FEDERAL RESERVE BANK *of* ATLANTA WORKING PAPER SERIES

Optimal Time-Consistent Taxation with Default

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Working Paper 2017-12
November 2017

Abstract: We study optimal time-consistent distortionary taxation when the repayment of government debt is not enforceable. The government taxes labor income or issues noncontingent debt in order to finance an exogenous stream of stochastic government expenditures. The government can repudiate its debt subject to some default costs, thereby introducing some state-contingency to debt. We are motivated by the fact that domestic sovereign default is an empirically relevant phenomenon, as Reinhart and Rogoff (2011) demonstrated. Optimal policy is characterized by two opposing incentives: an incentive to postpone taxes by issuing more debt for the future and an incentive to tax more currently in order to avoid punishing default premia. A generalized Euler equation (GEE) captures these two effects and determines the optimal back-loading or front-loading of tax distortions.

JEL classification: D52, E43, E62, H21, H63

Key words: labor tax, sovereign default, Markov-perfect equilibrium, time-consistency, generalized Euler equation, long-term debt

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The author thanks Manuel Amador, Marco Basseto, Alessandro Dovis, Karen Kopecky, Federico Mandelman, Fernando Martin, and Mark Wright for comments, Pablo D'Erasmo for his discussion, conference participants at the 2015 SED Meetings in Warsaw, the 2015 PET Meetings in Luxembourg and the 2015 Econometric Society World Congress in Montreal, and seminar participants at the Federal Reserve Bank of Chicago. The views expressed here are the author's and not necessarily those of the Federal Reserve Bank of Atlanta or the Federal Reserve System. Any remaining errors are the author's responsibility.

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1 Introduction

This paper studies the optimal time-consistent allocation of tax distortions and the optimal issuance of debt in an environment where government debt can be defaulted on. We consider a government that has to finance an exogenous stream of stochastic government expenditures and maximizes the utility of the representative household. The government can use distortionary labor taxes or issue non-contingent debt. The government can default on its debt subject to a default cost. Our setup is fully time-consistent; neither tax nor debt promises are honored.

Our analysis builds on the notion of Markov-perfect equilibrium (MPE) of Klein et al. (2008). Optimal policy is time-consistent in the payoff-relevant state variables, which for our case are government debt and government expenditures. Furthermore, we model default as in the work of Arellano (2008) and Aguiar and Gopinath (2006), that builds on the debt repudiation setup of Eaton and Gersovitz (1981). This setup allows to observe default in equilibrium.

In most of the sovereign default literature, government debt is assumed to be held only by foreigners whereas domestic households are hand-to-mouth consumers. However, Reinhart and Rogoff (2011) find that, on average, domestic debt accounts for nearly two-thirds of total public debt for a large group of countries. We consider a closed economy in which domestic households hold government debt. Thus, our model takes into account that default events often involve default on debt held by domestic households. This assumption is supported by the empirical literature. While domestic default events are more difficult to identify than external default events, Reinhart and Rogoff (2011) document 68 cases of overt default on domestic debt since 1800. Moreover, often even when default is only on external debt (which we do not model), a significant portion of the external debt is held by domestic investors (Sturzenegger and Zettelmeyer 2006). For these reasons it is of interest to understand the tradeoffs governments face when considering whether to default on domestic households.

Our purpose it to analyze optimal tax-smoothing and debt issuance in such an environment. The lack of state-contingent insurance markets hinders the ability of the government to smooth taxes. Default can in principle make debt partially state-contingent. In particular, the government affects both the pricing kernel of the agent and the payoff of government debt. Default risk is reflected in equilibrium prices and alters the optimal allocation of tax distortions over states and dates.

The government has an incentive to default when either government debt or government expenditures are high. By defaulting the government can avoid high distortionary taxation. However, default entails either direct costs in terms of output losses, or indirect

costs, in terms of a limited functioning of the market of government debt after a default event. In particular, we follow Arellano (2008) and assume that the market for government debt pauses to function for a random number of periods after a default event.

Optimal policy is characterized in our model by a generalized Euler equation (GEE) that balances the dynamic costs and benefits that the government is facing. The average welfare loss that is incurred by an increase in debt issuance (since higher debt has to be accompanied with higher future taxes) has to be balanced with the benefits of relaxing the government budget and allowing less taxes today. Our GEE reflects the fact that interest rates increase when debt increases, due to a higher probability of default. However, higher debt can also lead to reduction in interest rates by increasing marginal utility. The increase in marginal utility is coming from the fact that future consumption decreases in the event of repayment. This second channel is particularly important in our setup because it reflects the interest rate manipulation through the pricing kernel that is essential for our time-consistent setup. Our government chooses a debt and tax policy for the future, that will find optimal to follow in the following periods.

We consider also the optimal taxation, debt management and default plan when the government can also issue long-term debt as in Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012). Long-term debt has been shown to capture better the mean and volatility of spreads in the sovereign default literature. Furthermore, the volatility in prices due to the long duration of debt provides an additional tool for absorbing fiscal risks, which is of particular interest for our optimal taxation exercise. We show that a richer version of our GEE goes through. Reflecting the longer maturity of debt, the GEE encapsulates now the *present value* of the two opposing price effects that stem from the option to default and the lack of commitment to old policies.

Related literature. A short, though undoubtedly incomplete, list of references should include Aiyagari et al. (2002), the basic paper that analyzes optimal taxation in incomplete markets. They solve for optimal policy under commitment and without default. In the time-consistent literature, Krusell et al. (2004) and Debortoli and Nunes (2013) analyze time-consistent taxation and debt in deterministic setups without default. Martin (2009) analyzes the joint determination of time-consistent fiscal and monetary policy.

The closest paper to ours is Pouzo and Presno (2014), which has been the first to consider the possibility of debt repudiation à la Arellano in the optimal taxation problem. These authors alter the Aiyagari et al. (2002) setup only in one dimension; they allow the government to default but they retain a notion of commitment.[1] In their setup, the government

[1] Pouzo and Presno (2014) consider also the possibility of secondary markets in the event of default, a feature we do not share.

cannot commit to repay debt, but as long as the government decides to repay, it honors the marginal utility promises of the plan devised in previous periods and commits therefore to the tax sequence and the evolution of interest rates. In contrast, we treat debt and taxes symmetrically and derive the fully time-consistent policy in terms of the payoff -relevant endogenous state variable which is debt. Another paper of interest is D'Erasmo and Mendoza (2016), who study optimal domestic sovereign default in an quantitative model with heterogenous agents.

2 A two-period economy

To make things concrete, we will start our analysis with a two-period version of our model, $t = 0, 1$, and proceed in a later section with an infinite horizon economy.

The only uncertainty in the economy is coming from exogenous government expenditure shocks $g \in G$ at $t = 1$ with probability $\pi(g)$. There is a representative household that consumes c, works h, pays linear taxes τ on its labor income and trades in government debt. Government debt b is non-state contingent and trades at price q. At $t = 1$ the government can default on its promise to repay subject to an output loss. If the government defaults, it runs a balanced budget. Thus, government debt is a security that provides one unit of consumption next period at each state g for which the government is not defaulting.

Notation. Let D denote the set of shocks g at $t = 1$ for which the government is defaulting. Let $A \equiv D^c$ denote the repayment set. We will not specify yet what these sets depend on, since the representative household is a a price-taker. Let $d(g)$ be an indicator variable that takes the value 1 if the government defaults and zero otherwise, so $d(g) = 1, g \in D$ and $d(g) = 0$ if $g \in A$.

Resource constraints Output is produced by labor. The resource constraint at $t = 0$ reads

$$c_0 + g_0 = h_0 \tag{1}$$

At $t = 1$ we have

$$g \in A: \quad c(g) + g = h(g) \tag{2}$$

$$g \in D: \quad c(g) + g = zh(g), \tag{3}$$

where $z < 1$. We model the cost of default as an adverse technology shock.

Household. The household is deriving utility from consumption and leisure. The total amount of leisure is unity. Its preferences are

$$u(c_0, 1 - h_0) + \beta \sum_g \pi(g) u(c(g), 1 - h(g)). \tag{4}$$

We assume that initial debt is zero. The household's budget constraint at $t = 0$ reads

$$c_0 + qb = (1 - \tau_0) w_0 h_0 \tag{5}$$

At $t = 1$, at the household's budget constraints read

$$g \in A: \quad c(g) = (1 - \tau(g)) w(g) h(g) + b \tag{6}$$
$$g \in D: \quad c(g) = (1 - \tau(g)) w(g) h(g). \tag{7}$$

Note that labor taxes depend on the realization of the shock g.

Government. Similarly, the government budget constraint at $t = 0$ reads

$$0 = \tau_0 w_0 h_0 - g_0 + qb$$

and at $t = 1$ we

$$g \in A: \quad \tau(g) w(g) h(g) - g = b$$
$$g \in D: \quad \tau(g) w(g) h(g) - g = 0.$$

Firms. Competitive firms maximize profits given the linear technology and the default costs. The equilibrium wage is $w_0 = 1$ and $w(g) = 1, g \in A$, $w(g) = z < 1, g \in D$.

Household's problem. Given $\{q, \tau_0, \tau(g), w_0, w(g), D\}$ the household is choosing $\{c_0, h_0, c(g), h(g), b\}$ to maximize (4) subject to (5-7). The labor supply condition at $t = 0$ is

$$\frac{u_{l0}}{u_{c0}} = 1 - \tau_0,$$

whereas at $t = 1$ we have

$$\begin{aligned} g \in A: \qquad & \frac{u_l(g)}{u_c(g)} = 1 - \tau(g) \\ g \in D: \qquad & \frac{u_l(g)}{u_c(g)} = (1 - \tau(g))z \end{aligned}$$

Note that we have already used the equilibrium wages in these conditions. Furthermore, the Euler equation with respect to b is

$$\begin{aligned} q &= \beta \sum_g \pi(g) \frac{u_c(c(g), 1 - h(g))}{u_c(c_0, 1 - h_0)} (1 - d(g)) \\ &= \beta \sum_{g \in A} \pi(g) \frac{u_c(c(g), 1 - h(g))}{u_c(c_0, 1 - h_0)}. \end{aligned} \tag{8}$$

The Euler equation depicts the possibility of government default. If the default set is empty, $D = \emptyset$, then (8) simplifies to the standard Euler equation with risk-free debt.

Competitive equilibrium. The definition of the competitive equilibrium given government policy (b, τ, D) is obvious.

3 Optimal policy in the two-period economy

The government is choosing the optimal amount of taxes, debt and to default or repay. We will analyze optimal policy in two stages using backwards induction:

- At $t = 1$, given issued debt b, the government is deciding to default or not and how much to tax. The government takes into account the optimal reaction of the household to the tax rate, so it acts as a Stackelberg leader within the period.
- At $t = 0$, the government is choosing $\{b, \tau_0\}$, taking into account the decision of the household in the current period *and* the government's optimal decisions next period.

3.1 Default and repayment sets

Let u^d and u^r denote the equilibrium utility of default and repayment respectively. Define the default set, which depends on b,

$$D(b) \equiv \{g \in G | u^d > u^r\}, \tag{9}$$

and the repayment set

$$A(b) \equiv D(b)^c = \{g \in G | u^d \leq u^r\}. \tag{10}$$

In the definition of the sets we assume that if the government is indifferent about repaying or defaulting, then it is repaying. The default and repayment sets depend on the amount of debt through the default and repayment allocation.

Default allocation. The default allocation at g is determined by the following equations,

$$\begin{aligned} c + g &= zh \\ \frac{u_l}{u_c} &= (1-\tau)z \\ \tau z h &= g. \end{aligned}$$

The first is the resource constraint (taking into account the default costs), the second the labor supply and the third the balanced budget requirement. From these equations we get the default consumption-labor allocation and the default tax rate as functions of g, $\{c^d(g), h^d(g), \tau^d(g)\}$. The equilibrium utility of default is $u^d = u(c^d(g), 1 - h^d(g))$. Note that we can use the primal approach of Lucas and Stokey (1983) and eliminate the tax rate through the labor supply condition. This leads to a system of consumption and labor only,

$$\begin{aligned} c + g &= zh \\ \Omega(c,h) &= 0, \end{aligned} \tag{11}$$

where

$$\Omega(c,h) \equiv u_c(c, 1-h)c - u_l(c, 1-h)h. \tag{12}$$

Ω stands for consumption net of after-tax labor income, in marginal utility units. Equivalently, it is equal to government surplus in marginal utility units. For future reference, note that

$$\frac{\Omega_c}{u_c} = 1 - \epsilon_{cc} - \epsilon_{ch} \tag{13}$$

$$\frac{\Omega_h}{u_l} = -1 - \epsilon_{hh} - \epsilon_{hc} \tag{14}$$

where $\epsilon_{cc} \equiv -u_{cc}c/u_c, \epsilon_{ch} \equiv u_{cl}h/u_c, \epsilon_{hh} \equiv -u_{ll}h/u_l, \epsilon_{hc} = u_{cl}c/u_l$, the own and cross elasticities of the marginal utility of consumption and the marginal disutility of labor.

Repayment allocation. If the government repays, $g \in A$, we have

$$\begin{aligned} c + g &= h \\ \frac{u_l}{u_c} &= 1 - \tau \\ \tau h &= g + b \end{aligned}$$

which determines the repayment allocation and the repayment tax rate as functions of the debt and the shock, $\{c^r(b,g), h^r(b,g), \tau^r(b,g)\}$, and the repayment utility $u^r \equiv u(c^r(b,g), 1 - h^r(b,g))$. As before, the above system can be reduced to

$$\begin{aligned} c + g &= h \\ \Omega(c,h) &= u_c(c, 1-h)b, \end{aligned} \tag{15}$$

where the second constraint expresses the budget of the government in terms of consumption, labor and debt.

3.2 Properties of the default decision

We will make now two claims about the structure of the default set. We will see later the proofs.

Property 1. *If $b' > b$, then $D(b) \subseteq D(b')$. Default sets increase in debt.*

Property 2. *Let $g' > g$. If $g \in D(b)$, then $g' \in D(b)$. Default incentives increase with adverse fiscal shocks.*

Property 1 will be easy to prove. We will see later about property 2. Assume for the moment that the first property is true. Define the following borrowing limits:

$$\begin{aligned}\bar{b} &\equiv \inf\{b|D(b) = G\} \\ \underline{b} &\equiv \sup\{b|D(b) = \emptyset\}.\end{aligned}$$

$\underline{b}$ is the maximum amount of debt so that the government is repaying with certainty. $\bar{b}$ is the amount of debt above which the government is defaulting with probability 1. We have $\underline{b} \leq \bar{b}$. Furthermore, if $b \in (\underline{b}, \bar{b})$, then $D(b) \neq \emptyset$ and $A(b) \neq \emptyset$, so for intermediate values of debt there is always a shock for which the government defaults and a shock for which the government repays.

Lemma 1. *The utility of repayment is decreasing in debt. If $b' > b$ then $u(c^r(b', g), 1 - h^r(b', g)) < u(c^r(b, g), 1 - h^r(b, g))$.*

Proof. To be written.

□

Corollary 1. *Property 1 holds.*

Proof. Let $b' > b$ and $g \in D(b)$. Then $u^d(c^d(g), 1 - h^d(g)) > u(c^r(b, g), 1 - h^r(b, g)) > u(c^r(b', g), 1 - h^r(b', g))$, thus $g \in D(b')$. □

So property 1 is based on the fact that if debt increases, the government has to increase taxes, which leads to a reduction in utility.

Lemma 2. *The government never defaults if $b = 0$ for any value of the shock g, $u(c^r(0, g), 1 - h^r(0, g)) \geq u(c^d(g), 1 - h^d(g)) \forall g \Rightarrow D(0) = \emptyset$.*

Proof. The lemma seems obvious since the government would never default and incur the default costs. It needs some elaboration though. To be written.

□

Corollary 2. $\underline{b} \geq 0$.

Threshold. Define now $\omega(g)$ as the amount of debt for which the government is indifferent between repaying and defaulting at g,

$$u(c^d(g), 1 - h^d(g)) = u(c^r(\omega(g), g), 1 - h^r(\omega(g), g)) \tag{16}$$

Note that this equation has a solution in $[\underline{b}, \bar{b}]$ (which is unique since the utility of repayment is decreasing in b). Since the repayment utility is decreasing in debt, we have $d(g) = 1$ if $b > \omega(g)$ and $d(g) = 0$ if $b \leq \omega(g)$. Furthermore, the monotonicity of the threshold in g is equivalent to property 2.

Lemma 3. *Let $\omega(g') \leq \omega(g)$ for $g' > g \Leftrightarrow$ Property 2 holds.*

Proof. 1.($\Rightarrow$) Let $g \in D(b)$. This implies that $b > \omega(g) \geq \omega(g')$. Therefore, $g' \in D(b)$.

2. ($\Leftarrow$) Rephrase property 2 as follows: if $g' \in A(b)$ then $g \in A(b)$ for $g' > g$ (if I repay for the bad shock, I repay for the good shock). Assume now that $\omega(g') > \omega(g)$. Since

$$u(c^d(g'), 1 - h^d(g')) = u(c^r(\omega(g'), g'), 1 - h^r(\omega(g'), g')),$$

we have $g' \in A(\omega(g'))$. This implies that $g \in A(\omega(g'))$ by property 2 . Thus,

$$u(c^d(g), 1 - h^d(g)) = u(c^r(\omega(g), g), 1 - h^r(\omega(g), g)) \leq u(c^r(\omega(g'), g), 1 - h^r(\omega(g'), g)) \Rightarrow \omega(g) \geq \omega(g')$$

which is a contradiction. □

Thus, property 2 is equivalent to a non-increasing debt threshold in government expenditures. Its validity is not clear. We will show later that it holds if utility is linear in consumption. Furthermore, it holds numerically. To understand where this property depends on, note that if we could show that the difference in utility $\Delta u \equiv u^d - u^r$ is increasing in g, i.e. if

$$u(c^d(g'), 1 - h^d(g')) - u(c^r(b, g'), 1 - h^r(b, g')) > u(c^d(g), 1 - h^d(g)) - u(c^r(b, g), 1 - h^r(b, g)), g' > g,$$

then property 2 follows immediately.[2] It is easy to show that default and repayment utility fall if g increases. Δu increasing in g is stronger: it means that the loss in default utility due to larger g is *smaller* in absolute value than the loss in repayment utility.

3.3 Problem at $t = 0$

At $t = 0$ the government is choosing (c_0, h_0, b) to maximize

[2] An increasing Δu is the exact condition for a decreasing threshold. Use the implicit function theorem in the threshold equation to see that.

$$u(c_0, 1-h_0) + \beta \left[\sum_{g \in D(b)} \pi(g) u(c^d(g), 1-h^d(g)) + \sum_{g \in A(b)} \pi(g) u(c^r(b,g), 1-h^r(b,g)) \right]$$

subject to

$$\begin{aligned} \Omega(c_0, h_0) + \beta \big[\sum_{g \in A(b)} \pi(g) u_c(c^r(b,g), 1-h^r(b,g)) \big] b &= 0 \\ c_0 + g_0 &= h_0 \end{aligned}$$

Assume now that the shock follows a continuous distribution with density $f(g)$ and support $[\underline{g}, \bar{g}]$. Furthermore, assume that 2 is true and that the threshold is *strictly* decreasing in g. Then we can rewrite the default decision in terms of $\omega^{-1}(b)$, which is the level of g for which the government is indifferent between repayment and default for a particular level of b. Apparently, we have

$$u(c^d(\omega^{-1}(b)), 1-h^d(\omega^{-1}(b))) = u(c^r(b, \omega^{-1}(b)), 1-h^r(b, \omega^{-1}(b))) \tag{17}$$

The repayment and default sets become respectively $A(b) = [\underline{g}, \omega^{-1}(b)]$ and $D(b) = (\omega^{-1}(b), \bar{g}]$ for $b \in [\underline{b}, \bar{b}]$. In the analysis later we will also assume that ω^{-1} is differentiable, i.e. that the implicit function theorem applies to (17). The purpose of these assumptions is to derive an optimality condition for the optimal debt issuance of the government. We will not make them in any numerical treatment of the problem.

Given the structure of the default sets the optimization problem becomes: choose (c_0, h_0, b) to maximize

$$u(c_0, 1-h_0) + \beta \left[\int_{\underline{g}}^{\omega^{-1}(b)} u(c^r(b,g), 1-h^r(b,g)) f(g) dg + \int_{\omega^{-1}(b)}^{\bar{g}} u(c^d(g), 1-h^d(g)) f(g) dg \right]$$

subject to

$$\Omega(c_0, h_0) + \beta \big[\int_{\underline{g}}^{\omega^{-1}(b)} u_c(c^r(b,g), 1-h^r(b,g)) f(g) dg \big] b = 0 \tag{18}$$

$$c_0 + g_0 = h_0 \tag{19}$$

The government is taking into account how increasing debt affects the equilibrium price q. Higher debt affects the equilibrium price by both increasing the default region (reducing therefore the price) and by increasing the agent's marginal utility since repayment consumption falls (which increases the equilibrium price). The marginal utility effect is not present in Arellano (2008) due to risk-neutral foreign lenders.

Analysis. Assign the multiplier Φ on the implementability constraint (18) and λ on the resource constraint (19). First-order necessary conditions for (c_0, h_0) are

$$\begin{aligned} c_0: &\quad u_{c0} + \Phi\Omega_{c0} = \lambda_0 \\ h_0: &\quad -u_{l0} + \Phi\Omega_{h0} = -\lambda_0 \end{aligned}$$

which delivers the familiar wedge expression

$$\frac{u_{l0} - \Phi\Omega_{h0}}{u_{c0} + \Phi\Omega_{c0}} = 1.$$

This expression can be rewritten in terms of the tax rate τ_0 by using (13) and (14) as

$$\tau_0 = \frac{\Phi(\epsilon_{cc} + \epsilon_{ch} + \epsilon_{hh} + \epsilon_{hc})}{1 + \Phi(1 + \epsilon_{hh} + \epsilon_{hc})}. \tag{20}$$

Optimal debt issuance. Turn now to the optimal choice of b. Use Leibnitz's rule to get the following first-order condition:

$$\begin{aligned} &\frac{d\omega^{-1}}{db} f(\omega^{-1}(b)) \big[u\big(c^r(b, \omega^{-1}(b)), 1 - h^r(b, \omega^{-1}(b))\big) - u\big(c^d(\omega^{-1}(b)), 1 - h^d(\omega^{-1}(b))\big)\big] \\ &+ \int_{\underline{g}}^{\omega^{-1}(b)} \big[u_c^r \frac{\partial c^r}{\partial b} - u_l^r \frac{\partial h^r}{\partial b}\big] f(g) dg \\ &+ \Phi\Big\{ \int_{\underline{g}}^{\omega^{-1}(b)} u_c^r f(g) dg + b\big[u_c(c^r(b, \omega^{-1}(b)), 1 - h^r(b, \omega^{-1}(b))) f(\omega^{-1}(b)) \frac{d\omega^{-1}}{db} \\ &+ \int_{\underline{g}}^{\omega^{-1}(b)} \big(u_{cc}^r \frac{\partial c^r}{\partial b} - u_{cl}^r \frac{\partial h^r}{\partial b}\big) f(g) dg\big] \Big\} = 0 \end{aligned}$$

This expression can be simplified as follows. The terms in the first and second line correspond to the change in expected utility triggered by an increase in debt. An increase in debt

has two effects on expected utility. At first it reduces the repayment region, by decreasing the threshold value, $d\omega^{-1}/db < 0$. Second, it decreases expected utility because an increase in debt decreases the utility of repayment. The term in the first line corresponds to utility differential due to the reduction in the repayment region. This utility differential is equal to *zero* at the threshold value of spending $\omega^{-1}(b)$, where the government is indifferent between repayment and defaulting.

GEE in the two-period economy. Thus, the optimality condition reduces to

$$\underbrace{-\int_{\underline{g}}^{\omega^{-1}(b)} [u_c^r \frac{\partial c^r}{\partial b} - u_l^r \frac{\partial h^r}{\partial b}] f(g) dg}_{\text{I}} = \Phi \Big\{ \underbrace{\int_{\underline{g}}^{\omega^{-1}(b)} u_c^r f(g) dg}_{II(+)}$$

$$+ b \Big[\underbrace{u_c(c^r(b, \omega^{-1}(b)), 1 - h^r(b, \omega^{-1}(b))) f(\omega^{-1}(b)) \frac{d\omega^{-1}}{db}}_{III(-)} + \underbrace{\int_{\underline{g}}^{\omega^{-1}(b)} (u_{cc}^r \frac{\partial c^r}{\partial b} - u_{cl}^r \frac{\partial h^r}{\partial b}) f(g) dg}_{IV(+)} \Big] \Big\} \tag{21}$$

The LHS denotes the expected marginal utility loss due to an increase in debt. There is a utility cost because repayment utility falls with an increased amounted of debt due to the increase in taxes next period, $\partial u^r/\partial b = u_c^r \frac{\partial c^r}{\partial b} - u_l^r \frac{\partial h^r}{\partial b} = (u_c^r - u_l^r)\partial c^r/\partial b < 0$, since $\partial c^r/\partial b = \partial h^r/\partial b < 0$ and $u_c^r > u_l^r$ (we need to tax in order to repay). This is term I. The RHS denotes the welfare benefit of increasing debt, which comes form relaxing the budget constraint of the government and allowing less taxes today ($\Phi > 0$). The right-hand side is essentially the welfare benefit of the marginal revenue of debt issuance. The government would never find it optimal to issue a level of debt that would deliver a negative marginal revenue, so the right-hand side is positive. This essentially will imply a stricter borrowing limit than $\bar{b}$. The right-hand side has three terms:

- Term II is proportional to the price q. By issuing debt by one unit, the government gets revenue proportional to q.
- The third and fourth term essentially correspond loosely to $q'(b)$. The government takes into account that increasing debt will affect the price of debt through two channels:
    1. Term III: by increasing debt the government reduces the repayment region, which decreases the prices. Term III is *negative* ($d\omega^{-1}/db < 0$).
    2. Term IV: By increasing debt, the repayment consumption and labor become lower. As a result, marginal utility increases (and recall that $u_{cl} \geq 0$), so the price *in-*

creases. Term IV is *positive*. The higher the curvature in c, the more important we expect this term to be quantitatively.

Note that (21) is a generalized Euler equation (GEE) with default.

3.4 Quasi-linear example

Consider now an example with quasi-linear utility

$$u = c - \frac{1}{2}h^2. \tag{22}$$

This utility allows a simpler characterization of the default set. In particular we prove property 2 for this utility function and prove also differentiability of ω^{-1}. Furthermore, the absence of the marginal utility channel implies equilibrium prices are determined only by the probability of repayment. In other words, the planner is manipulating the equilibrium price of government debt only through the size of the repayment region and not through repayment consumption. This element eliminates time-inconsistency issues in the infinite horizon problem so it is of limited interest for us. Nevertheless, it provides an easier interpretation of (21) by eliminating term IV.

Proposition 1. *("Default in the quasi-linear case") Assume the period utility function (22). Define $\lambda \equiv z^2 < 1$ and assume that the shocks are not too large, $g < 1/4\lambda$ and that the debt position is not too large, $b + g < 1/4$. Then,*

1. *The default allocation is $h^d = z(1-\tau_d)$ and $c^d = zh^d - g$. The default tax rate and respective utility are is*

$$\begin{aligned} \tau_d(g) &= \frac{1-\sqrt{1-4g/\lambda}}{2} \\ u^d(g) &= \frac{1}{2}\lambda(1-\tau_d^2) - g. \end{aligned}$$

2. *The repayment allocation is $h^r = 1-\tau_r$, $c^r = h^r - g$. The repayment tax rate and respective utility are*

$$\begin{aligned} \tau_r(b,g) &= \frac{1-\sqrt{1-4(b+g)}}{2} \\ u^r(b,g) &= \frac{1}{2}(1-\tau_r^2) - g \end{aligned}$$

Note that for $b > g(1/\lambda - 1) > 0$ we have $\tau_r > \tau_d$.

3. *Default and repayment decision:*

$$d(g) = 1 \quad \textit{if} \quad \tau_r^2 > \lambda\tau_d^2 + 1 - \lambda$$
$$d(g) = 0 \quad \textit{if} \quad \tau_r^2 \leq \lambda\tau_d^2 + 1 - \lambda$$

4. *The threshold $\omega(g)$ or $\omega^{-1}(b)$ is defined implicitly $\tau_r^2(b,g) = \lambda\tau_d^2(g) + 1 - \lambda$. The threshold is monotonically decreasing,*

$$\frac{d\omega^{-1}(b)}{db} = -\frac{\tau_r(1-2\tau_d)}{\tau_r - \tau_d} < 0$$

Note also that the slope of the threshold is larger than unity in absolute value, $\frac{d\omega^{-1}(b)}{db} < -1$.

Note that the indifference condition that determines the threshold requires that the square repayment tax is a weighted average of the square default tax and unity ($\lambda < 1$). Therefore, for the government to be indifferent between repayment and default the repayment tax rate has to be *greater* than the default tax rate ($\tau_r > \tau_d$). Thus, even if the repayment tax is larger than the default tax, which superficially would lead to the false conclusion that the government has to default, the government may still want to repay. The reason behind that is coming from the fact that default entails output costs that reduce utility. Thus, for a given government expenditure shock, debt and the associated repayment tax has to be sufficiently high to lead to default. Therefore, a *necessary* (and not sufficient) condition for indifference (or defaulting) is that $b > g(1/\lambda - 1)$. The formula also shows that if $\lambda = z^2 = 1$, so if there are *zero* default costs, the government will never issue any debt. The government defaults if $\tau_r > \tau_d \Rightarrow b > 0$ and repays if $b \leq 0$.

Equilibrium price and debt issuance. In the quasi-linear case we can write the equilibrium price as a function of b,

$$\begin{aligned} q(b) &= \beta Prob(\text{repayment}) = \beta F(\omega^{-1}(b)), \quad b \in [\underline{b}, \bar{b}] \\ q'(b) &= \beta f(\omega^{-1}(b))\frac{d\omega^{-1}}{db} < 0. \end{aligned}$$

Note that $q(b) = 0$ for $b > \bar{b}$ and $q(b) = \beta$ for $b < \underline{b}$. Let $R(b) \equiv q(b)b$ denote the revenues from debt issuance. Expressing consumption and labor in terms of the tax ate, the implementability constraint simplifies to

$$\tau_0(1-\tau_0) - g_0 + q(b)b = 0, \tag{23}$$

which furnishes an initial tax rate as function of debt revenue $\tau_0(b, g_0) = \frac{1-\sqrt{1-4(g_0-R(b))}}{2}$, as long as the initial expenditures *adjusted* for any revenue from debt issuance are not too large, $g_0 - R(b) < 1/4$. The larger the revenue from debt issuance, the smaller the initial tax rate, which shows the tradeoffs that the government is facing.

The optimality equation with respect to debt (21) for the quasi-linear case simplifies to[3]

$$\beta \int_{\underline{g}}^{\omega^{-1}(b)} \tau_r \frac{\partial \tau_r}{\partial b} f(g)dg = \Phi[q + bq'(b)] = \Phi q(1-\epsilon),$$

where

$$\epsilon \equiv -\frac{q'(b)b}{q} = -b\frac{f(\omega^{-1}(b))}{F(\omega^{-1}(b))}\frac{d\omega^{-1}}{db} > 0,$$

the elasticity of the equilibrium price with respect to b. The right-hand side depicts the social value of the marginal revenue from debt-issuance, $\Phi R'(b)$. The marginal revenue from debt issuance has to be positive, otherwise issuing more debt will have only a welfare effect loss, since it is associated with higher taxes, as the left-hand side of the optimality equation shows. Therefore, we need $\epsilon(b) < 1$ in order to have a positive marginal revenue from debt issuance. Note that this implies a stricter upper bound for borrowing, above which the government never borrows.

In particular, let b^* the level of debt for which the revenue from debt issuance is maximal. If the solution is in $(\underline{b}, \bar{b})$, this corresponds to $\epsilon(b^*) = 1$. We expect that for $b < b^*$ we have $R'(b) > 0$. A sufficient condition for this (and for which we have a unique maximum) is that the price elasticity of debt is strictly increasing in b, $\epsilon'(b) > 0$. This obviously depends on the assumptions on the cumulative distribution function of the shocks and the slope of the threshold $d\omega^{-1}/db$. If it is true, we can restrict attention to the quadrant where $\tau_0 < 1/2$ and $b < b^*$. Furthermore, b^* has to be as follows.

Lemma 4. *The revenue-maximal level of debt satisfies* $b^* \in [\underline{b}, \bar{b})$. *If* $\lim_{b \to \underline{b}^+} R'(b) > 0$, *then*

[3] We multiply with β in order to express the condition in terms of the price q.

$b^* \in (\underline{b}, \bar{b})$.

Proof. We cannot have $b^* \geq \bar{b}$ since revenue is zero for this interval. Furthermore, we cannot have $b^* < \underline{b}$ since marginal revenue for this interval is positive. Therefore $b^* \in [\underline{b}, \bar{b})$. There is the possibility though that the maximum revenue is at the lower boundary point, $\underline{b}$. The right derivative of the revenue schedule is

$$\lim_{b \to \underline{b}^+} R'(b) = \beta + \beta f(\bar{g}) \frac{d\omega^{-1}(\underline{b})}{db} \underline{b} > 0,$$

according to the claim. Unless $f(\bar{g}) = 0$, there is a downward jump in marginal revenue. If the marginal revenue though still remains positive, then we cannot have an optimum at $\underline{b}$. Thus, $b^* \in (\underline{b}, \bar{b})$.

□

An interior b^*, which corresponds to $\epsilon(b^*) = 1$, implies that there is the *possibility* of an equilibrium with default (the planner may still optimally choose amounts of debt below $\underline{b}$). We would never have an equilibrium with default if $b^* = \underline{b}$. This could happen if the reduction in prices was such so that the marginal revenue at $\underline{b}^+$ is negative. This would make $\underline{b}$ a local maximum and if revenue falls for larger debt, a global maximum. So there is the possibility for an equilibrium without default only if the price schedule is extremely *steep*.

Remark 1. *Even if $b^* > \underline{b}$, optimal debt issuance does not necessarily entail default. The planner may run a deficit at the initial period to be financed by debt that matures at $t = 1$, but he may find it optimal to issue $b \leq \underline{b}$. In that case there is no default in equilibrium. The optimality condition captures these tradeoffs. The larger the default costs (low z), the more probable this scenario. Furthermore, this possibility depends also on the initial level of government expenditures g_0. If they are too small, then the planner may run a small deficit that does not require an optimal debt that falls in the default region. At the extreme, when $g_0 = 0$, the planner runs a surplus at the initial period and uses the proceeds to lend to the private sector, $b < 0$.*

3.5 Numerical illustrations for the quasi-linear case

Calibration of shocks: $\underline{g} = 0, \bar{g} = 0.2$. To get an idea of their size, note that the first-best output it unity, so government expenditures vary from 0 till 20% of first-best output. We use $2,000$ gridpoints and a uniform distribution. Furthermore, we set $\beta = 0.95, z = 0.99$.

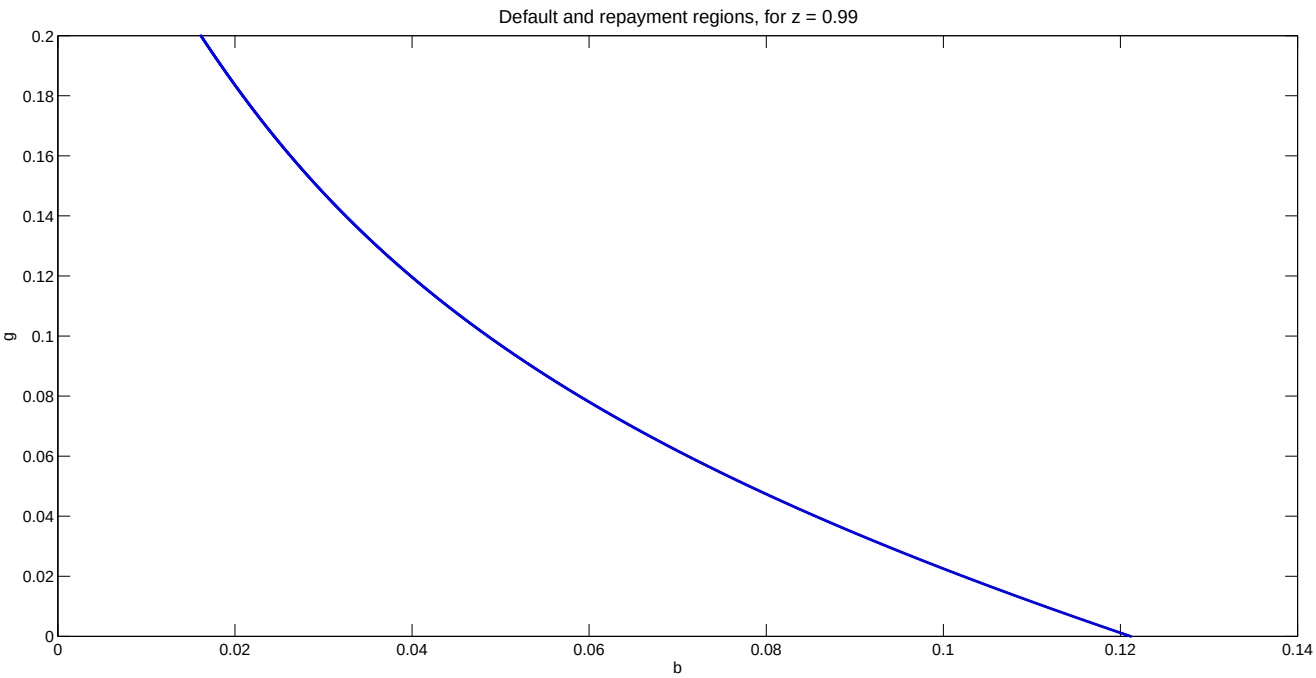

Figure 1: The figure plots $\omega^{-1}(b)$. For each level of debt the government is defaulting if $g > \omega^{-1}(b)$.

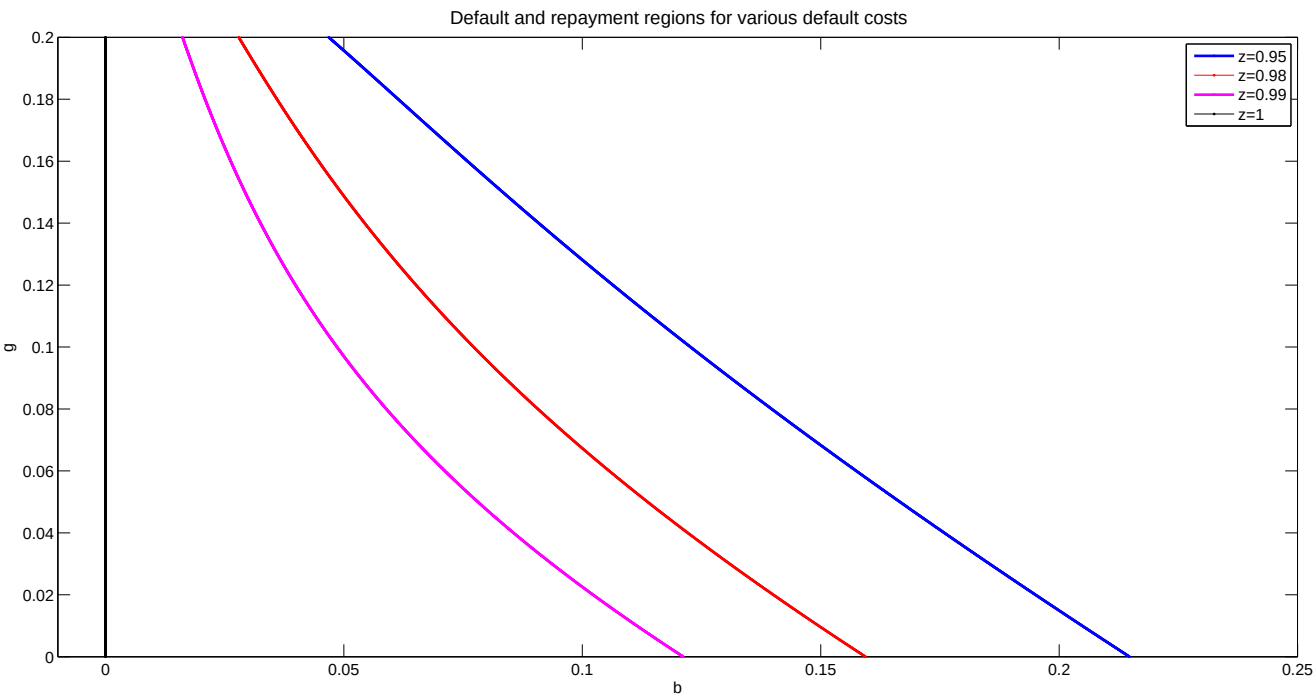

Figure 2: The figure plots $\omega^{-1}(b)$ for varying default costs.

Default and repayment regions. Figure 1 depicts the default/repayment region for the base-line calibration. Note the monotonically decreasing threshold ω^{-1}. To understand the impact of default costs z, on the deault/repayment regions, figure 2 plots the corresponding sets for varying default costs. The case of no default costs $z = 1$ corresponds to a vertical line at $b = 0$, i.e. the government defaults with certainty if there is positive debt and repays with certainty if $b \leq 0$. Besides this extreme case (for which ω^{-1} is not well-defined), note that an increase of default costs shifts the threshold curve to the right (thus for a given level of government expenditures, the government can sustain a larger debt without defaulting). Furthermore, the threshold becomes *flatter*.

Equilibrium price and revenues. Figure 3 plots the equilibrium price q and the corresponding revenues $R(b)$. For $b \leq \underline{b}$ the price is β, whereas for $b > \bar{b}$ we have $q = 0$. Note that the level of debt for which revenues are maximal, b^* is larger than $\underline{b}$.

Optimal debt issuance. As noted in remark 1, the *optimal* debt that the government issues is not necessarily large enough so that it entails default. For the particular calibration I use $g_0 = \bar{g}$, and it turns out that $b > \underline{b}$, as figure 4 shows. The probability of default is 20.85%. Figure 5 depicts how the optimal debt issuance depends on the initial shock. For each level of g_0 we calculate the optimal debt. As noted earlier, at the extreme where the initial shock is zero, the government is lending to the private sector, $b < 0$. There is a positive relationship between the initial shock and optimal b. Note that there is a region of initial shocks for which the optimal debt issuance is always at $\underline{b}$.

4 Infinite horizon economy

Consider now an infinite horizon model, where the government can default on issued debt. The uncertainty is coming from government expenditures shocks g_t that take values in G, with probability of the partial history g^t equal to $\pi_t(g^t)$. We assume that there is no uncertainty at the initial period, so $\pi_0(g_0) \equiv 1$.

We use $d_t = 1$ to denote default and $d_t = 0$ when there is repayment. The resource constraint in the economy when the government does not default is

$$c_t + g_t = h_t. \tag{24}$$

If the government defaults there are default costs that are captured as a technology shock.

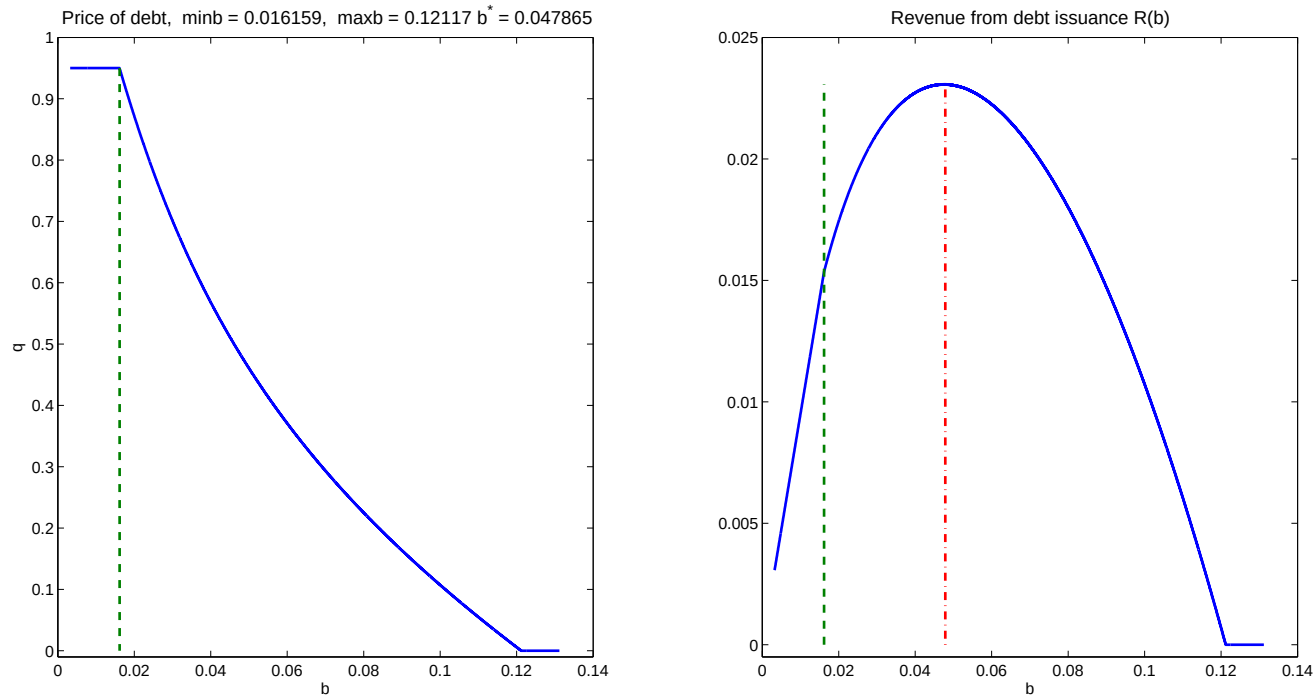

Figure 3: The left graph plots the price of debt, $q(b)$. The right graph plots the revenues from debt issuance.

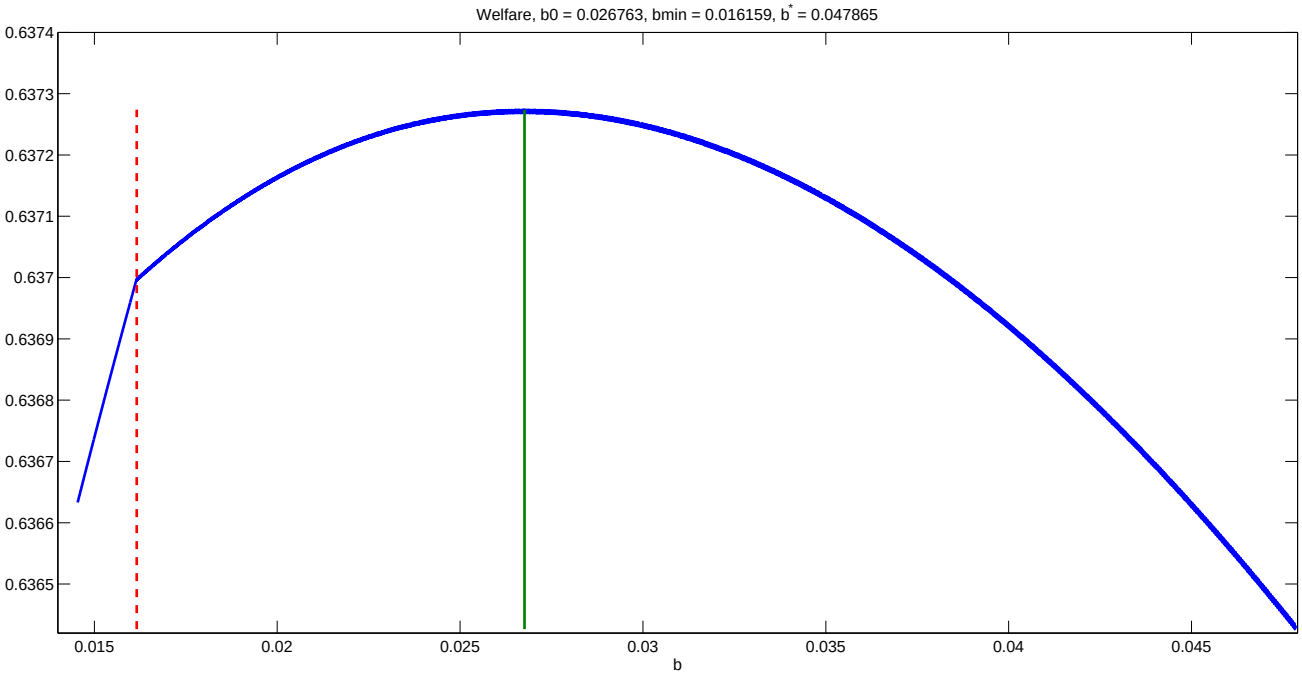

Figure 4: The graph depicts expected discounted utility at $t = 0$ when $g_0 = \bar{g}$. The red vertical line denotes $\underline{b}$. The upper level of debt is b^*. The optimal debt issuance b_0 is larger than $\underline{b}$. The probability of default is 0.2085.

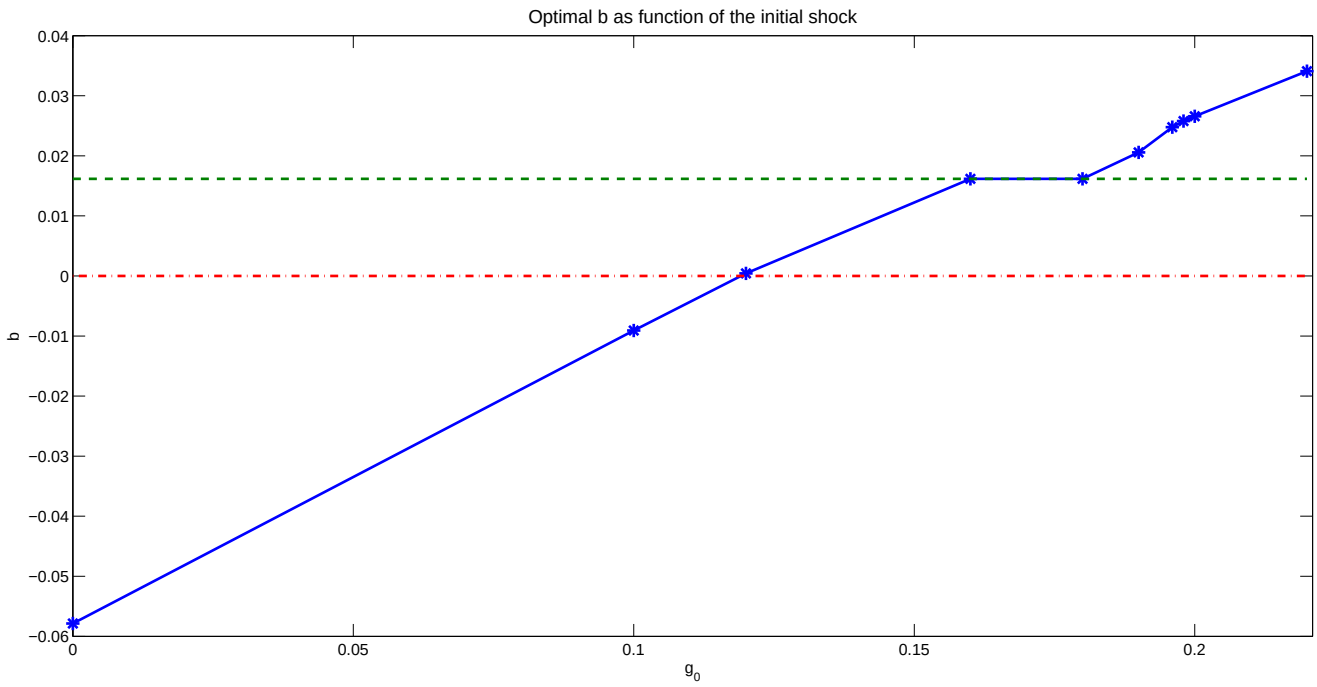

Figure 5: The graph depicts the optimal debt issuance as function of the initial shock g_0. The green dashed line depicts $\underline{b}$. Any level of optimal debt that is smaller or equal than $\underline{b}$ entails repayment with certainty. We set the initial shock $g_0 = \kappa\bar{g}$ with $\kappa \in \{0, 0.5, 0.6, 0.8, 0.9, 0.95, 0.98, 0.99, 1, 1.1\}$.

The resource constraint in the event of default is

$$c_t + g_t = zh_t,$$

where $z < 1$. We will explore also more elaborate default costs as in Arellano (2008).

Household. The household's preferences are

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t).$$

The household trades with the government a discount bond that gives one unit of consumption next period at any state of the world where the government is not defaulting and zero in the event of default. The price of the bond is q_t. The household pays a linear tax τ_t on labor income $w_t h_t$. The household's budget constraint when the government is not defaulting reads

$$c_t + q_t b_{t+1} \leq (1 - \tau_t) w_t h_t + b_t.$$

Note that the bond position b_{t+1} is function of information at time t. Furthermore, the household may have some initial debt b_0.

Default entails direct and indirect costs. The direct ones are in terms of output losses due to the negative technology shock. The indirect costs are coming from the exclusion from the market for government debt. When the government defaults at time t, b_t is wiped out and the household is also excluded from the market for new government debt at the same period. This can be thought of as a collapse of the market. At every period after default, the household can enter the market with probability α or stay excluded with probability $1 - \alpha$. When $\alpha = 1$, the implicit cost of default is small, since the exclusion lasts only one period, whereas when $\alpha = 0$, the cost is large, since the government has to run a balanced budget forever. In the open economy literature as in Arellano (2008), α is calibrated in order to match the average duration of exclusion from international markets. The international market justification is obviously not relevant for the closed economy, thus our market "collapse" interpretation.

Therefore, the household's budget constraint in the event of default, or for any period where there is exclusion from the market is

$$c_t = (1 - \tau_t) w_t h_t.$$

The household is also subject to some borrowing limits that we assume that are large enough so that they do not bind.

Wages. Note that in equilibrium the wage rate is $w_t = 1$ if $d_t = 0$ and $w_t = z$ if $d_t = 1$ or if there is exclusion after a default event. Note that the assumption that the direct output costs are relevant also for any period that the household is excluded from the market for government debt compounds the implicit default cost.

Government. The budget constraint of the government in the event of repayment is

$$B_t = \tau_t w_t h_t - g_t + q_t B_{t+1}.$$

$B_t > 0$ means that the government borrows and $B_t < 0$ that the government lends.

If there is default or for any period after a default event for which there is exclusion, the government runs a balanced budget,

$$\tau_t w_t h_t = g_t.$$

Equilibrium. A competitive equilibrium with taxes and default is a price-tuple $\{q_t, w_t\}$, a government policy $\{\tau_t, d_t, B_t\}$, and a household's allocation and bond holdings $\{c_t, h_t, b_t\}$ such that 1) Given prices and government policies, the household maximizes his utility subject to the budget constraint. 2) Given wages, firms maximize profits. 3) Prices and government policies are such so that markets clear: the resource constraint and the government budget constraint hold. Furthermore, the bond market clears, $b_t = B_t$.

Remark 2. *We have used different notation for government debt and then imposed the equilibrium condition $b_t = B_t$. This is really redundant. Furthermore, given $b_t = B_t$ and the rest of equilibrium conditions, the government budget constraint is redundant.*

Optimality conditions. The labor supply condition is

$$\frac{u_{lt}}{u_{ct}} = (1 - \tau_t) w_t.$$

The Euler equation for government bonds is

$$q_t = \beta E_t \frac{u_{c,t+1}}{u_{ct}} (1 - d_{t+1}).$$

The household is aware of the default decision of the government but is not able to affect it. The equation shows that the equilibrium price of debt is zero, if the government defaults with certainty. If the government repays with certainty, then it reduces to the standard Euler equation without default.

5 Markov-perfect policy

The policymaker decides how much to tax, how much debt to issue and if he will repay or not. His objective is to maximize the utility of the representative household. The constraints are the optimality conditions, the budget, and resource constraints coming from the competitive equilibrium. We are using the primal approach of Lucas and Stokey (1983) to

eliminate tax rates and equilibrium prices. We are assuming a Markov-perfect timing protocol as in Klein et al. (2008), so the solution to the policy problem will be time-consistent in the payoff-relevant state variables.[4]

Our Markov-perfect equilibrium (MPE) has two state variables, government debt B and the exogenous shock g, which we also assumes that is Markov. Let $V^r(B,g)$ denote the value function if the government decides to repay and $V^d(g)$ the value function if the government defaults. The value function of the government is

$$V(B,g) = \max\{V^r(B,g), V^d(g)\}.$$

Value of default. When the government default the consumption and labor allocation is $(c^d(g), h^d(g))$ for each value of the shock g is determined by the resource constraint, the labor supply condition and the balanced budget requirement. Thus, it has to satisfy

$$\begin{aligned} \Omega(c,h) &= 0 \\ c+g &= zh, \end{aligned}$$

as in the two-period model. Given (c^d, h^d) we can immediately deduce the default tax rate, $\tau^d(g) = 1 - u_l^d/(zu_c^d)$. The value of default is

$$V^d(g) = u(c^d(g), 1-h^d(g)) + \beta \sum_{g'} \pi(g'|g)\left[\alpha V(0,g') + (1-\alpha)V^d(g')\right]$$

Note that if $\alpha = 0$, i.e. if the market for government debt seized to exist forever after a default event, and if G is finite, we could calculate immediately the value of "autarky" as

$$\mathbf{V^d} = (I - \beta\Pi)^{-1}\mathbf{u^d}.$$

Boldface variables denote vector columns, I the identity matrix and Π the transition matrix of the shocks.

Default decision. Define the default set as

[4]See Bassetto (2005) for a careful analysis of the timing protocols underlying policy design.

$$D(B) \equiv \{g \in G | V^d(g) > V^r(B,g)\}$$

and the repayment set as the complement of $D(B)$,

$$A(B) \equiv D(B)^c = \{g \in G | V^d(g) \leq V^r(B,g)\}.$$

Given an amount of debt B at the beginning of the period, the default set denotes the set of values of g for which the government decides to default, so $d(B,g) = 1$ if $g \in D(B)$. The repayment set corresponds to $d(B,g) = 0$ if $g \in A(B)$. Given the default and repayment set we have

$$V(B,g) = V^r(B,g), g \in A(B) \quad \text{or} \quad V(B,g) = V^d(g), g \in D(B).$$

Value of repayment. In a Markov-perfect equilibrium the planner takes into account at the current period that he will follow an optimal policy from next period onward, given the value of debt next period. To capture this requirement, let $\mathcal{C}(B,g)$ and $\mathcal{H}(B,g)$ denote the consumption and labor policy functions in the event of repayment. They satisfy $\mathcal{C}(B,g) + g = \mathcal{H}(B,g)$. The "current" planner takes into account that by choosing debt B', he affects the consumption-labor choice of the "future" planner though $\mathcal{C}$ and $\mathcal{H}$. The value of repayment is

$$V^r(B,g) = \max_{c,h,B'} u(c, 1-h) + \beta \sum_{g'} \pi(g'|g) V(B', g')$$

subject to

$$\begin{aligned} u_c(c, 1-h)B &\leq \Omega(c,h) + \beta B' \sum_{g' \in A(B')} \pi(g'|g) u_c(\mathcal{C}(B',g'), 1 - \mathcal{H}(B',g')) \\ c + g &= h \\ c &\geq 0, h \in [0,1] \end{aligned}$$

We have used the Euler equation and the labor supply condition in order to rewrite the budget constraint of the household in terms of allocations. Taking into account the optimal policy functions of next period has a bite only in the case of curvature in the utility

function. If the utility was linear in consumption, so if there was no room for manipulation of interest rates, $\mathcal{C}$ and $\mathcal{H}$ would not be relevant and the commitment solution would be time-consistent.

Note that given the definition of the default sets we can rewrite the problem as

$$V^r(B,g) = \max_{c,h,B'} u(c,1-h) + \beta\Big[\sum_{g'\in A(B')} \pi(g'|g)V^r(B',g') + \sum_{g'\in D(B')} \pi(g'|g)V^d(g')\Big]$$

subject to

$$\begin{aligned} u_c(c,1-h)B &\leq \Omega(c,h) + \beta B' \sum_{g'\in A(B')} \pi(g'|g)u_c(\mathcal{C}(B',g'),1-\mathcal{H}(B',g')) \\ c+g &= h \\ c &\geq 0, h\in[0,1] \end{aligned}$$

MPE requirement. Let $c(B,g)$, $h(B,g)$ and $B'(B,g)$ be the policy functions of the above problem. The Markov-perfect requirement is that $c(B,g) = \mathcal{C}(B,g)$ and $h(b,g) = \mathcal{H}(B,g)$.[5] Note that there may be multiple solutions for the policy functions. We are going to focus on the MPE that is the limit of a finite horizon problem. So we are going to solve for T periods and increase T till there is no difference in the policy and value functions.

6 Analysis

We can get two lemmata.

Lemma 5. *The value of repayment is decreasing in B.*

Proof. This is obvious since for $B_1 < B_2$ the constraint correspondence increases, and therefore the repayment value is larger at B_1, $V^r(B_1,g) \geq V^r(B_2,g)$. □

Since the repayment value decreases in debt we have property 1 of the two-period model, $B_1 > B_2 \Rightarrow D(B_2) \subseteq D(B_1)$.

We can define as in the two period model the upper and lower debt limit,

$$\begin{aligned} \bar{B} &\equiv \inf\{B|D(B) = G\} \\ \underline{B} &\equiv \sup\{B|D(B) = \emptyset\}. \end{aligned}$$

[5] A more precise MPE requirement would be that $\mathcal{C}(B,g)$ and $\mathcal{H}(B,g)$ are maximizers of the stated problem in order to account for the existence of multiple solutions. This is for example what Klein et al. (2008) do.

Lemma 6. *$V(0,g) = V^r(0,g), \forall g$. If the government has no debt, it does not default. Thus $D(0) = \emptyset$ and $\underline{B} \geq 0$.*

Proof. To be written. □

As in the two period model, let $\omega(g)$ denote the amount of debt given the value of spending g such that the government is indifferent between defaulting and repaying,

$$V^d(g) = V^r(\omega(g), g).$$

The government defaults if $B > \omega(g)$ and repays if $B \leq \omega(g)$. We only need Claim I for that.

Assume now that property 2 of the two-period model is true, i.e. that if $g \in D(B)$ then $g' \in D(B)$ for $g' > g$, which as we saw is equivalent to a monotonically decreasing threshold. If it is strictly decreasing we can define $\omega^{-1}(B)$ as the value of government spending that makes the government indifferent between repaying or defaulting given B, so the government defaults if $g > \omega^{-1}(B)$ and repays if $g \leq \omega^{-1}(B)$ and we obviously have

$$V^d(\omega^{-1}(B)) = V^r(B, \omega^{-1}(B)).$$

Assume again a continuous distribution of shocks in $[\underline{g}, \bar{g}]$ with conditional density $f(g'|g)$. We can write the the value function of repayment as

$$V^r(B,g) = \max_{c,h,B'} u(c, 1-h) + \beta\Big[\int_{\underline{g}}^{\omega^{-1}(B')} V^r(B', g')f(g'|g)dg' + \int_{\omega^{-1}(B')}^{\bar{g}} V^d(g')f(g'|g)dg'\Big]$$

subject to

$$\begin{aligned} u(c, 1-h)B &\leq \Omega(c,h) + \beta B' \int_{\underline{g}}^{\omega^{-1}(B')} u_c(\mathcal{C}(B', g'), 1 - \mathcal{H}(B', g'))f(g'|g)dg' \\ c + g &= h \end{aligned}$$

6.1 Optimal tax rate

We will assume now differentiability and take first-order conditions. This is only to develop intuition for the tradeoffs that the government is facing. We will not make any differentiabil-

ity assumption in our numerical treatment of the problem. Note that non-differentiabilities arise from two sources: a) the default decision b) the MPE requirement.

Assign multiplier Φ and λ on the implementability and resource constraint respectively. The first-order conditions with respect to consumption and labor are

$$\begin{aligned} c: &\quad u_c + \Phi[\Omega_c - u_{cc}B] = \lambda \\ h: &\quad -u_l + \Phi[\Omega_h + u_{cl}B] = -\lambda \end{aligned}$$

Eliminating λ we get

$$\frac{u_l - \Phi[\Omega_h + u_{cl}B]}{u_c + \Phi[\Omega_c - u_{cc}B]} = 1 \tag{25}$$

Given the resource constraint and (25) we can write c, h as functions of (Φ, B, g). Debt has two effects: a direct one through B and an indirect one though Φ since at the optimum $\Phi = \Phi(B, g)$. Furthermore, we can derive the optimal tax rate as[6]

$$\tau = \frac{\Phi(\epsilon_{cc}(1 - B/c) + \epsilon_{ch} + \epsilon_{hh} + \epsilon_{hc}(1 - B/c))}{1 + \Phi(1 + \epsilon_{hh} + \epsilon_{hc}(1 - B/c))}. \tag{26}$$

This expressions shows the dependence of the tax rate on the marginal cost of taxation, captured by Φ, on debt and the particular elasticities of the period utility function. For the constant Frisch elasticity case (39) it takes the form

$$\tau = \frac{\Phi(\rho(1 - B/c) + \phi_h)}{1 + \Phi(1 + \phi_h)}. \tag{27}$$

6.2 Generalized Euler equation

Consider now the optimality condition with respect to B'.

[6]Bear in mind also the two non-negativity conditions from the positivity of λ,

$$\begin{aligned} 1 + \Phi[1 - \epsilon_{cc}(1 - B/c) - \epsilon_{ch}] &> 0 \\ 1 + \Phi(1 + \epsilon_{hh} + \epsilon_{hc}(1 - B/c)) &> 0 \end{aligned}$$

$$-\frac{\partial}{\partial B'}\int_{\underline{g}}^{\bar{g}} V(B',g')f(g'|g)dg' = \Phi\Big\{\int_{\underline{g}}^{\omega^{-1}(B')} u_c(\mathcal{C}(B',g'),1-\mathcal{H}(B',g'))f(g'|g)dg'$$
$$+B'\Big[u_c(\mathcal{C}(B',\omega^{-1}(B')),1-\mathcal{H}(B',\omega^{-1}(B')))f(\omega^{-1}(B')|g)\frac{d\omega^{-1}}{dB'}$$
$$+\int_{\underline{g}}^{\omega^{-1}(B')}[u'_{cc}\frac{\partial\mathcal{C}}{\partial B'}-u'_{cl}\frac{\partial\mathcal{H}}{\partial B'}]f(g'|g)dg'\Big]\Big\}$$

Note that $\partial\mathcal{C}/\partial B' = \partial\mathcal{H}/\partial B'$ and that

$$\begin{aligned}\frac{\partial}{\partial B'}\int_{\underline{g}}^{\bar{g}} V(B',g')f(g'|g)dg' &= f(\omega^{-1}(B')|g)\frac{d\omega^{-1}}{dB'}[V^r(B',\omega^{-1}(B'))-V^d(\omega^{-1}(B'))]\\ &\quad+\int_{\underline{g}}^{\omega^{-1}(B')}\frac{\partial V^r(B',g')}{\partial B'}f(g'|g)dg'\\ &= \int_{\underline{g}}^{\omega^{-1}(B')}\frac{\partial V^r(B',g')}{\partial B'}f(g'|g)dg'.\end{aligned}$$

Thus, we have

Proposition 2. *("GEE") The generalized Euler equation in an environment with incomplete markets and default takes the form*

$$-\int_{\underline{g}}^{\omega^{-1}(B')}\frac{\partial V^r(B',g')}{\partial B'}f(g'|g)dg' = \Phi\Big\{\int_{\underline{g}}^{\omega^{-1}(B')} u_c(\mathcal{C}(B',g'),1-\mathcal{H}(B',g'))f(g'|g)dg'$$
$$+B'\Big[u_c(\mathcal{C}(B',\omega^{-1}(B')),1-\mathcal{H}(B',\omega^{-1}(B')))f(\omega^{-1}(B')|g)\frac{d\omega^{-1}}{dB'}$$
$$+\int_{\underline{g}}^{\omega^{-1}(B')}[u'_{cc}-u'_{cl}]\frac{\partial\mathcal{C}}{\partial B'}f(g'|g)dg'\Big]\Big\} \tag{28}$$

Each term of the GEE has *exactly the same* interpretation as in the two-period model. The GEE equates the marginal cost of increasing debt, with the marginal benefit coming from the relaxation of the government budget constraint at the current period. The relaxation of the government budget constraint is coming from increasing debt revenue and being able therefore to decrease the current tax rate. The marginal revenue expression reflects the way the default region increases with increased debt, a fact which decreases equilibrium prices, and the way equilibrium prices *increase* due to the increase of marginal utility, in the case of $\frac{\partial\mathcal{C}}{\partial B'} < 0$.

The envelope condition under the differentiability assumption takes the form

$$\frac{\partial V^r}{\partial B} = -\Phi u_c, \tag{29}$$

which allows the rewriting of the GEE (28) in terms of the multipliers on the implementability constraint. Recall that these multipliers capture essentially the welfare cost of the lack of lump-sum taxes, and therefore they capture tax-distortions. For that reason, we will call Φ the *excess burden of taxation*. The GEE becomes

$$\begin{aligned}
&\int_{\underline{g}}^{\omega^{-1}(B')} u_c' \Phi(B', g') f(g'|g) dg' = \Phi \Big\{ \int_{\underline{g}}^{\omega^{-1}(B')} u_c(\mathcal{C}(B', g'), 1 - \mathcal{H}(B', g')) f(g'|g) dg' \\
&+ B' \Big[u_c(\mathcal{C}(B', \omega^{-1}(B')), 1 - \mathcal{H}(B', \omega^{-1}(B'))) f(\omega^{-1}(B')|g) \frac{d\omega^{-1}}{dB'} \\
&+ \int_{\underline{g}}^{\omega^{-1}(B')} [u_{cc}' - u_{cl}'] \frac{\partial \mathcal{C}}{\partial B'} f(g'|g) dg' \Big] \Big\}.
\end{aligned} \tag{30}$$

This form of the GEE is potentially helpful in order to contrast our analysis with Aiyagari et al. (2002), Pouzo and Presno (2014) and Debortoli and Nunes (2013). In order to see that, turn into sequence notation and note that since $d_{t+1} = 1$ if $g_{t+1} \in (\omega^{-1}(B_{t+1}), \bar{g}]$, we can write the GEE as

$$\begin{aligned}
E_t(1 - d_{t+1}) u_{c,t+1} \Phi_{t+1} \quad = \quad & \Phi_t \Big\{ E_t(1 - d_{t+1}) u_{c,t+1} + B_{t+1} \cdot \big[u_{c,t+1} f(\omega^{-1}(B_{t+1})|g_t) \frac{d\omega^{-1}}{dB'} \\
& + E_t(1 - d_{t+1})(u_{cc,t+1} - u_{cl,t+1}) \frac{\partial \mathcal{C}}{\partial B'} \big] \Big\}
\end{aligned}$$

Define the positive random variable $m_{t+1} \equiv \frac{(1-d_{t+1})u_{c,t+1}}{E_t(1-d_{t+1})u_{c,t+1}}$, which by construction integrates to unity, $E_t m_{t+1} = 1$. This random variable determines the *default-and-risk-adjusted measure*, with conditional density $f^m(g_{t+1}|g_t) \equiv m_{t+1} \cdot f(g_{t+1}|g_t)$. By dividing the GEE over expected marginal utility over the repayment region, we can rewrite the GEE as

$$E_t m_{t+1} \Phi_{t+1} = \Phi_t \Big\{ 1 + B_{t+1} \cdot \big[f^m(\omega^{-1}(B_{t+1})|g_t) \frac{d\omega^{-1}}{dB'} + E_t m_{t+1} \frac{u_{cc,t+1} - u_{cl,t+1}}{u_{c,t+1}} \frac{\partial \mathcal{C}}{\partial B'} \big] \Big\} \tag{31}$$

This form of the GEE makes clearer the connection to the tax-smoothing literature. At first, note that the left-hand side denotes the *average* tax distortions with respect to the default-and-risk-adjusted measure. The right-hand side, as previously, denotes the change in the revenue from debt issuance due to a larger position (the first term that is unity in (31))

and to the pricing effects that a larger position entails, due to the lack of repayment enforcement and the lack of commitment to tax policies designed in the past. Consider for example the setup of Aiyagari et al. (2002). Since there is no default and no commitment, and incomplete markets, the respective tax-smoothing condition is $E_t x_{t+1} \Phi_{t+1} = \Phi_t$, where $x_{t+1} \equiv u_{c,t+1}/E_t u_{c,t+1}$, the *risk-adjusted* change of measure. Thus, the planner tries to keep on average (with respect to the risk-adjusted measure) tax distortions *constant*. In contrast, here the planner does not try to make average distortions constant due to the lack of commitment to the two dimensions of policy, debt repayment and tax rates. If the reduction in interest rates due to the marginal utility channel is larger than the increase in interest rates due to default risk, then it is optimal for the planner to tax less today, issue more debt and postpone on average distortions for the future. This translates to a positive drift in the excess burden according to the default-and-risk-adjusted measure, $E_t m_{t+1} \Phi_{t+1} \geq \Phi_t$, when $B_{t+1} > 0$. The opposite will happen if the default force is stronger, i.e. if default premia increase so much that they dominate the decrease in interest rates due to high marginal utility. Then the planner on average wants to decrease the excess burden over time, so he taxes more today and less on average in the future. This negative drift in the excess burden should also materialize as a negative drift in debt.[7]

7 Long-term debt

Consider now the possibility of long-term debt with exponentially decaying coupons as in Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012).[8] Coupons decay at the rate δ, so the coupon payments are $(1-\delta)^i, i \geq 0$. Assume that the government chooses every period to default or not on all current and future debt obligations and that the same costs (direct and indirect in form of the exclusion from debt markets) apply as in the case with short-term debt ($\delta = 1$). Under this assumption the definition of the default value function remains the same as previously.

To calculate the value of repayment, we need to modify the policy problem. Let B_t denote the debt obligation of the government at the beginning of the period, which consists of the coupon payments that correspond to debt issued in all previous periods. This takes the form of

[7] Debortoli and Nunes (2013) and Krusell et al. (2004) analyze interest rate manipulation in deterministic setups through the marginal utility channel, so they do not consider uncertainty and default. Their respective GEE is the deterministic version of (31). In Pouzo and Presno (2014), the manipulation of prices through the marginal utility channel is not present due to the commitment to previously designed tax policies, so their respective GEE exhibits only the default term.

[8] See also Woodford (2001) for the analysis of non-Ricardian fiscal regimes with this type of long-term debt.

$$B_t = b_t + (1-\delta)b_{t-1} + (1-\delta)^2 b_{t-2} + ... = b_t + (1-\delta)B_{t-1},$$

where b_t are the holdings of the security at end of period $t-1$, b_{t-1} the previous period (so it provides a coupon of$(1-\delta)$, etc. Thus, the household's budget constraint in the event of repayment reads

$$c_t + q_t(B_{t+1} - (1-\delta)B_t) = (1-\tau_t)w_t h_t + B_t$$

The Euler equation with defaultable long-term debt becomes

$$q_t = \beta E_t \frac{u_{c,t+1}}{u_{ct}}(1-d_{t+1})\left[1 + (1-\delta)q_{t+1}\right], \tag{32}$$

which becomes

$$Q_t = \beta E_t (1-d_{t+1})\left[u_{c,t+1} + (1-\delta)Q_{t+1}\right] \tag{33}$$

if we define Q_t as the price of long-term debt in *marginal utility* units, $Q_t \equiv u_{ct} \cdot q_t$.

7.1 Value of repayment

The policy-maker that cannot commit takes into account that the future policy-maker will follow an optimal policy. Let $\mathcal{C}(B', g')$, $\mathcal{H}(B', g')$ and $\mathcal{K}(B', g')$ denote the consumption, labor and debt policy function next period. The value of repayment is defined as

$$V^r(B, g) = \max_{c,h,B'} u(c, 1-h) + \beta E_{g'|g} V(B', g')$$

subject to

$$\begin{aligned} [u_c(c, 1-h) + (1-\delta)Q(B', g)]B &\leq \Omega(c, h) + Q(B', g)B' \\ c + g &= h \\ c &\geq 0, h \in [0, 1] \end{aligned}$$

where the price of long-term debt satisfies the following recursion,

$$Q(B',g) = \beta E_{g'|g}(1-d(B',g'))\Big[u_c(\mathcal{C}(B',g'),1-\mathcal{H}(B',g')) + (1-\delta)Q\big(\mathcal{K}(B',g'),g'\big)\Big]$$

Note that the fact that debt is not short-lived, requires the specification of next period's debt policy function, in order to determine the price of the long-term asset. The Markov-perfect requirement is obviously $c(B,g) = \mathcal{C}(B,g), h(B,g) = \mathcal{H}(B,g)$ and $B'(B,g) = \mathcal{K}(B,g)$.

7.2 Analysis

The optimality conditions with respect to c, h are the same as in the case with short-term debt, leading to the same formula for the optimal wedge and tax rate, (25) and (26). Consider the optimal debt issuance, that is governed by the following equation:

$$-\beta\frac{\partial}{\partial B'}E_{g'|g}V(B',g') = \Phi\cdot[Q(B',g) + \frac{\partial Q}{\partial B'}\left(B' - (1-\delta)B\right)] \tag{34}$$

As previously, the left-hand side denotes the cost of issuing more debt, whereas the right-hand side the welfare benefit of the marginal revenue from debt issuance. Note that a change in price affects also the price of the remaining coupons that are to be paid in the future, due to the long-term nature of debt. The respective envelope condition with long-term debt is

$$\frac{\partial V^r(B,g)}{\partial B} = -\Phi[u_c + (1-\delta)Q]. \tag{35}$$

To get a simpler expression, assume as previously that default sets have a "nice" structure, $A(B') = [\underline{g}, \omega^{-1}(B')]$ and that the conditional density of shocks is $f(g'|g)$. The derivative of price with respect to debt is

$$\begin{aligned}\frac{\partial Q}{\partial B'} \;=\; & \beta\Big[\underbrace{\{u_c(\mathcal{C}(B',\omega^{-1}(B')),1-\mathcal{H}(B',\omega^{-1}(B'))) + (1-\delta)Q(\mathcal{K}(B',\omega^{-1}(B')),\omega^{-1}(B'))\}f(\omega^{-1}(B')|g)\frac{d\omega^{-1}}{dB'}}_{\text{decrease in prices due to increased default prob. (-)}} \\ & + \underbrace{\int_{\underline{g}}^{\omega^{-1}(B')}(u'_{cc}-u'_{cl})\frac{\partial \mathcal{C}}{\partial B'}f(g'|g)dg'}_{\text{increase in price due to increase in MU (+)}} + (1-\delta)\underbrace{\int_{\underline{g}}^{\omega^{-1}(B')}\frac{\partial Q'}{\partial B''}\frac{\partial \mathcal{K}}{\partial B'}f(g'|g)dg'}_{\text{future price change}}\Big]\end{aligned} \tag{36}$$

The derivative of the price with respect to debt exhibits again the two forces that we previously identified: the decrease in prices due to an increased default probability (the first term in the right-hand side of (36)) and the increase in prices due to increased marginal

utility (the second term in the right-hand side). Debt though is long-lived so there are capital gains and losses and the derivative of the current price is determined also by the (properly discounted) derivative of the future price Q' with respect to debt, taking into account how the current debt issuance will affect future debt issuance, $\partial\mathcal{K}/\partial B'$.

As in the case with short-term debt, we will write the GEE in terms of the multipliers on the implementability constraint, or else in terms of the excess burden, in order to understand the implied tax and debt decisions over states and dates.

Proposition 3. *("GEE with long-term debt")*

1. *Define the default-and-long-term-debt-adjusted change of measure as*

$$n_{t+1} \equiv \frac{(1-d_{t+1})\left[u_{c,t+1} + (1-\delta)Q_{t+1}\right]}{E_t(1-d_{t+1})\left[u_{c,t+1} + (1-\delta)Q_{t+1}\right]} \geq 0 \quad \textit{with} \quad E_t n_{t+1} = 1.$$

Define also the price <u>*semi-elasticity*</u> *with respect to debt as* $\eta_t \equiv \frac{\partial Q_t}{\partial B_{t+1}} \frac{1}{Q_t}$. *Then the GEE takes the form*

$$E_t n_{t+1} \Phi_{t+1} = \Phi_t \cdot \left[1 + (B_{t+1} - (1-\delta)B_t)\eta_t\right] \tag{37}$$

2. *The semi-elasticity* η_t *follows the recursion*

$$\begin{aligned} \eta_t \quad = \quad & \left[f^n(\omega^{-1}(B_{t+1})|g_t)\frac{d\omega^{-1}_{t+1}}{dB_{t+1}} + E_t n_{t+1}\frac{u_{cc,t+1} - u_{cl,t+1}}{u_{c,t+1} + (1-\delta)Q_{t+1}}\frac{\partial\mathcal{C}_{t+1}}{\partial B_{t+1}}\right] \\ & + (1-\delta)E_t n_{t+1}\frac{Q_{t+1}}{u_{c,t+1} + (1-\delta)Q_{t+1}}\frac{\partial\mathcal{K}_{t+1}}{\partial B_{t+1}}\eta_{t+1} \end{aligned} \tag{38}$$

where $f^n(g_{t+1}|g_t) \equiv n_{t+1} \cdot f(g_{t+1}|g_t)$, *the default-and-long-term-debt-adjusted conditional density. Thus,* η_t *captures the present value of the two opposing price effects,*

$$\begin{aligned} \eta_t \quad = \quad & E_t^n \sum_{i=1}^{\infty}(1-\delta)^{i-1}\left(\prod_{j=1}^{i-1}\frac{Q_{t+j}}{u_{c,t+j} + (1-\delta)Q_{t+j}} \cdot \frac{\partial\mathcal{K}_{t+j}}{\partial B_{t+j}}\right) \cdot \left[f^n(\omega^{-1}(B_{t+i})|g_{t+i-1})\frac{d\omega^{-1}_{t+i}}{dB_{t+i}}\right. \\ & \left. + \frac{u_{cc,t+i} - u_{cl,t+i}}{u_{c,t+i} + (1-\delta)Q_{t+i}}\frac{\partial\mathcal{C}_{t+i}}{\partial B_{t+i}}\right], \end{aligned}$$

where E_t^n *denotes expectation according to the default-and-long-term-debt-adjusted measure.*

Thus, with long-term debt the entire stream of the future opposing pricing effects affect the elasticity of prices with respect to debt and therefore the revenue schedule of the government– thus its decisions to postpone or not taxes for the future. Obviously, for $\delta = 1$ the above formulas reduce to the case with short-term debt and the GEE becomes the same as (31).

8 Numerical results

For our basic numerical exercise we are using a utility function with constant Frisch elasticity,

$$U = \frac{c^{1-\rho} - 1}{1-\rho} - a_h \frac{h^{1+\phi_h}}{1+\phi_h}. \tag{39}$$

At this stage we are are abstaining from simulations and merely use some standard values for the parameters in order to illustrate the main forces of the model, as captured by the policy functions for consumption, debt issuance and default. Our future goal is to have a calibration that considers particular characteristics of a country.

We set $(\beta, \phi_h, \rho) = (0.9, 1, 2)$. The labor disutility parameter is set to $a_h = 19.2901$, so that the household works 40% of its time at the first-best. For the government expenditure shocks we assume that they are i.i.d. and that they follow a uniform distribution, $g \sim U[\underline{g}, \bar{g}]$, where $\underline{g} = 0$ and $\bar{g} = 0.08$. The maximum amount of g corresponds to 20% of first-best output. In the future we are also going to use a persistent specification of shocks and perform the computation for the case of long-term debt.

The probability of re-entry is set to unity, $\alpha = 1$. We allow asymmetric default costs, i.e. it is less costly to default when government expenditures are high. We use a linear specification, $z(g) = z(\underline{g}) + \lambda_z(g - \underline{g})$, where $(z(\underline{g}), \lambda_z) = (0.98, 0.2375)$.

There are several issues with the numerical computation of the problem. These issues have to do with non-convexities which lead to discontinuous policy functions and touch upon the Markov-perfect nature of policy.

8.1 Deterministic setup: non-convexities and discontinuities

If we shut off uncertainty and the option to default, the setup reduces to a deterministic Lucas and Stokey (1983) economy where the policymaker has no commitment. The GEE (31) reduces to

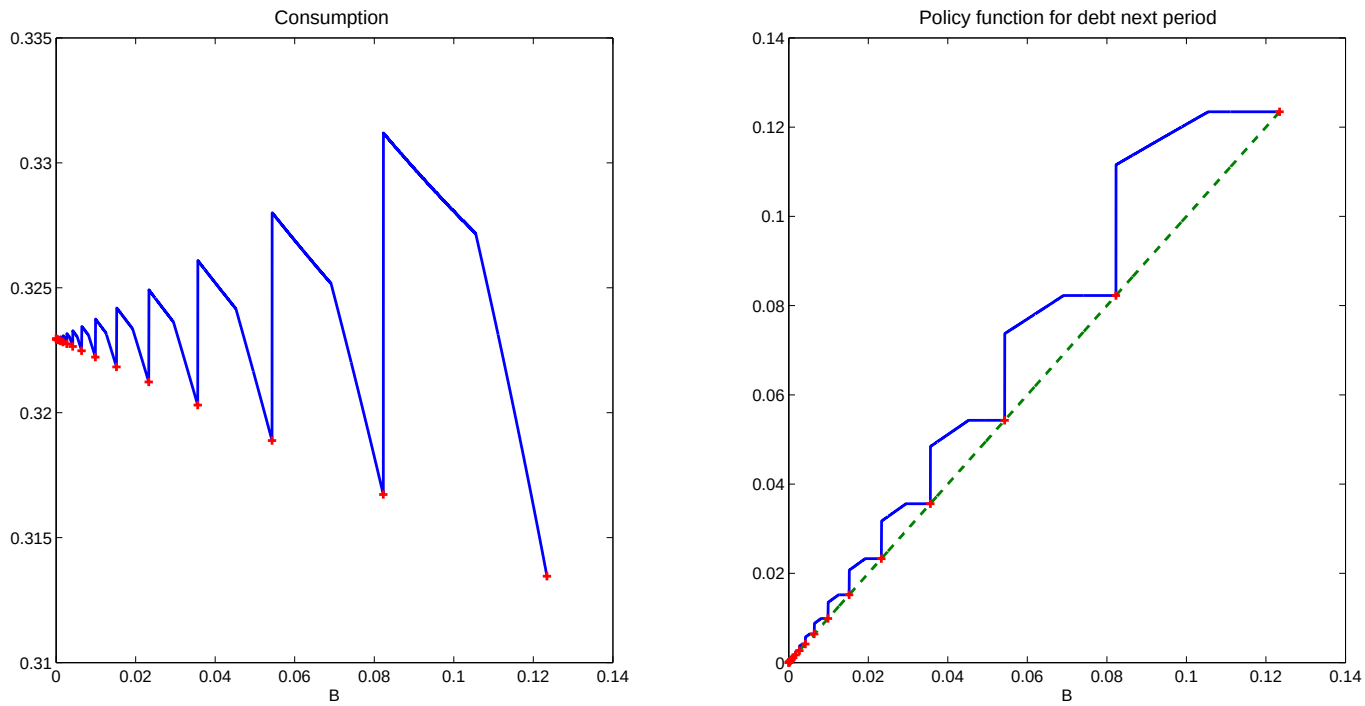

Figure 6: Consumption and debt policy function in the Krusell et al. (2004) economy. The crosses '+' correspond to steady states. At the right of each of these points, there is a jump upwards in the debt and consumption policy function.

$$\Phi_{t+1} = \Phi_t \cdot [1 + \frac{u_{cc,t+1} - u_{cl,t+1}}{u_{c,t+1}} \frac{\partial \mathcal{C}}{\partial B_{t+1}} \cdot B_{t+1}].$$

Krusell et al. (2004) have shown that the non-convexities associated with the Markov-perfect assumption in this economy introduce serious discontinuities in the policy functions for consumption and debt.[9] The GEE is then valid only locally. Figure 6 displays the severity of these discontinuities. The debt policies are continuous from the left. At each jump, if current debt increases marginally, the current planner has an incentive to issue a large amount of debt for next period, reducing a lot the interest rate (since the consumption of the *future* policymaker will jump downwards and therefore marginal utility will jump upwards) and allowing therefore to tax less currently– which is why also current consumption jumps upwards as well.

[9] Debortoli and Nunes (2013) allow government spending to provide utility, an assumption which retrieves continuity of the policy functions by altering the reaction functions of the future policymaker. See also Occhino (2012). This assumption is not necessarily useful when we are interested in analyzing the incentives to default, because with endogenous government spending there is typically a negative drift in taxes and debt. As a result, we conjecture that the government would never accumulate an amount of debt that it would default upon with some positive probability.

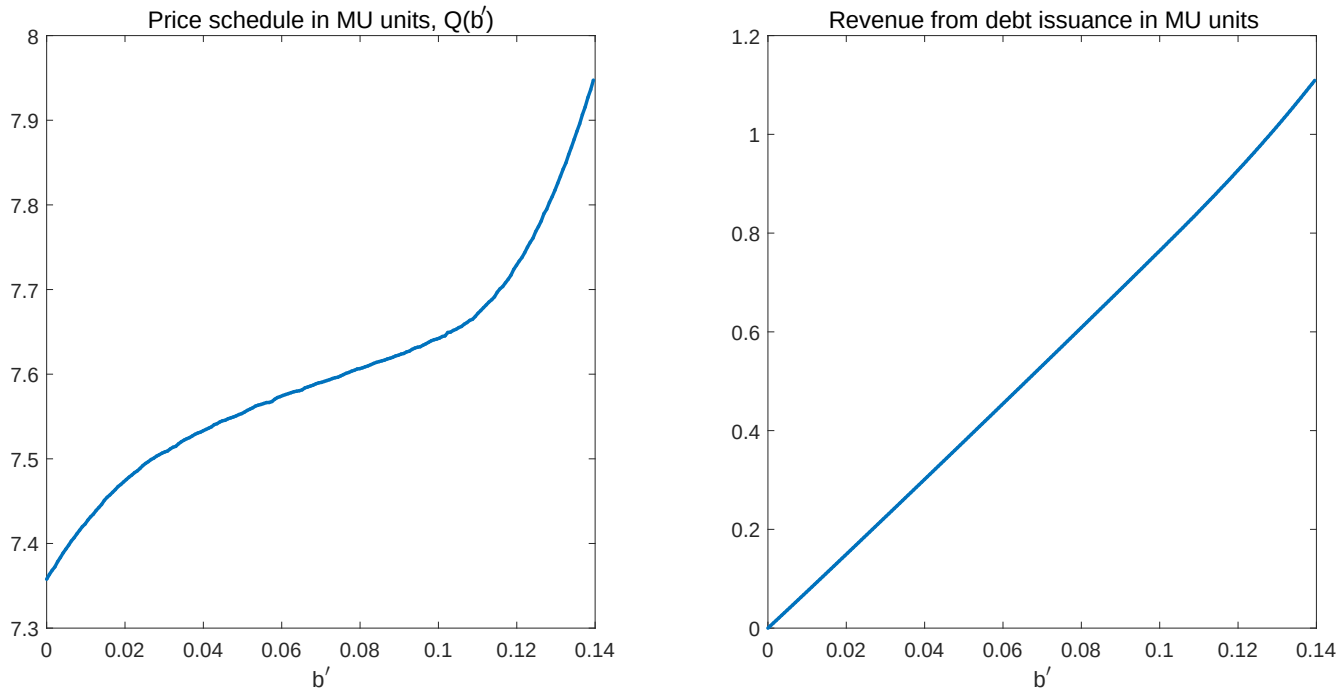

Figure 7: The left panel depicts average marginal utility ("price"), in a world with Markov-perfect policy under uncertainty and without default. The right panel plots the respective revenue from debt issuance.

8.2 Stochastic setup

When we turn into a stochastic setup, there is a possibility for elimination of the jumps in the policy functions, due to the smoothing effect of uncertainty. To see that in a heuristic way, drop for simplicity the option to default and consider the average marginal utility over the entire support of government expenditures, $E_t u_c(\mathcal{C}(B_{t+1}, g_{t+1}))$. This object captures equilibrium prices and is the source of discontinuity in the implementability constraint, through the actions of the future policymaker $\mathcal{C}$. Assume for the sake of the argument that each policy functions of consumption for next period (index by g'), is discontinuous in B'. As long as the points where the policy functions have jumps are countable and not the same across shocks g', average marginal utility will smooth out the jumps, leading to a continuous price function of debt. At a more fundamental level, uncertainty partially "convexifies" the constraint set.[10]

8.2.1 No option to default

To see the smoothing effect of uncertainty numerically consider first the Markov-perfect policy without the option to default. The GEE (whenever valid) takes the form

[10] A formal introduction of lotteries in debt would perform the same role. We are abstaining from this currently.

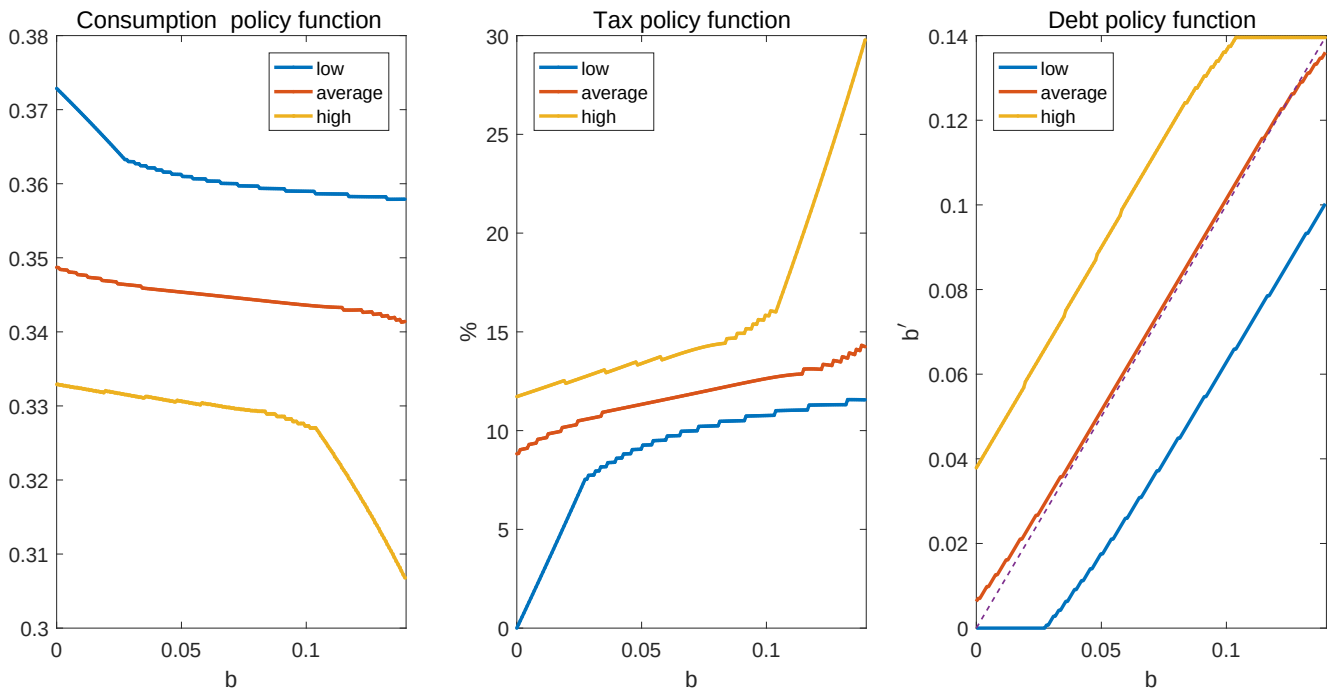

Figure 8: Policy functions for consumption, taxes and debt in a world with Markov-perfect policy under uncertainty and without default. The policy functions are drawn for three different level of government expenditures.

$$E_t x_{t+1} \Phi_{t+1} = \Phi_t \cdot \left[1 + E_t x_{t+1} \frac{u_{cc,t+1} - u_{cl,t+1}}{u_{c,t+1}} \frac{\partial \mathcal{C}}{\partial B_{t+1}} \cdot B_{t+1}\right], \quad x_{t+1} \equiv \frac{u_{c,t+1}}{E_t u_{c,t+1}}$$

In that case, we have only the *average* interest-rate effects emerging from the Markov-perfect policy assumption through $\mathcal{C}$. More debt for tomorrow is reducing the interest rate by increasing next period's average marginal utility, giving an incentive to the planner to increase average distortions tomorrow relative to today, i.e. giving an incentive to back-load tax distortions.

The left panel in figure 7 depicts the effect of Markov-perfect policy on equilibrium prices. An increase in debt for next period, reduces consumption and increases average marginal utility, reducing the interest rate. Note that uncertainty has smoothed out the jumps in policy functions, leading to a continuous and increasing pricing schedule. The right panel depicts the respective revenue from debt issuance, which, since there is not default risk, is always increasing. Figure 8 depicts the respective policy functions for consumption, taxes and debt.

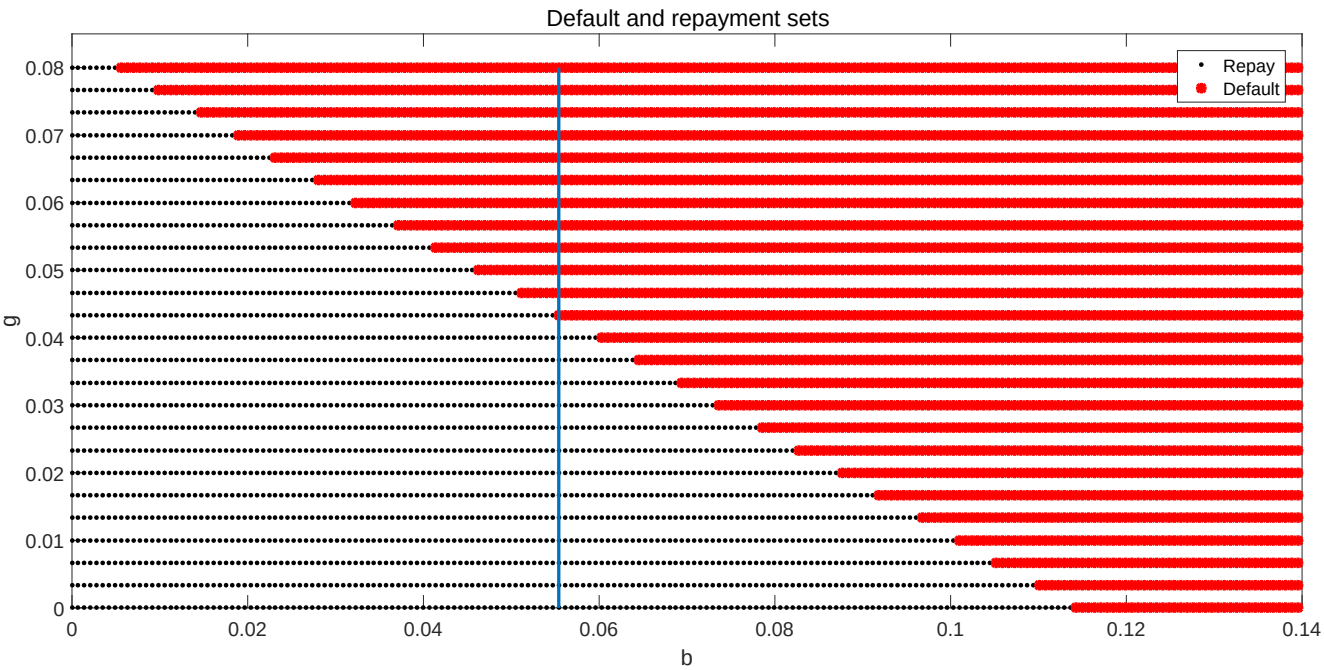

Figure 9: Default and repayment sets in the infinite horizon economy. The vertical line is plotted at the maximizer of the debt Laffer curve.

8.2.2 Default option

We turn now to the full-blown model with the option to default. Figure 9 plots the default and repayment sets. Both of our claims are valid, i.e. the default set is increasing in debt and the country defaults more in "bad" times (high government expenditure shocks). Figure 10 plots the price schedule and the debt Laffer curve in an economy with Markov-perfect policy and default. In contrast to the findings of figure 7, the price schedule starts to decrease when we enter the region where there is positive probability of default, i.e. the negative price effect of an increased default probability is larger than the positive price effect through the reaction function of the future policymaker, $\mathcal{C}$. Furthermore, the amount of debt that corresponds to the maximum of the debt Laffer curve is larger than the maximum amount of debt for which the government repays with certainty. As a result, there is a region for which there is risky debt in equilibrium.[11] Lastly, figure 11 plots the respective policy functions for consumption, taxes and debt in the event of repayment and default.

[11] See Arellano (2008) for a discussion of the importance of the maximum of the debt Laffer curve in an open economy without distortionary taxation.

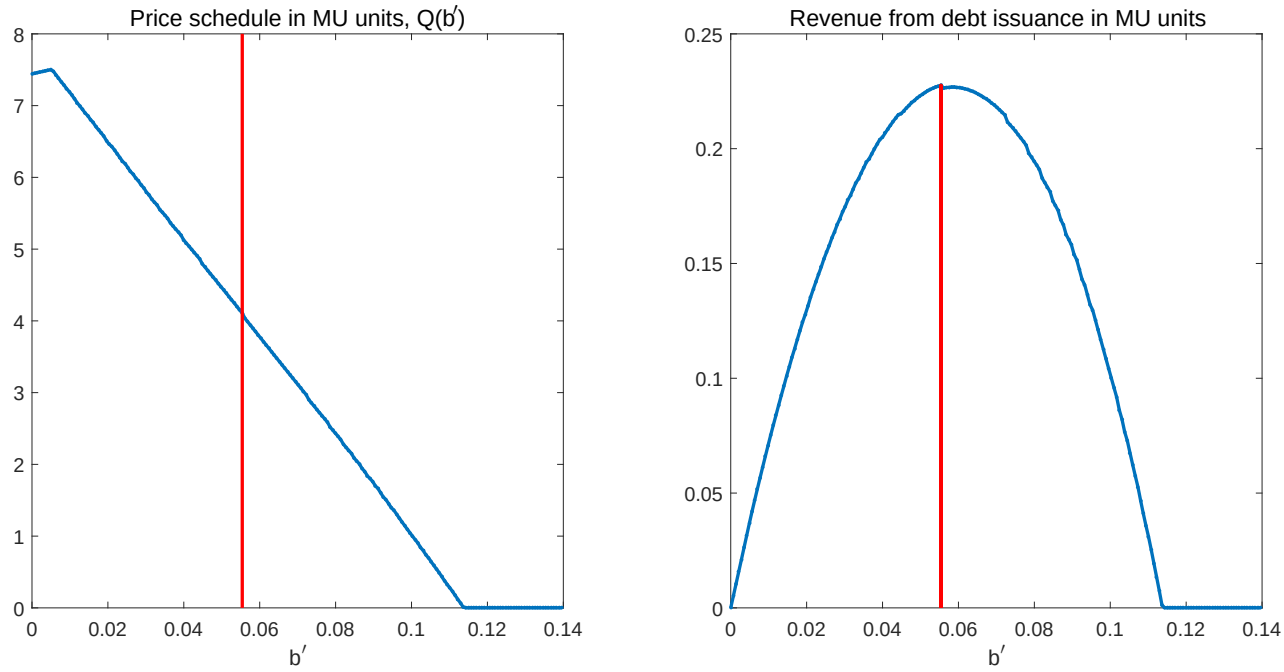

Figure 10: Price schedule and debt Laffer curve in the economy with default. Note that the price is increasing initially in debt (as in the economy without default) and then it starts decreasing when we enter the region where there is positive probability of default.

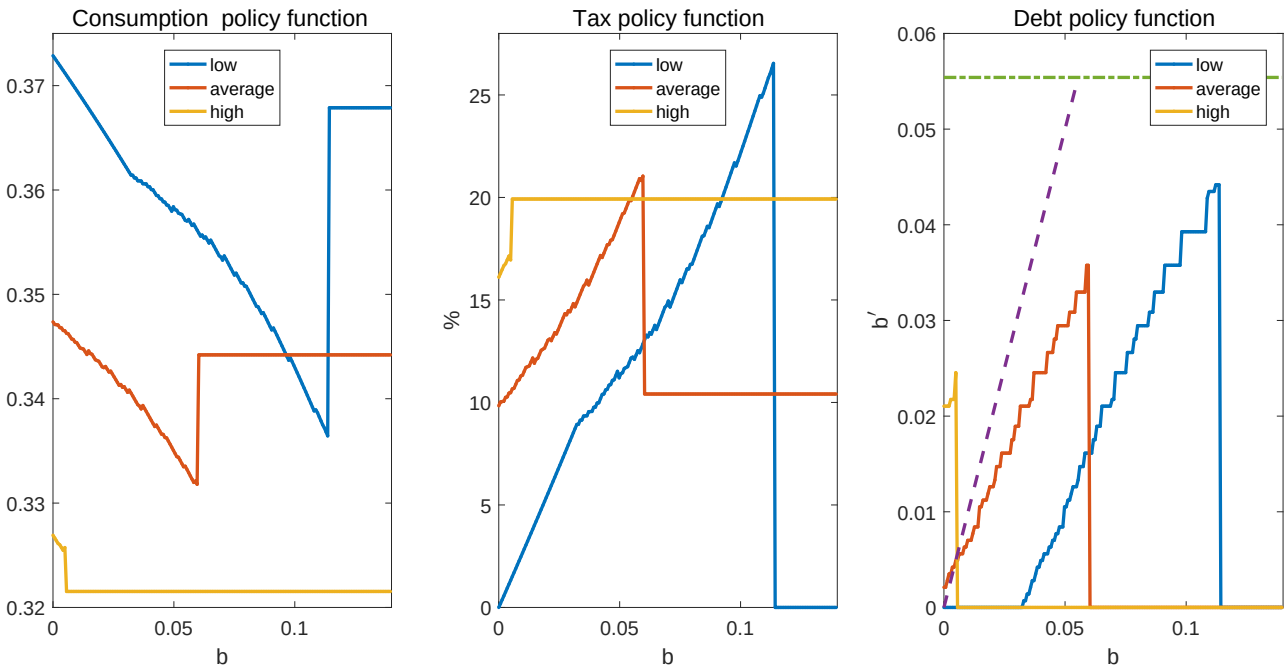

Figure 11: Policy functions for consumption, taxes and debt in the full-blown model with default. The policy functions are drawn for three different level of government expenditures.

9 Concluding remarks

In this paper we analyze optimal distortionary taxation in a setup where policymakers can commit to neither repaying debt nor to the taxation and debt scheme devised in the past. We want to understand how this double absence of commitment alters the basic tax-smoothing and debt issuance prescriptions. We are motivated by the fact that domestic sovereign default is an empirically relevant phenomenon, as Reinhart and Rogoff (2011) showed. Ultimately, we want to use our optimal policy exercise in order to evaluate various fiscal rules that have been proposed.

A Proof of proposition 3

Proof. Note that the derivative of the value function takes the form

$$\frac{\partial}{\partial B'} E_{g'|g} V(B', g') = \int_{\underline{g}}^{\omega^{-1}(B')} \frac{\partial V^r(B', g')}{\partial B'} f(g'|g) dg' = E_t(1 - d_{t+1}) \frac{\partial V^r_{t+1}}{\partial B_{t+1}}.$$

Update the envelope condition (35) one period and rewrite (34) in sequential form as

$$\beta E_t(1 - d_{t+1})[u_{c,t+1} + (1 - \delta)Q_{t+1}]\Phi_{t+1} = \Phi_t \Big[Q_t + (B_{t+1} - (1 - \delta)B_t) \frac{\partial Q_t}{\partial B_{t+1}} \Big]$$

Divide over Q_t and remember that Q_t satisfies the recursion (33). Use that fact in order to express the expectation in term of n_{t+1} to finally get (37). To get the recursion for η_t, divide (36) over Q_t and write it in sequential form as

$$\begin{aligned} \eta_t \quad = \quad & \beta \Big[\frac{\big(u_{c,t+1}(B_{t+1}, \omega^{-1}(B_{t+1})) + (1 - \delta)Q_{t+1}(\mathcal{K}(B_{t+1}, \omega^{-1}(B_{t+1})), \omega^{-1}(B_{t+1}))\big)}{Q_t} f(\omega^{-1}(B_{t+1})|g_t) \frac{d\omega^{-1}}{dB_{t+1}} \\ & + E_t \frac{(1 - d_{t+1})}{Q_t} [u_{cc,t+1} - u_{cl,t+1}] \frac{\partial \mathcal{C}}{\partial B_{t+1}} + (1 - \delta) E_t \frac{(1 - d_{t+1})}{Q_t} Q_{t+1} \frac{\partial \mathcal{K}_{t+1}}{\partial B_{t+1}} \eta_{t+1} \Big] \end{aligned}$$

Use again (33) and the definition of n_{t+1} to finally get (38). □

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