+\gamma_2} E_4 y^{-\gamma_1}$, for $1 < y_{**}^E < y < \frac{1-L}{l}$ where

$$
\begin{aligned}
E_4 &= E_2 + \frac{\left(y_{**}^E\right)^{\gamma_1}}{(\eta_1+\gamma_1)} \left[\begin{array}{c} \eta_1\left(K_3+\overline{K}_8-K_7\right) + (\eta_1-1)\left(K_4+\overline{K}_9\right) y_{**}^E \\ +(\eta_1+\gamma_3)\kappa U_2\left(y_{**}^E\right)^{-\gamma_3} + (\eta_1-\eta_3)\kappa U_3\left(y_{**}^E\right)^{\eta_3} \end{array} \right], \\
E_2 &= \frac{1}{\eta_1+\gamma_1} \left[\begin{array}{c} \eta_1\left(K_1+\overline{K}_6-K_3-\overline{K}_8\right) - (\eta_1-1)\left(K_4+\overline{K}_9-K_2-\overline{K}_7\right) \\ +(\eta_1-\eta_3)\kappa(U_1-U_3) - (\eta_1+\gamma_3)\kappa U_2 \end{array} \right].
\end{aligned}
$$

To prove the increasing monotonicity of $W_E(y)$, we only need to prove the coefficient of $y^{-\gamma_1}$ in $W_E(y)$ is negative, or $E_4 < 0$. It is straightforward to verify that $E_2 = 0$. As a result,

$$
\begin{aligned}
& (\eta_1+\gamma_1) E_4 \left(y_{**}^E\right)^{-(\gamma_1+1)} \\
=\; & -\left(\theta\delta l + \kappa\delta\overline{K}_4\right)\chi_1 + \left(\eta_1+\gamma_3 - \frac{\kappa\delta\eta_1}{\rho+\phi+\delta(1+\theta+\kappa)}\right) U_2\left(y_{**}^E\right)^{-(\gamma_3+1)} \\
& + \left(\eta_1-\eta_3 - \frac{\kappa\delta\eta_1}{\rho+\phi+\delta(1+\theta+\kappa)}\right) U_3\left(y_{**}^E\right)^{\eta_3-1} \\
<\; & \left[-\left(\theta\delta l + \kappa\delta\overline{K}_4\right)\chi_1 + \left(\eta_1-\eta_3 - \frac{\kappa\delta\eta_1}{\rho+\phi+\delta(1+\theta+\kappa)}\right) U_3\left(\frac{1-L}{l}\right)^{\eta_3-1}\right]\left(\frac{y_{**}^E}{(1-L)/l}\right)^{\eta_3-1} \\
<\; & 0.
\end{aligned}
$$

where we used the following fact based on a similar argument as in in the proof of (i) that under Condition (20)

$$
\begin{aligned}
& -\left(\theta\delta l + \kappa\delta\overline{K}_4\right)\chi_1 + \left(\eta_1-\eta_3 - \frac{\kappa\delta\eta_1}{\rho+\phi+\delta(1+\theta+\kappa)}\right) U_3\left(\frac{1-L}{l}\right)^{\eta_3-1} \\
<\; & \frac{1}{\gamma_1}\left[\eta_3(\eta_1-\eta_3) + \gamma_1\left(\eta_1-\eta_3 - \frac{\kappa\delta\eta_1}{\rho+\phi+\delta(1+\theta+\kappa)}\right)\right] U_3\left(\frac{1-L}{l}\right)^{\eta_3-1} \\
=\; & 0.
\end{aligned}
$$

(vi) In Case F where $\overline{r} \leq \overline{\rho} + \theta\delta(1-L-ly) + \kappa\delta(1-U(y))$, the function $W(y) = W_F(y) = \frac{\left[\begin{array}{c} \eta_1\left(K_3+\overline{K}_8\right) + \gamma_2 K_8 + (\eta_1-1)\left(K_4+\overline{K}_9\right) y \\ +(\eta_1+\gamma_3) U_2 y^{-\gamma_3} + (\eta_1-\eta_3) U_3 y^{\eta_3} \end{array} \right]}{(\eta_1+\gamma_2)} + \frac{\eta_1+\gamma_1}{\eta_1+\gamma_2} F_2 y^{-\gamma_1}$ for $y \in \left(1, \frac{1-L}{l}\right)$where

$F_2 = E_2 = 0$. Under Condition (20), we have

$$(\eta_1 + \gamma_2) W_F'(y) = (\eta_1 - 1)\left(K_4 + \overline{K}_9\right) - \gamma_3 (\eta_1 + \gamma_3) U_2 y^{-\gamma_3 - 1} + \eta_3 (\eta_1 - \eta_3) U_3 y^{\eta_3 - 1}$$
$$> \left[(\eta_1 - 1)\left(K_4 + \overline{K}_9\right) + \eta_3 (\eta_1 - \eta_3) U_3 \left(\frac{1-L}{l}\right)^{\eta_3 - 1}\right] \left(\frac{y}{(1-L)/l}\right)^{\eta_3 - 1} > 0.$$

(vii) In Case G where $\overline{r} \geq \overline{\rho} + \theta\delta(1 - L - l) + \kappa\delta(1 - U(1))$, the function $W(y) = W_G(y) = W_A(y)$ for $y \geq \frac{1-L}{l}$. The proof of (i) applies here too.

(viii) In Case H where $\overline{\rho} + \kappa\delta\left(1 - U\left(\frac{1-L}{l}\right)\right) \leq \overline{r} < \overline{\rho} + \theta\delta(1 - L - l) + \kappa\delta(1 - U(1))$, the function $W(y) = W_H(y) = W_E(y)$ for $y \geq \frac{1-L}{l}$. The proof of (v) applies here too. ■

Proof. [Proof of Theorem 1] The equilibrium threshold y_* is determined by the condition $V(y_*; y_*) = 1$. Define $W(y_*) \equiv V(y_*; y_*)$. Here we prove that there always exists a unique y_* such that $W(y_*) = 1$. To simplify notation, we replace y_* by y and express $W(y_*)$ as $W(y)$ throughout the proof. It is easy to show that under the parameter restriction (9), $W_C(0) < W_B(0) < 1$, $W_A(\infty) > 1$, and $W_E(\infty) > 1$.

Denote by $y_{**} = \max\{y : R(y; y_*) = \overline{r}\}$ the maximum fundamental value that is associated with the max rate. That is, the constraint of the max rate is binding if and only if $y \leq y_{**}$. It is straightforward to see that in Case B or Case F, y_{**} coincides with y_* (i.e., $y_{**} = y_*$), and in Case C, $y_{**}^C \equiv \frac{\overline{\rho}+\phi-\overline{r}}{\phi} \leq 1$. For the other cases, y_{**} is determined by $f(y_{**}) = 0$ where the function $f(\cdot)$ is defined as

$$f(y) = \overline{\rho} + \phi(1 - y)^+ + \theta\delta(1 - L - ly)^+ + \kappa\delta(1 - U(y)) - \overline{r}.$$

Then from Lemma 1, $f(y)$ is continuous and strictly decreasing. Furthermore, under the parameter restrictions (8) and (9), we have $f(0) > 0$ and $f\left(\frac{1-L}{l}\right) \leq 0.$, implying that $f(y_{**}) = 0$ has a unique solution $y_{**} \in (0, \frac{1-L}{l}]$. It is straightforward to check that $W_B\left(y_{**}^C\right) = W_C\left(y_{**}^C\right)$, $W_A(y_{**}) = W_B(y_{**})$, $W_F(y_{**}) = W_E(y_{**})$, and $W_B(1) = W_F(1)$.

We now prove the existence of the unique threshold y_* by considering all the possible max rates $\overline{r}$. Under the restriction (6), $\overline{\rho} + \phi < \overline{\rho} + \theta\delta(1 - L - l) + \kappa\delta(1 - U(1))$. There are three possibilities.

(i) Consider the possibility where $\overline{r} \geq \overline{\rho} + \theta\delta(1 - L - l) + \kappa\delta(1 - U(1))$, implying $f(1) \leq 0$ and $y_{**} \in (0, 1]$. Based on the strict monotonicity of W_A and W_B, as well as $y_{**} \leq 1$, we have

$$W_B(0) < W_A(y_{**}) = W_B(y_{**}) \leq W_A(1) < W_A(\infty).$$

If $W_A(1) < 1$, then Case D or Case G holds (note $W_A(\infty) > 1$) where $W_A(y) = 1$ has a unique root $y > 1$, depending on whether $W_A\left(\frac{1-L}{l}\right) \geq 1$ or not. Otherwise, if $W_A(1) \geq 1$, depending on whether $W_A(y_{**}) = W_B(y_{**}) < 1$ or not, either Case A holds where $W_A(y) = 1$ has a unique root $y \in (y_{**}, 1]$, or Case B holds where $W_B(y) = 1$ has a unique root $y \in (0, y_{**}]$.

(ii) Consider the possibility where $\overline{\rho} + \phi \leq \overline{r} < \overline{\rho} + \theta\delta(1 - L - l) + \kappa\delta(1 - U(1))$, implying

$f(1) > 0$ and $y_{**} \in (1, \frac{1-L}{l}]$. Based on the strict monotonicity of W_B, W_E, and W_F, and $y_{**} > 1$, we know

$$W_B(1) = W_F(1) < W_F(y_{**}) = W_E(y_{**}).$$

If $W_E(y_{**}) < 1$, then Case E or Case H holds (note $W_E(\infty) > 1$) where $W_E(y) = 1$ has a unique root $y > y_{**}$, depending on whether $W_E\left(\frac{1-L}{l}\right) \geq 1$ or not. Otherwise, if $W_E(y_{**}) = W_F(y_{**}) \geq 1$, depending on whether $W_B(1) = W_F(1) < 1$ or not, either Case F holds where $W_F(y) = 1$ has a unique root $y \in (1, y_{**}]$, or Case B holds where $W_B(y) = 1$ has a unique root $y \in (0, 1]$.

(iii) Consider the possibility where $\overline{r} < \overline{\rho} + \phi$, implying $0 < y_{**}^C \leq 1$ and $y_{**} \in (1, \frac{1-L}{l}]$. Based on the strict monotonicity of W_B and W_F, as well as $y_{**} > 1$, we have

$$W_C(0) < W_B\left(y_{**}^C\right) = W_C\left(y_{**}^C\right) < W_B(1) = W_F(1) < W_F(y_{**}) = W_E(y_{**}).$$

If $W_C\left(y_{**}^C\right) \geq 1$, then Case C holds (note $W_C(0) < 1$) where $W_C(y) = 1$ has a unique solution $y \in (0, y_{**}^C]$. Otherwise, if $W_C\left(y_{**}^C\right) < 1$, by the same argument used in Possibility (ii), we can prove that Case B holds if $W_B(1) \geq 1$, or Case E or Case H holds if $W_B(1) < 1$ and $W_E(y_{**}) < 1$, or Case F holds if $W_B(1) < 1$ and $W_E(y_{**}) \geq 1$. ■

Proof. [Proof of Proposition 1] (i) We first prove $\frac{dy_*}{d\overline{r}} < 0$. By the implicit function theorem, $\frac{dy_*}{d\overline{r}} = -\frac{\partial W/\partial \overline{r}}{\partial W/\partial y_*}$. We have shown in Lemma 3 that $\partial W/\partial y > 0$. Therefore, we only need to show that $\partial W/\partial \overline{r} > 0$ for each of functions $W_A(y), \cdots, W_H(y)$ in order to prove the claim. From Lemma 2, we have

$$\begin{aligned}
\frac{\partial W_A(y)}{\partial \overline{r}} &= \frac{y^{-\gamma_1}}{(\eta_1 + \gamma_2)} \frac{\partial\left[\eta_1 (K_1 - K_7)\left(y_{**}^A\right)^{\gamma_1} + (\eta_1 - 1) K_2 \left(y_{**}^A\right)^{\gamma_1+1}\right]}{\partial \overline{r}} \\
&= \frac{(\gamma_1 + 1) y^{-\gamma_1}\left(y_{**}^A\right)^{\gamma_1}}{(\eta_1 + \gamma_2)}\left[\frac{\eta_1}{\rho + \phi + \delta(1+\theta)} - \frac{\eta_1 - 1}{\rho + \phi + \delta(1+\theta) - \mu}\right] \\
&> 0.
\end{aligned}$$

When $\overline{r} < \overline{\rho} + \phi$, for $y < y_{**}^C$, from Lemma 2, we have

$$\begin{aligned}
\frac{\partial W_C(y)}{\partial \overline{r}} &= \frac{\partial \left[\frac{\eta_1 K_1 + \gamma_2 K_5}{\eta_1+\gamma_2} + \frac{\gamma_2(K_8-K_5)-(\gamma_2+1)K_6 y_{**}^C}{(\eta_1+\gamma_2)(y_{**}^C)^{\eta_2}} y^{\eta_2}\right]}{\partial \overline{r}} \\
&= \frac{1}{\eta_1+\gamma_2}\left[\frac{\eta_1}{\rho+\phi+\delta(1+\theta)} + \frac{\gamma_2}{\rho+\phi}\right] + \frac{(\eta_2-1)\, y^{\eta_2}\left(y_{**}^C\right)^{-\eta_2}}{\eta_1+\gamma_2}\left[\frac{\gamma_2}{\rho+\phi} - \frac{\gamma_2+1}{\rho+\phi-\mu}\right] \\
&> \frac{1}{\eta_1+\gamma_2}\left[\frac{\eta_1}{\rho+\phi+\delta(1+\theta)} + \frac{\gamma_2}{\rho+\phi}\right] + \frac{\eta_2-1}{\eta_1+\gamma_2}\left[\frac{\gamma_2}{\rho+\phi} - \frac{\gamma_2+1}{\rho+\phi-\mu}\right] \\
&= \frac{1}{\eta_1+\gamma_2}\frac{\eta_1}{\rho+\phi+\delta(1+\theta)} > 0.
\end{aligned}$$

When $\overline{r} < \overline{\rho} + \theta\delta(1-L-l)$, from Lemma 2, we have

$$\begin{aligned}
\frac{\partial W_E(y)}{\partial \overline{r}} &= \frac{y^{-\gamma_1}}{\eta_1+\gamma_2}\frac{\partial\left[\eta_1(K_3-K_7)\left(y_{**}^E\right)^{\gamma_1} + (\eta_1-1)K_4\left(y_{**}^E\right)^{1+\gamma_1}\right]}{\partial \overline{r}} \\
&= \frac{(\gamma_1+1)\, y^{-\gamma_1}\left(y_{**}^E\right)^{\gamma_1}}{\eta_1+\gamma_2}\left[\frac{\eta_1}{\rho+\phi+\delta(1+\theta)} - \frac{(\eta_1-1)}{\rho+\phi+\delta(1+\theta)-\mu}\right] > 0.
\end{aligned}$$

Lastly, $\frac{\partial W_B(y)}{\partial \overline{r}} = \frac{\partial W_F(y)}{\partial \overline{r}} = \frac{\eta_1}{\eta_1+\gamma_2}\frac{1}{\rho+\phi+\delta(1+\theta)} > 0$.

(ii) Next, we prove $\frac{dy_*}{d\Delta} < 0$. By the similar argument as in (i), we only need to show that $\partial W/\partial\Delta > 0$ for each of functions $W_A(y), \cdots, W_H(y)$. From Lemma 2, we have

$$\begin{aligned}
\frac{\partial W_A(y)}{\partial \Delta} &= \left[\begin{array}{c} \frac{\eta_1}{\eta_1+\gamma_2}\frac{1}{\rho+\phi+\delta(1+\theta)} + \frac{\gamma_2}{\eta_1+\gamma_2}\frac{1}{\rho+\phi} \\ -\frac{(\gamma_1+1)y^{-\gamma_1}\left(y_{**}^A\right)^{\gamma_1}}{(\eta_1+\gamma_2)}\left(\frac{\eta_1}{\rho+\phi+\delta(1+\theta)} - \frac{\eta_1-1}{\rho+\phi+\delta(1+\theta)-\mu}\right)\end{array}\right] \\
&> \left[\begin{array}{c} \frac{\eta_1}{\eta_1+\gamma_2}\frac{1}{\rho+\phi+\delta(1+\theta)} + \frac{\gamma_2}{\eta_1+\gamma_2}\frac{1}{\rho+\phi} \\ -\frac{\gamma_1+1}{\eta_1+\gamma_2}\left(\frac{\eta_1}{\rho+\phi+\delta(1+\theta)} - \frac{\eta_1-1}{\rho+\phi+\delta(1+\theta)-\mu}\right)\end{array}\right] = \frac{\gamma_2}{\eta_1+\gamma_2}\frac{1}{\rho+\phi} > 0;
\end{aligned}$$

and when $\overline{r} < \overline{\rho} + \phi$, for $y < y_{**}^C$, $\frac{\partial W_C(y)}{\partial \Delta} = \frac{(\eta_2-1)y^{\eta_2}\left(y_{**}^C\right)^{-\eta_2}}{\eta_1+\gamma_2}\left(\frac{\gamma_2+1}{\rho+\phi-\mu} - \frac{\gamma_2}{\rho+\phi}\right) > 0$; and furthermore, when $\overline{r} < \overline{\rho} + \theta\delta(1-L-l)$, we have

$$\begin{aligned}
\frac{\partial W_E(y)}{\partial \Delta} &= \left[\begin{array}{c} \frac{\eta_1}{\eta_1+\gamma_2}\frac{1}{\rho+\phi+\delta(1+\theta)} + \frac{\gamma_2}{\eta_1+\gamma_2}\frac{1}{\rho+\phi} \\ -\frac{(\gamma_1+1)y^{-\gamma_1}\left(y_{**}^E\right)^{\gamma_1}}{\eta_1+\gamma_2}\left(\frac{\eta_1}{\rho+\phi+\delta(1+\theta)} - \frac{(\eta_1-1)}{\rho+\phi+\delta(1+\theta)-\mu}\right)\end{array}\right] \\
&> \left[\begin{array}{c} \frac{\eta_1}{\eta_1+\gamma_2}\frac{1}{\rho+\phi+\delta(1+\theta)} + \frac{\gamma_2}{\eta_1+\gamma_2}\frac{1}{\rho+\phi} \\ -\frac{\gamma_1+1}{\eta_1+\gamma_2}\left(\frac{\eta_1}{\rho+\phi+\delta(1+\theta)} - \frac{(\eta_1-1)}{\rho+\phi+\delta(1+\theta)-\mu}\right)\end{array}\right] = \frac{\gamma_2}{\eta_1+\gamma_2}\frac{1}{\rho+\phi} > 0.
\end{aligned}$$

Lastly, $\frac{\partial W_B(y)}{\partial \Delta} = \frac{\partial W_F(y)}{\partial \Delta} = \frac{\gamma_2}{\eta_1+\gamma_2}\frac{1}{\rho+\phi} > 0$.

(iii) Lastly, we prove by contradiction that $\lim_{\Delta\to 0} y_* = \infty$. Suppose $lim_{\overline{\rho}\searrow\rho} y_* = y_0 < \infty$. Consider two cases: $\overline{r} > \rho + \theta\delta\left(1 - L - l\right)$ and $\overline{r} \leq \rho + \theta\delta\left(1 - L - l\right)$.

(iii-A) If $\overline{r} > \rho + \theta\delta\left(1 - L - l\right)$, then $lim_{\overline{\rho}\searrow\rho} W_A\left(y_*\right) = 1 = lim_{\overline{\rho}\searrow\rho} W_A\left(y_0\right)$. However, note that when $\overline{\rho}$ tends to ρ, K_7 and K_8 tends to 1. Therefore, we have

$$
\begin{aligned}
& lim_{\overline{\rho}\searrow\rho} W_A\left(y_0\right) \\
= \; & 1 + lim_{\overline{\rho}\searrow\rho} \frac{\eta_1\left(K_7 - 1\right) + \gamma_2\left(K_8 - 1\right)}{\eta_1 + \gamma_2} + \frac{\eta_1\left(K_1 - K_7\right) + \left(\eta_1 - 1\right) K_2 y_{**}^A}{\left(\eta_1 + \gamma_2\right)\left(y_{**}^A\right)^{-\gamma_1}} y_0^{-\gamma_1} \\
= \; & 1 + \frac{\eta_1\left(K_1 - K_7\right) + \left(\eta_1 - 1\right) K_2 y_{**}^A}{\left(\eta_1 + \gamma_2\right)\left(y_{**}^A\right)^{-\gamma_1}} y_0^{-\gamma_1} < 1,
\end{aligned}
$$

which is a contradiction.

(iii-B) If $\overline{r} \leq \rho + \theta\delta\left(1 - L - l\right)$, then $lim_{\overline{\rho}\searrow\rho} W_B\left(1\right) < 1$ because

$$
\begin{aligned}
& W_B\left(1\right) - 1 = \frac{\eta_1\left(K_1 - 1\right) + \gamma_2\left(K_8 - 1\right) + \left(\eta_1 - 1\right) K_2}{\eta_1 + \gamma_2} \\
\to \; & \frac{\eta_1}{\eta_1 + \gamma_2}\left(\frac{\overline{r} + \delta\left(1 + \theta L\right)}{\rho + \phi + \delta\left(1 + \theta\right)} - 1\right) + \frac{\eta_1 - 1}{\eta_1 + \gamma_2}\frac{\phi + \theta\delta l}{\rho + \phi + \delta\left(1 + \theta\right) - \mu} \\
= \; & -\frac{\eta_1}{\eta_1 + \gamma_2}\left(\frac{\left(\rho + \theta\delta\left(1 - L - l\right)\right) - \overline{r}}{\rho + \phi + \delta\left(1 + \theta\right)}\right) \\
& -\frac{\phi + \theta\delta l}{\eta_1 + \gamma_2}\left(\frac{\eta_1}{\rho + \phi + \delta\left(1 + \theta\right)} - \frac{\eta_1 - 1}{\rho + \phi + \delta\left(1 + \theta\right) - \mu}\right) \\
< \; & 0.
\end{aligned}
$$

Similarly as before, we can prove that in this case, $lim_{\overline{\rho}\searrow\rho} y_* = \infty$. We can prove it by contradiction. Suppose $lim_{\overline{\rho}\searrow\rho} y_* = y_0 < \infty$. Then $lim_{\overline{\rho}\searrow\rho} W_E\left(y_*\right) = 1 = lim_{\overline{\rho}\searrow\rho} W_E\left(y_0\right)$. However,

$$
lim_{\overline{\rho}\searrow\rho} W_E\left(y_0\right) = 1 + \left[\frac{\eta_1 + \gamma_1}{\eta_1 + \gamma_2} E_2 + \frac{\eta_1\left(K_3 - K_7\right) + \left(\eta_1 - 1\right) K_4 y_{**}^E}{\left(\eta_1 + \gamma_2\right)\left(y_{**}^E\right)^{-\gamma_1}}\right] y_0^{-\gamma_1} < 1,
$$

which is a contradiction. ■

Proof. [Proof of Proposition 2] There are only three possibilities: $y_* < 1$, $1 \leq y_* \leq \frac{1-L}{l}$, and $y_* > \frac{1-L}{l}$:

1. If $y_* \leq 1$, then the value function is given by

$$
V\left(y; y_*\right) = \begin{cases} \left(K_1 + \overline{K}_6\right) + \left(K_2 + \overline{K}_7\right) y + U_1 y^{\eta_3} + A_1 y^{\eta_1}, & \text{if } y \in \left(0, y_*\right] \\ K_5 + K_6 y + A_2 y^{-\gamma_2} + A_3 y^{\eta_2}, & \text{if } y \in \left(y_*, 1\right] \\ K_{10} + A_4 y^{-\gamma_2}, & \text{if } y \in \left(1, \infty\right) \end{cases}
$$

2. If $1 < y_* \leq \frac{1-L}{l}$, then the value function is given by

$$V(y;y_*) = \begin{cases} \left(K_1+\overline{K}_6\right)+\left(K_2+\overline{K}_7\right)y+U_1y^{\eta_3}+B_1y^{\eta_1}, & \text{if } y\in(0,1] \\ \left[\begin{array}{c} \left(K_3+\overline{K}_8\right)+\left(K_4+\overline{K}_9\right)y+U_2y^{-\gamma_3} \\ +U_3y^{\eta_3}+B_2y^{-\gamma_1}+B_3y^{\eta_1} \end{array}\right], & \text{if } y\in(1,y_*] \\ K_{10}+B_4y^{-\gamma_2}, & \text{if } y\in(y_*,\infty) \end{cases}$$

3. If $y_* > \frac{1-L}{l}$, then the value function is given by

$$V(y;y_*) = \begin{cases} \left(K_1+\overline{K}_6\right)+\left(K_2+\overline{K}_7\right)y+U_1y^{\eta_3}+C_1y^{\eta_1}, & \text{if } y\in(0,1] \\ \left[\begin{array}{c} \left(K_3+\overline{K}_8\right)+\left(K_4+\overline{K}_9\right)y+U_2y^{-\gamma_3} \\ +U_3y^{\eta_3}+C_2y^{-\gamma_1}+C_3y^{\eta_1} \end{array}\right], & \text{if } y\in(1,\frac{1-L}{l}] \\ \left(K_9+\overline{K}_{10}\right)+U_4y^{-\gamma_3}+C_4y^{-\gamma_1}+C_5y^{\eta_1}, & \text{if } y\in(\frac{1-L}{l},y_*] \\ K_{10}+C_6y^{-\gamma_2}, & \text{if } y\in(y_*,\infty) \end{cases}$$

where the unknown coefficients $A_1,\cdots,C_6$ are determined through the value matching and smooth pasting conditions.

Similarly as the proof of Proposition 1, to prove $\frac{dy_*}{d\kappa}>0$, we only need to prove $\frac{\partial W(y)}{\partial\kappa}<0$ for Case A, Case B, and Case C.

For simplicity, we only provide the proof for Case A: $\frac{\partial W_A(y)}{\partial\kappa}<0$ for $0<y\leq 1$. The proof for the other two cases is similar. In Case A,

$$\begin{aligned} W_A(y) &= \left[\begin{array}{c} \frac{\eta_1\left(K_1+\overline{K}_6\right)+\gamma_2K_5}{\eta_1+\gamma_2}+\frac{(\eta_1-1)\left(K_2+\overline{K}_7\right)+(1+\gamma_2)K_6}{\eta_1+\gamma_2}y \\ +\frac{\gamma_2(K_{10}-K_5)-(1+\gamma_2)K_6}{\eta_1+\gamma_2}y^{\eta_2}+\frac{\eta_1-\eta_3}{\eta_1+\gamma_2}U_1y^{\eta_3} \end{array}\right] \\ &= I_A+II_Ay+III_Ay^{\eta_2}+IV_Ay^{\eta_3} \end{aligned}$$

Note that $\frac{\partial IV_A}{\partial\kappa}\Big|_{\kappa=0}=\frac{U_1}{\eta_1+\gamma_2}\frac{\partial\eta_1}{\partial\kappa}\Big|_{\kappa=0}<0$. Below we consider the first three terms. First, it is straightforward to show that

$$\begin{aligned} \frac{\partial\left(K_1+\overline{K}_6\right)}{\partial\kappa} &= \frac{\partial\left(K_3+\overline{K}_8\right)}{\partial\kappa}=-\frac{\delta^2(1-L)}{\left(\rho+\phi+\delta(1+\theta+\kappa)\right)^2}, \\ \frac{\partial\left(K_2+\overline{K}_7\right)}{\partial\kappa} &= \frac{\partial\left(K_4+\overline{K}_9\right)}{\partial\kappa}=\frac{\delta^2 l}{\left(\rho+\phi+\delta(1+\theta+\kappa)-\mu\right)^2}, \\ \frac{\partial\left(K_9+\overline{K}_{10}\right)}{\partial\kappa} &= 0. \end{aligned}$$

Therefore,

$$\left.\frac{\partial W_A(y)}{\partial\kappa}\right|_{\kappa=0} < \left[\begin{array}{c}\left(\frac{\gamma_2(K_1+K_2y)}{(\eta_1+\gamma_2)^2}-\frac{\gamma_2(K_{10}-K_5)-(1+\gamma_2)K_6}{(\eta_1+\gamma_2)^2}y^{\eta_2}\right)\frac{\partial\eta_1}{\partial\kappa}\\ +\frac{\eta_1}{\eta_1+\gamma_2}\left(-\frac{\delta^2(1-L)}{(\rho+\phi+\delta(1+\theta))^2}+\frac{\delta^2 l}{(\rho+\phi+\delta(1+\theta)-\mu)^2}y\right)\end{array}\right].$$

Because $\gamma_2(K_{10}-K_5)-(1+\gamma_2)K_6=\phi\left(\frac{\gamma_2}{\rho+\phi}-\frac{1+\gamma_2}{\rho+\phi-\mu}\right)<0$ and $y\leq 1$, we have

$$\left.\frac{\partial W_A(y)}{\partial\kappa}\right|_{\kappa=0} \leq \left[\begin{array}{c}\frac{\gamma_2(K_1+K_2)-(\gamma_2(K_{10}-K_5)-(1+\gamma_2)K_6)}{(\eta_1+\gamma_2)^2}\frac{\partial\eta_1}{\partial\kappa}\\ +\frac{\eta_1}{\eta_1+\gamma_2}\left(\frac{\delta^2 l}{(\rho+\phi+\delta(1+\theta)-\mu)^2}-\frac{\delta^2(1-L)}{(\rho+\phi+\delta(1+\theta))^2}\right)\end{array}\right]$$

Therefore, for $\left.\frac{\partial W_A(y)}{\partial\kappa}\right|_{\kappa=0}<0$, it is sufficient to have

$$K_1+K_2 < \left[\begin{array}{c}\frac{1}{\gamma_2}\left(\gamma_2(K_{10}-K_5)-(1+\gamma_2)K_6\right)\\ +\frac{\eta_1(\eta_1+\gamma_2)\delta^2}{\gamma_2}\left(\frac{1-L}{(\rho+\phi+\delta(1+\theta))^2}-\frac{l}{(\rho+\phi+\delta(1+\theta)-\mu)^2}\right)\end{array}\right]$$

which imposes an upper bound on $\overline{r}$.

In Case B,

$$\begin{aligned} W_B(y) &= \left[\begin{array}{c}\frac{\eta_1\left(K_3+\overline{K}_8\right)+\gamma_2K_{10}}{\eta_1+\gamma_2}+\frac{(\eta_1-1)\left(K_4+\overline{K}_9\right)}{\eta_1+\gamma_2}y+\frac{\eta_1\left(\overline{K}_3-\overline{K}_1\right)+(\eta_1-1)\left(\overline{K}_4-\overline{K}_2\right)}{\eta_1+\gamma_2}y^{-\gamma_1}\\ +\frac{(\eta_1+\gamma_3)U_2}{\eta_1+\gamma_2}y^{-\gamma_3}+\frac{(\eta_1-\eta_3)U_3}{\eta_1+\gamma_2}y^{\eta_3}\end{array}\right]\\ &= I_B+II_By+III_By^{-\gamma_1}+IV_By^{-\gamma_3}+V_By^{\eta_3}\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial W_B}{\partial\kappa}\right|_{\kappa=0} &= \left[\begin{array}{c}\frac{\partial\frac{\eta_1}{\eta_1+\gamma_2}}{\partial\kappa}\left(K_3+K_4y+\left(\overline{K}_3-\overline{K}_1+\overline{K}_4-\overline{K}_2\right)y^{-\gamma_1}\right)\\ +\frac{\eta_1}{\eta_1+\gamma_2}\frac{\partial\left(K_3+\overline{K}_8\right)}{\partial\kappa}+\frac{\eta_1-1}{\eta_1+\gamma_2}\frac{\partial\left(K_4+\overline{K}_9\right)}{\partial\kappa}y\\ -\frac{\eta_1\left(\overline{K}_3-\overline{K}_1\right)+(\eta_1-1)\left(\overline{K}_4-\overline{K}_2\right)}{\eta_1+\gamma_2}\log(y)\frac{\partial\gamma_1}{\partial\kappa}y^{-\gamma_1}\end{array}\right]\\ &= \left[\begin{array}{c}\frac{\gamma_2}{(\eta_1+\gamma_2)^2}\frac{\partial\eta_1}{\partial\kappa}\left(K_3+K_4y+\left(\overline{K}_3-\overline{K}_1+\overline{K}_4-\overline{K}_2\right)y^{-\gamma_1}\right)\\ -\delta^2\frac{\eta_1}{\eta_1+\gamma_2}\frac{1-L}{(\rho+\phi+\delta(1+\theta+\kappa))^2}+\delta^2\frac{\eta_1-1}{\eta_1+\gamma_2}\frac{l}{(\rho+\phi+\delta(1+\theta+\kappa)-\mu)^2}y\\ -\frac{\eta_1\left(\overline{K}_3-\overline{K}_1\right)+(\eta_1-1)\left(\overline{K}_4-\overline{K}_2\right)}{\eta_1+\gamma_2}\log(y)\,y^{-\gamma_1}\frac{\partial\eta_1}{\partial\kappa}\end{array}\right]\\ &< \frac{\gamma_2}{(\eta_1+\gamma_2)^2}\frac{\partial\eta_1}{\partial\kappa}\left(K_3+K_4\frac{1-L}{l}+\left(\overline{K}_3-\overline{K}_1+\overline{K}_4-\overline{K}_2\right)\left(\frac{1-L}{l}\right)^{-\gamma_1}\right)\\ &\quad -\delta^2\frac{\eta_1}{\eta_1+\gamma_2}\frac{1-L}{(\rho+\phi+\delta(1+\theta+\kappa))^2}+\delta^2\frac{\eta_1-1}{\eta_1+\gamma_2}\frac{1-L}{(\rho+\phi+\delta(1+\theta+\kappa)-\mu)^2}\end{aligned}$$

where we used the fact $1 < y \leq \frac{1-L}{l}$,

$$\begin{aligned} \overline{K}_3 - \overline{K}_1 + \overline{K}_4 - \overline{K}_2 &= \frac{\phi}{\rho + \phi + \overline{\theta}\delta} - \frac{\phi}{\rho + \phi + \overline{\theta}\delta - \mu} < 0 \\ \eta_1 \left(\overline{K}_3 - \overline{K}_1 \right) + (\eta_1 - 1) \left(\overline{K}_4 - \overline{K}_2 \right) &= \phi \left(\frac{\eta_1}{\rho + \phi + \overline{\theta}\delta} - \frac{\eta_1 - 1}{\rho + \phi + \overline{\theta}\delta - \mu} \right) > 0 \end{aligned}$$

Therefore, for $\left. \frac{\partial W_B(y)}{\partial \kappa} \right|_{\kappa=0} < 0$, it is sufficient to have $\frac{\eta_1}{(\rho+\phi+\delta(1+\theta))^2} > \frac{\eta_1 - 1}{(\rho+\phi+\delta(1+\theta)-\mu)^2}$ and

$$K_3 + K_4 \frac{1-L}{l} < \left[\begin{array}{c} \left(\frac{\phi}{\rho+\phi+\overline{\theta}\delta-\mu} - \frac{\phi}{\rho+\phi+\overline{\theta}\delta} \right) \left(\frac{1-L}{l} \right)^{-\gamma_1} \\ + \frac{\delta^2 (1-L)}{\frac{\partial \eta_1}{\partial \kappa}} \frac{\eta_1 + \gamma_2}{\gamma_2} \left(\frac{\eta_1}{(\rho+\phi+\delta(1+\theta))^2} - \frac{\eta_1 - 1}{(\rho+\phi+\delta(1+\theta)-\mu)^2} \right) \end{array} \right]$$

which imposes an upper bound on μ and $\overline{r}$.

In Case C,

$$\begin{aligned} W_C(y) &= \frac{\eta_1 \left(K_9 + \overline{K}_{10} \right) + \gamma_2 K_{10}}{(\eta_1 + \gamma_2)} \\ &\quad + \left[- \frac{\eta_1 \left(K_3 + \overline{K}_8 - K_1 - \overline{K}_6 \right) + (\eta_1 - 1) \left(K_4 + \overline{K}_9 - K_2 - \overline{K}_7 \right)}{\eta_1 + \gamma_2} \right] y^{-\gamma_1} \\ &\quad + \frac{\eta_1 \left(K_3 + \overline{K}_8 - K_9 - \overline{K}_{10} \right) + (\eta_1 - 1) \left(K_4 + \overline{K}_9 \right) \left(\frac{1-L}{l} \right)}{(\eta_1 + \gamma_2)} \left(\frac{y}{(1-L)/l} \right)^{-\gamma_1} \\ &\quad + \frac{(\eta_1 + \gamma_3) U_4}{(\eta_1 + \gamma_2)} y^{-\gamma_3} - \frac{(\eta_1 + \gamma_3) U_2 + (\eta_1 - \eta_3)(U_3 - U_1)}{\eta_1 + \gamma_2} y^{-\gamma_1} \\ &\quad + \frac{(\eta_1 + \gamma_3)(U_2 - U_4) \left(\frac{1-L}{l} \right)^{-\gamma_3} + (\eta_1 - \eta_3) U_3 \left(\frac{1-L}{l} \right)^{\eta_3}}{(\eta_1 + \gamma_2) \left(\frac{1-L}{l} \right)^{-\gamma_1}} y^{-\gamma_1} \\ &= I_C + II_C y^{-\gamma_1} + III_C \left(\frac{y}{(1-L)/l} \right)^{-\gamma_1} + IV_C \end{aligned}$$

where we used the facts

$$\frac{\partial \left(K_4 + \overline{K}_9 - K_2 - \overline{K}_7 \right)}{\partial \kappa} = \frac{\partial \left(K_3 + \overline{K}_8 - K_1 - \overline{K}_6 \right)}{\partial \kappa} = \frac{\partial \left(K_9 + \overline{K}_{10} \right)}{\partial \kappa} = 0$$

$$\begin{aligned} \eta_1 (K_3 - K_1) + (\eta_1 - 1)(K_4 - K_2) &= \phi \chi_1 > 0 \\ \eta_1 (K_3 - K_9) + (\eta_1 - 1) K_4 \left(\frac{1-L}{l} \right) &= -\theta\delta (1-L) \chi_1 < 0 \end{aligned}$$

$$
\begin{aligned}
K_1 - K_3 + K_2 - K_4 &= \frac{\phi}{\rho + \phi + \delta\,(1+\theta+\kappa) - \mu} - \frac{\phi}{\rho + \phi + \delta\,(1+\theta+\kappa)} > 0 \\
K_3 - K_9 + K_4\left(\frac{1-L}{l}\right) &= \frac{\theta\delta\,(1-L)}{\rho + \phi + \delta\,(1+\theta+\kappa) - \mu} - \frac{\theta\delta\,(1-L)}{\rho + \phi + \delta\,(1+\theta+\kappa)} > 0
\end{aligned}
$$

and

$$
\begin{aligned}
& \eta_1 \frac{\partial\left(K_3 + \overline{K}_8\right)}{\partial\kappa} + (\eta_1 - 1)\frac{\partial\left(K_4 + \overline{K}_9\right)}{\partial\kappa}\left(\frac{1-L}{l}\right) \\
= \; & -\delta^2\,(1-L)\left(\frac{\eta_1}{\left(\rho+\phi+\delta\,(1+\theta+\kappa)\right)^2} - \frac{\eta_1 - 1}{\left(\rho+\phi+\delta\,(1+\theta+\kappa)-\mu\right)^2}\right)
\end{aligned}
$$

thus

$$
\begin{aligned}
& \left.\frac{\partial W_C}{\partial\kappa}\right|_{\kappa=0} \\
< \; & \frac{\partial\frac{\eta_1}{\eta_1+\gamma_2}}{\partial\kappa}\left(K_9 + (K_1 - K_3 + K_2 - K_4)\left(\frac{1-L}{l}\right)^{-\gamma_1} + \left(K_3 - K_9 + K_4\frac{1-L}{l}\right)\right) \\
& -\delta^2\,(1-L)\,\frac{\frac{\eta_1}{\left(\rho+\phi+\delta(1+\theta+\kappa)\right)^2} - \frac{\eta_1-1}{\left(\rho+\phi+\delta(1+\theta+\kappa)-\mu\right)^2}}{(\eta_1+\gamma_2)}\left(\frac{y}{(1-L)\,/l}\right)^{-\gamma_1}
\end{aligned}
$$

Therefore, for $\left.\frac{\partial W_C(y)}{\partial\kappa}\right|_{\kappa=0} < 0$, it is sufficient to have $\frac{\eta_1}{\left(\rho+\phi+\delta(1+\theta+\kappa)\right)^2} > \frac{\eta_1-1}{\left(\rho+\phi+\delta(1+\theta+\kappa)-\mu\right)^2}$ and

$$
\begin{aligned}
& \frac{\gamma_2}{\eta_1+\gamma_2}\frac{\partial\eta_1}{\partial\kappa}\left[K_9 + (K_1 - K_3 + K_2 - K_4)\left(\frac{1-L}{l}\right)^{-\gamma_1} + K_3 - K_9 + K_4\frac{1-L}{l}\right] \\
< \; & \delta^2\frac{(1-L)^{1+\gamma_1}}{l^{\gamma_1}}\left[\frac{\eta_1}{\left(\rho+\phi+\delta\,(1+\theta+\kappa)\right)^2} - \frac{\eta_1-1}{\left(\rho+\phi+\delta\,(1+\theta+\kappa)-\mu\right)^2}\right] y^{-\gamma_1}
\end{aligned}
$$

■

Table 1: Run-Induced Externalities

Table 1 summarizes the payoffs of the current creditors and future creditors depending on the decision of the current creditors to run or roll over.

Choice of current creditors	Run			Rollover
Liquidity Provision	NO	YES		
		Failed	Survived	Survived
Probability	$\kappa\delta dt$	$\theta\delta dt$	$1-\kappa\delta dt-\theta\delta dt$	1
Payoff of current creditors	$U(y)$	$L(y)$	1	$V(y)$
Payoff of future creditors	$U(y)$	$L(y)$	$V(y)$	$V(y)$

Table 2: Calibration and Estimation Results

Panel A of this table reports calibrated values for some of the model's primitive parameters. In Panel B, the estimated values for the other primitive paramters are reported. Standard errors, reported in parentheses, are constructed from Monte Carlo simulations. Each simulation begins by randomly generating 119 weekly data of the underlying fundamental process. We then reestimate the key parameters with maximum likelihood using these data. We repeat this procedure 1000 times to construct the standard errors. In Panel C, we compute the equilibrium rollover thresholds based on the estimation results, including the equilibrium threshold in the benchmark model in HX where the interest rate is fixed at $\bar{r}$. In Panel D, the estimated value of a liquidity backstop is reported. These parameters are estimated using the quasi-maximum likelihood (QML) method. The data are the VRDO rate as well as weekly repo rate from May 1991 to October 2012, and the ARS rate from May 2006 and December 2009.

		VRDO	ARS
Panel A: Calibration			
max rate	$\bar{r}$	0.12	same
cash flow rate	r	0.0239	same
avg. maturity	$1/\phi$	25	same
tax-adj. riskless rate	ρ	0.0195	same
avg. duration	$1/\delta$	1/12	same
recovery rate	α	0.5	same
Panel B: Estimation			
drift	μ	0.024	same
	$(s.e.)$	(0.0107)	
volatility	σ	0.217	same
	$(s.e.)$	(0.0884)	
liquidity premium	Δ	0.0001	same
	$(s.e.)$	(0.0005)	
default intensity	θ	0.0111	same
	$(s.e.)$	(0.0047)	
auction failure intensity	κ	0	0.0030
	$(s.e.)$	$(n.a.)$	(0.0033)
pricing error	v	0.0107	same
	$(s.e.)$	(0.0063)	
Panel C: Equilibrium Rollover Threshold (y_*)			
Eqm. threshold	y_*	0.403	0.569
Eqm. threshold (HX)	y_*^{HX}	0.829	
Panel D: Value of a Liquidity Backstop (Γ)			
Value of a liq. backstop	Γ	0.0014	

Figure 1: Average VRDO and ARS Interest Rates

Figure 1 plots the average interest rates in percent on the indexes of weekly resettable high-grade ARS (solid line) and VRDO (dashed line) between May 2006 and December 2009, maintained by the Securities Industry and Financial Markets Association (SIFMA).

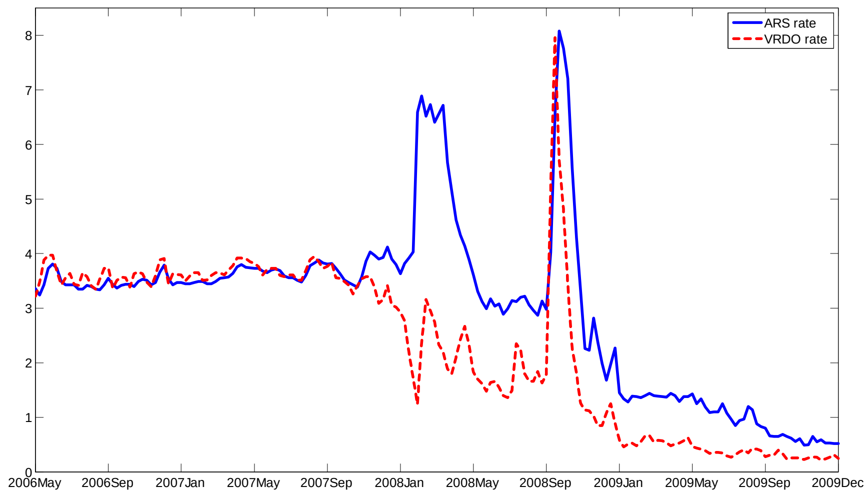

Figure 2: VRDO and ARS Annual Issuance Amounts (in Billion)

Figure 2 plots annual issuance amounts of VRDOs (green bars) and ARS (blue bars) in billions in the period between 1989 and 2011.

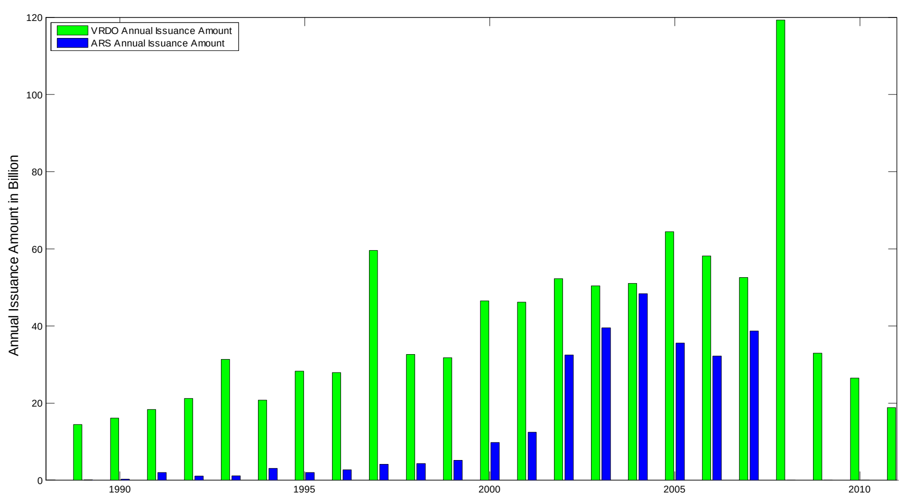

Figure 3: Models of VRDO and ARS

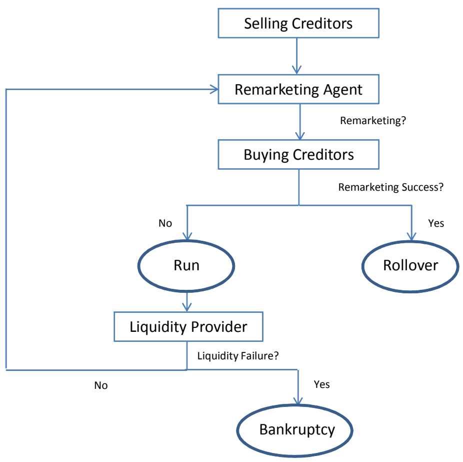

Figure 3A: VRDO

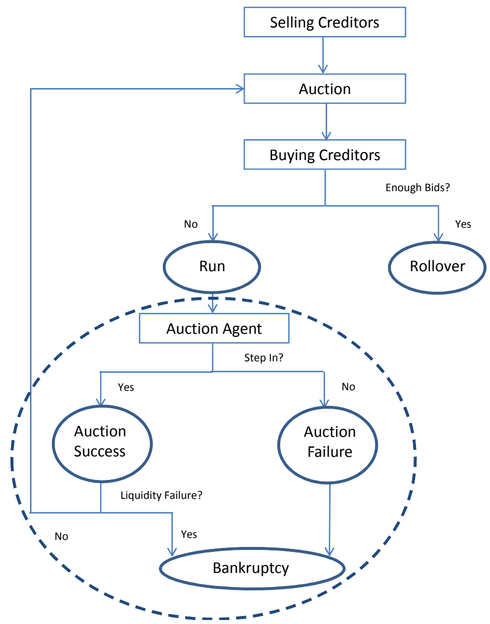

Figure 3B: ARS

Figure 4: Unconstrained Interest Rate Schedule

Figure 4 plots interest rate schedules in the absence of the maximum rate cap in three possible scenarios: $y_* \leq 1$, $1 < y_* \leq \frac{1-L}{l}$, and $y_* > \frac{1-L}{l}$.

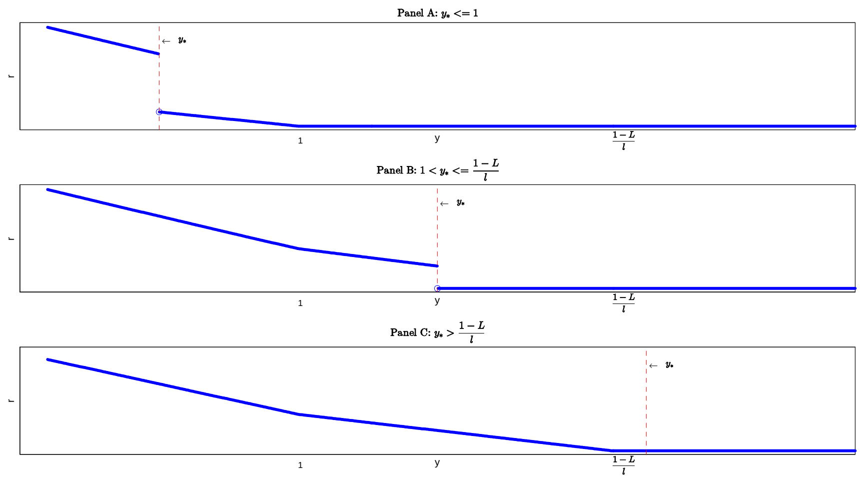

Figure 5: Constrained Interest Rate Schedule

Figure 5 plots interest rate schedules in the presence of the maximum rate cap in eight possible scenarios.

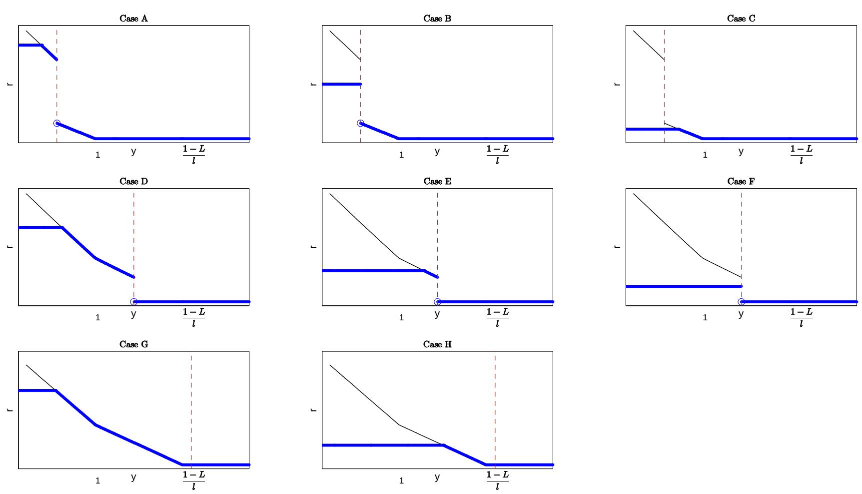

Figure 6: Structural Estimation Results

Panel A of Figure 6 plots the estimated fundamental process $\{y_t\}$ (blue solid line), the VRDO rollover threshold (blue dashed line), as well as the increased ARS rollover threshold following structural change in investors' beliefs (red dotted-dashed line). Panel B (or Panel C) of Figure 6 plots the actual and model-implied VRDO (or ARS) excess rates in blue solid and red dashed lines, respectively.

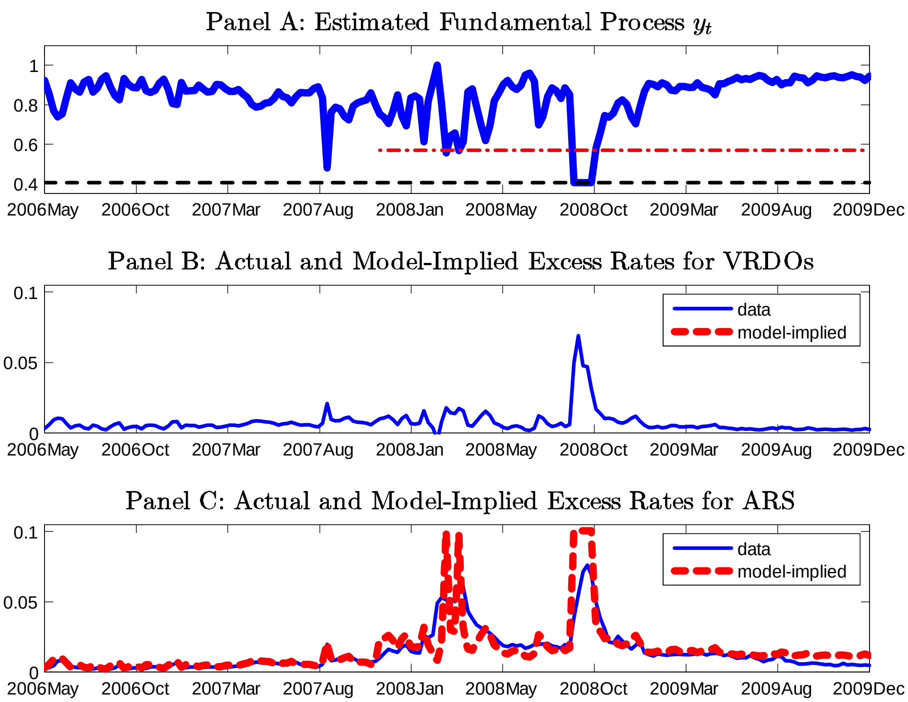

Figure 7: Model-Implied Run Probabilities

Figure 7 plots the model-implied run probabilities based on structural estimation results during the period between May 2006 and December 2009, for the VRDO market in Panel A and for the ARS market in Panel B.