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**Stare Down the Barrel and Center the Crosshairs:**
**Targeting the Ex Ante Equity Premium**

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# Stare Down the Barrel and Center the Crosshairs: Targeting the Ex Ante Equity Premium

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**Abstract:** The equity premium of interest in theoretical models is the extra return investors anticipate when purchasing risky stock instead of risk-free debt. Unfortunately, we do not observe this ex ante premium in the data; we only observe the returns that investors actually receive ex post, after they purchase the stock and hold it over some period of time during which random economic shocks affect prices. Over the past century U.S. stocks have returned roughly 6 percent more than risk-free debt, which is higher than warranted by standard economic theory; hence the "equity premium puzzle." In this paper we devise a method to simulate the distribution from which ex post equity premia are drawn, conditional on various assumptions about investors' ex ante equity premium. Comparing statistics that arise from our simulations with key financial characteristics of the U.S. economy, including dividend yields, Sharpe ratios, and interest rates, suggests a much narrower range of plausible equity premia than has been supported to date. Our results imply that the true ex ante equity premium likely lies very close to 4 percent.

JEL classification: G12, C13, C15, C22

Key words: equity risk premium, equity premium puzzle, Monte Carlo simulation

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# Stare Down the Barrel and Center the Crosshairs: Targeting the Ex Ante Equity Premium

Over the past century the average annual return to investing in the US stock market has been roughly 6% higher than the return to investing in risk-free US T-bills. Mehra and Prescott [1985] argue that consumption within the US has not been sufficiently volatile to warrant such a large premium on risky stocks relative to riskless bonds; hence the well known "equity premium puzzle."[1] The equity premium at issue in economic theory is the premium investors anticipate *ex ante,* at the moment they first make the decision to purchase stocks instead of risk-free debt. Conversely, the premium we observe in market data is the return investors actually received *ex post*, after they have held the stock for some time and nature has buffeted the economy with its random shocks.

To examine the equity premium puzzle, we devise a method to simulate the distribution from which ex post equity premia are drawn, conditional on various values for investors' ex ante equity premium. We calibrate our approach to S&P 500 dividends and US interest rates (not stock prices or returns) and then conduct statistical tests to confirm that with investors' true ex ante equity premium as low as, say, 2%, the economy could still reasonably produce an ex post premium of, say, 6%. This is consistent with the well-known observation that ex post equity premia are observed with error, and a large range of realized equity premia are consistent with any given ex ante equity premium. Once we confirm that our simulations produce sensible results, we examine the distributions of various key financial statistics that arise in our simulations, such as dividend yields, reward-to-risk ratios, and ex post equity premia. We consider the various distributions jointly, conditional on particular values of the ex ante equity premium, and compare those conditional distributions with actual realizations from the US economy. That is, given various characteristics of the US economic experience (such as low interest rates and high ex post equity premia as well as observed Sharpe ratios and dividend yields), what values of the ex ante equity premium are most plausible? We

[1] The equity premium literature is large, continuously growing, and much too vast to fully cite here. For recent reviews see Kocherlakota [1996] and Siegel and Thaler [1997].

determine that the range of ex ante equity premia most consistent with the US market data is very close to 4%.

Like us, previous authors have investigated the extent to which ex ante considerations may impact the realized equity premium. For example, Rietz [1988] investigated the effect that the fear of a serious, but never realized, depression would have on equilibrium asset prices and equity premia. Our work is distinct from his on at least two fronts. First, Rietz studies conditions necessary to obtain an ex ante equity premium as high as 6%; conversely, we develop a method for determining the probability of observing a 6% equity premium ex post even if the ex ante premium is as low as, say, 2%. Second, Rietz assumes the possibility of a catastrophic economic state modeled on the Great Depression in order to obtain large equity premia; conversely, we calibrate to post-WWII data during which there are no catastrophic states.

Jorion and Goetzmann [1999] take the approach of comparing the US stock market's performance with stock market experiences in many other countries. They find that, while some markets such as the US and Canada have done very well over the past century, other countries have not been so fortunate; average stock market returns from 1921 to 1996 in France, Belgium and Italy, for example, are all close to zero, while countries such as Spain, Greece and Romania have experienced negative returns. Jorion and Goetzmann do not conduct statistical tests because, first, the stock indices they consider are largely contemporaneous and returns from the various indices are not independent. Statistical tests would have to take into account the panel nature of the data and explicitly model covariances across countries. Second, many countries in the comparison pool are difficult to compare directly to the United States in terms of economic history and underlying data generating processes. (Economies like Egypt and Romania, for example may have equity premia generated from data generating processes that differ from the US.) Since in our paper we simulate many independent economies with the same data generating process as the US over the past half century, we avoid both of these issues and are hence able to narrow the range of plausible ex ante equity premia.

There are some recent papers that, like our paper, make use of fundamental information in examining the equity premium. However, these studies differ from ours in that they focus on estimating the ex post equity premium, while we consider the relationship between the ex ante equity premium and various financial statistics including the ex post equity premium. One such paper, Fama and French [2002], uses historical dividend yields and other fundamental information to calculate estimates of the equity premium which are smaller than previous estimates. Fama and French obtain point estimates of the ex post equity premium ranging from 2.55% (based on dividend growth rate fundamentals) to 4.78% (based on bias-adjusted earnings growth rate fundamentals), however these estimates have large standard errors. For example, for their point estimate of 4.32% based on non-bias-adjusted earnings growth rates, a 99% confidence interval stretches from approximately -1% to about 9%. Mehra and Prescott's initially troubling estimate of 6% is easily within this confidence interval and is in fact within one standard deviation of the Fama and French point estimate.

Another paper that similarly makes use of fundamental information to form lower estimates of the ex post equity premium is Claus and Thomas [2001]. The Claus and Thomas study covers a shorter time period relative to the Fama and French study – 14 years versus 50 years – yielding point estimates that are subject to at least as much variability as the Fama and French estimates. Given the large confidence intervals around the equity premium point estimates from these studies, conducting inference about the original Mehra and Prescott equity premium puzzle is challenging, even if fundamental sources of information such as dividends and interest rates are employed in the analysis.

The remainder of our paper proceeds as follows. The basic methodology used to conduct our simulation approach to estimating equity premia is presented in Section 1 below, along with important details on estimation of equity premia and the premia used in cash-flow discounting models. (An appendix to the paper provides a more detailed exposition of the technical aspects of our simulations, including calibration of key model parameters.) In

Section 2 we compare financial statistics that arise in our simulations with US market data, including dividend yields, Sharpe ratios, and interest rates. Our results imply only ex ante equity premia in a very narrow range are consistent with the US historical experience: an ex ante equity premium as low as 3.5% is likely too low, an ex ante equity premium as high as 4.5% appears too high, while an ex ante equity premium in the close vicinity of 4% seems just right. We present evidence supporting the robustness of our findings to changes in key parameters of our model in Section 3. Section 4 concludes.

# 1 Foundations

## 1.1 Basic Methodology

As noted in the introduction, the equity premium is the extra return, or premium, that investors demand to purchase risky stock instead of risk-free debt. We call this premium the ex ante equity premium (denoted $\pi_e$), and it is formally defined as the difference between the expected return on risky assets, $\mathcal{E}\{R\}$, and the expected risk-free rate, $\mathcal{E}\{r_f\}$:[2]

$$\pi_e \equiv \mathcal{E}\{R\} - \mathcal{E}\{r_f\}. \tag{1}$$

Empirically we do not observe this ex ante premium; we only observe the returns that investors actually receive ex post, after they have purchased the stock and held it over some period of time during which random economic shocks impact prices. Hence, the ex post equity premium is typically estimated using historical equity returns and risk-free rates. Define $\overline{R}$ as the average historical annual return on the S&P 500 and $\overline{r}_f$ as the average historical return on US T-bills. Then we can calculate the estimated ex post equity premium, $\hat{\pi}_e$, as follows:

$$\hat{\pi}_e \equiv \overline{R} - \overline{r}_f. \tag{2}$$

[2]See, for instance, Mehra and Prescott [1985, Equation 14].

Given that the world almost never unfolds exactly as one expects, there is no reason to believe that the stock return we estimate ex post is exactly the same as the return investors anticipated ex ante. It is therefore difficult to argue that just because we observe a 6% ex post equity premium in the US data, the premium that investors demand ex ante is also 6% and thus a puzzling challenge to economic theory. We therefore ask the question: if investors' true ex ante premium is $X\%$, what is the probability that the US economy could randomly produce an ex post premium of at least 6%? We can then argue whether or not the 6% ex post premium observed in the US data is consistent with various ex ante premium values, $X$, with which standard economic theory may be more compatible. We then go on to consider key financial statistics and yields from the US economy to investigate if an X% ex ante equity premium could likely be consistent with the combinations that have been observed, such as high Sharpe ratios and low dividend yields, low interest rates and high ex post equity premia, and so on. The use of both univariate and multivariate distributions of these statistics allows us to narrow substantially the range of equity premia consistent with the US market data. We calibrate to US data over 1952 through 1998, with the starting year of 1952 motivated by the U.S. Federal Reserve Board's adoption of a modern monetary policy regime in 1951.

A summary of the basic methodology we employ (detailed in Appendix 1) is as follows:

(a) Assume a value for the equity premium that investors demand when they first purchase stock (*e.g.* 2%). This assumed premium, appropriately bias-adjusted as described in Appendix 1, is added to the risk-free interest rate to determine the discount rate that an investor would rationally apply to a forecasted dividend stream in order to calculate the present-value-price of dividend-paying stock.

(b) Estimate econometric models for the time-series processes driving dividends and interest rates in the US economy, allowing for autocorrelation and covariation. Then use these models to Monte Carlo simulate a variety of potential paths for US dividends and interest rates. The simulated dividend and interest rate paths are of course different in each of these simulated economies because different sequences of random innovations are applied to the common stochastic processes in each case. However, the key drivers of the simulated economies themselves are all still identical to those of the US economy since all economies share common stochastic processes fitted to US data.

(c) Given the assumed equity premium investors demand ex ante (which is the same for all simulated economies in a given experiment), use a discounted-dividend model to calculate the fundamental stock returns (and hence ex post equity premia) that arise in each simulated economy. All economies have the same ex ante equity premium, and yet all economies have different ex post equity premia. Given the returns and ex post equity premia for each economy, as well as the means of the interest rates and dividend growth rates produced for each economy, we are able to calculate various other important characteristics, like Sharpe ratios and dividend yields.

(d) Examine the distribution of ex post equity premia, interest rates, dividend growth rates, Sharpe ratios, and dividend yields that arise conditional on various values of the ex ante equity premia. Comparing the performance of the US economy with intersections of the various univariate and multivariate distributions of these quantities and conducting joint hypothesis tests allows us to determine a narrow range of equity premia consistent with the US market data.

A large literature makes use of similar techniques in many asset pricing applications. Some of these papers (directly or indirectly) simulate stock prices and dividends under various assumptions to investigate price and dividend behavior.[3] However, these studies typically employ restrictions on the dividend and discount rate processes so as to obtain prices from some variant of the Gordon [1962] model and/or some log-linear approximating framework. For instance, the present value (price, defined as $P_0$) of an infinite stream of discounted future dividends can be simplified under the Gordon model as $P_0 = D_1/(r-g)$

[3]See, for example, Scott [1985], Kleidon [1986], West [1988a,b], Campbell [1991], Gregory and Smith [1991], Mankiw, Romer and Shapiro [1991], Hodrick [1992], Timmermann [1993,1995], Donaldson and Kamstra [1996] and Campbell and Shiller [1998].

where $D_1$ is the coming dividend, $r$ is the constant discount rate, and $g$ is the constant dividend growth rate. That is, by assuming constant $r$ and $g$, one can analytically solve for the price. If discount rates or dividend growth rates are in fact time-varying, the infinite stream of discounted future dividends cannot be simplified, and it is difficult or impossible to solve prices analytically without imposing other simplifying assumptions.

Rather than employ approximations to solve our price calculations analytically, we instead simulate the dividend growth and discount rate processes directly, and evaluate the expectation through Monte Carlo integration techniques. This approach is computationally burdensome, but it is the only way to evaluate prices, returns and other financial quantities without approximation error.[4] We also take extra care to calibrate our models to the time series properties of actual data. For example, dividend growth is strongly autocorrelated in the S&P 500 stock market data, counter to the assumption of a logarithmic random walk for dividends sometimes employed for tractability in other applications. Furthermore, interest rates are autocorrelated and cross-correlated with dividend growth rates. Thus we model these properties in our simulated dividend growth rates and interest rates.

# 2 Results

## 2.1 Univariate Conditional Distributions

It is well known that the ex post equity premium is estimated with error. See, for instance, Merton [1980], Gregory and Smith [1991], and Fama and French [1997]. Any particular realization of the equity premium is drawn from a distribution, implying that given key information about the distribution (like its mean and standard deviation), one can construct a confidence interval of statistically similar values and determine whether a particular estimate is outside the confidence interval. As mentioned above, an implication of statistics in a recent paper by Fama and French [2002] is that an approximate 99% confidence interval for

[4] There is still Monte Carlo simulation error, but that is random, unlike most types of approximation error, and it can also be measured explicitly and controlled to be very small, which we have done, as explained in Appendix 1.

the equity premium spans from -1% to 9%. It is our goal to narrow this range considerably.

Consider the following: conditional on a particular value of the *ex ante* equity premium, how unusual is an observed realization of the *ex post* equity premium? We answer this question by generating distributions of ex post equity premia based on various values of the ex ante equity premium.

Table 1 reports the mean of the ex post equity premium estimates from two thousand simulated economies as well as percentiles of the distribution of ex post equity premia based on various values of the ex ante equity premium.[5] Note that we estimate the ex post equity premia over 47 years of simulated data to mirror the 47 years of annual data we have available for the S&P 500, 1952 to 1998.[6] Each row of Table 1 reports statistics corresponding to a set of simulated economies based on a particular value of the ex ante equity premium. We consider ex ante equity premium values ranging from 2% to 6%, as shown in the first column. (An equity premium of 2% is the lowest equity premium we can assume in our simulations while maintaining the ability to produce auxiliary statistics, such as dividend growth rate to discount rate ratios and dividend yields, as will be discussed below, that are at least broadly consistent with observed US data.) The next column shows the mean of the ex post equity premium estimates across the simulated economies for each case. Note that the mean ex post equity premium for the simulated data roughly equals the assumed ex ante equity premium value in each case. For instance, from the bottom row we see that with an ex ante equity premium of 6%, the mean of the simulated economies' ex post equity premium estimates is equal to 6.04%. This is one way to confirm that our simulations are producing sensible results: on average the world unfolds ex post as it is assumed it will unfold ex ante. However, the individual simulated economies randomly deviate from this average, depending

[5] In conducting the experiments described in this paper, we found that two thousand simulations were sufficient to control Monte Carlo error to be very small. This number of simulations balances the need to control Monte Carlo error with computing constraints. (Each set of 2,000 simulations we conduct takes roughly a week to run, depending on the complexity of the case being considered, and we consider dozens of cases in this paper.)

[6] Findings reported in this paper are identical when estimating the equity premium over shorter investment horizons, such as one year.

on how each simulated economy randomly unfolded. The percentiles of the distribution for the case of a 6% equity premium indicate that a 90% confidence interval (covering from the $5^{th}$ percentile to the $95^{th}$ percentile) encloses premia of roughly 2.4% to 8.6%. We also see that 1% of the economies produced ex post premia greater than about 9.2% even though the ex ante premium was only 6%. Similarly, 1% of the economies produced ex post premia less than -.1%. The median premium (the $50^{th}$ percentile) is 6.3%, revealing the slight skew in the simulated distribution of premia.

The other rows of Table 1 confirm that in each case our simulated economies produce ex post equity premia with an average equal to the value of the ex ante premium assumed. We see in the first row of the table that with an ex ante equity premium as low as 2%, a 6% ex post equity premium is not significantly unusual at the 1% level; *i.e.* a 6% ex post equity premium is less than the $99^{th}$ percentile of about 6.2% shown in the first row of the table. Other values of the ex ante equity premium between 2% and 6%, of course, are similarly consistent with an ex post equity premium of 6%. Therefore, results from Table 1 suggest, as has previous research, that the 6% premium we observe in US data may be simply a "lucky" outcome, not a true puzzle to challenge generally accepted economic theory.[7]

Panel A of Figure 1 contains probability distribution functions (PDFs) for the Table 1 rows corresponding to an assumed ex ante equity premium of 2% (indicated in Figure 1 with a plain line), 3% (line with diamonds), 4% (line with asterisks), and 6% (line with pound signs). The historic US ex post equity premium of 6% is denoted with a vertical column of dots. Consistent with the discussion of Table 1 above, a 6% ex post (estimated) equity premium is not located in the extreme tail of any of the distributions, even for the case of an ex ante (true) equity premium as low as 2%. Notice that the PDFs are skewed and include negative realizations of the ex post equity premium. Of further interest, these plots show

[7]The ex post equity premium estimated over the last 47 years of S&P 500 data is 7.6%. We focus our analysis on an ex post equity premium of 6% because 6% is the estimate most often cited in the equity premium literature, and also because 6% is the ex post equity premium estimate that emerges based on a longer (100 year) span of S&P 500 data. The use of an ex post equity premium of 7.6% in place of 6% suggests the true ex ante equity premium is slightly higher than described here (around 4.5%). Otherwise inferences are unchanged.

that the distributions change shape slightly as the premium drops, becoming fatter (larger variance, lower peaked).

Next we consider the distribution of the dividend yield (dividend divided by price, $D/P$) produced in our simulations. Recall that each of our two thousand simulated economies produced a single (47 year) estimate of the ex post equity premium, hence Table 1 and Panel A of Figure 1 conveyed the distributions of *mean* ex post equity premium estimates. In contrast, for financial statistics like the dividend yield (and several others to be described below), each simulated economy produces a set of *annual* time series observations for which we can consider higher moments. By considering not only the distribution of the mean across simulated economies, but also the distribution of the standard deviation, skewness, and kurtosis of key financial quantities produced in our simulations, we can determine with greater refinement the ability of our simulated data to match characteristics of the US economy. For instance, market returns (to be discussed below) are well known to be kurtotic. Thus, it is interesting to examine the degree to which our simulations are able to produce kurtotic returns and to look at the distribution of kurtosis across our simulated economies.

In Table 2 we present the distribution of the first four moments of the dividend yields produced in our simulations, conditional on values of the ex ante equity premium ranging from 2% to 6% as shown in the left-most column. In the column labeled "S&P 500 $D/P$" we report the first four moments of the actual annual S&P 500 dividend yields. The mean S&P 500 dividend yield is 3.789, the standard deviation is 1.06, the skew is 0.843, and the kurtosis is 3.144. (These S&P 500 moments are replicated in the rows corresponding to each assumed value of the ex ante equity premium to facilitate comparisons with the distributions of the first four moments of the *simulated* dividend yields.) The mean and percentiles of the first four moments of the simulated dividend yields are reported in the remaining columns of the table. Notice that for the 2% ex ante equity premium case, the mean S&P 500 dividend yield of 3.789 exceeds the $99^{th}$ percentile, suggesting the first moment of S&P 500 dividend yields is inconsistent with an ex ante equity premium of 2%. Similarly, for the case of the 6%

ex ante equity premium, the mean S&P 500 dividend yield is less than the $1^{st}$ percentile of the simulated distribution, suggesting the true ex ante equity premium is not likely as high as 6%. For ex ante equity premia greater than 2.5% and less than 6%, the mean S&P 500 dividend yield is between the $5^{th}$ and $95^{th}$ percentiles in each case. Each of the observed higher moments of the S&P 500 dividend yields (the standard deviation, skew, and kurtosis) lie within the 90% confidence interval implied by the $5^{th}$ and $95^{th}$ percentiles for all values of the ex ante equity premia considered. Thus moments of the S&P 500 dividend yield other than the mean do not imply bounds on plausible values of the ex ante equity premium, but Table 2 results based on the mean dividend yield suggest the true ex ante equity premium is less than 6% and greater than 2%.

The simulated mean dividend yield distributions summarized in the first row of the 2%, 3%, 4%, and 6% sections of Table 1, are plotted in Panel B of Figure 1. The 3.8% dividend yield observed over the last 47 years of S&P 500 data is represented by a column of dots, and PDFs based on the different values of the ex ante equity premium are represented by the same sets of symbols that were used in Panel A. Consistent with what we observed in Table 2, notice that the dividend yield realized in the US is in the extreme left tail of the PDF marked with pound signs (representing the distribution of mean dividend yields from simulations based on a 6% ex ante equity premium). Thus, we conclude the observed S&P 500 dividend yield is unlikely to have been produced by an economy with a true ex ante equity premium of 6%. Likewise, the observed dividend yield is also unusual relative to the PDF marked by the plain line (representing the distribution of mean dividend yields from simulations based on a 2% ex ante equity premium), suggesting the observed mean dividend yield was not likely generated by an economy in which the true ex ante equity premium is 2%. Values of the ex ante equity premium between 2% and 6% are more consistent with the observed mean dividend yield. Overall, results suggest that given the mean dividend yield observed for the S&P 500 over the past 47 years, it is implausible that the ex ante equity premium is as low as 2% or as high as 6%.

In Panel C of Figure 1 we consider distributions of simulated (nominal) arithmetic mean returns relative to the 13.4% mean return observed for the S&P 500 during 1952-1998, and in Panel D we consider return volatility. Table 3 reports the corresponding summary statistics. (PDFs in Panel C are based on select Table 3 rows labeled "Mean" and plots in Panel D are based on Table 3 rows marked "$\sigma$.") While the observed S&P 500 mean return shown in Panel C of Figure 1 is in the upper tail of some of the PDFs in Panel C, percentiles from Table 3 confirm that for all of our simulations, the observed S&P 500 return is not unusual at a 1% level of significance for any ex ante equity premium values in the range of 2% through 6%. Furthermore, the observed S&P 500 mean return is within a 90 percent confidence interval for all the simulations based on ex ante equity premia of 4% and higher. The observed standard deviation of S&P 500 returns, which is about 15%, is well within the standard deviation distributions for all the ex ante equity premium cases we consider, suggesting that we are able to closely replicate the observed mean and volatility of S&P 500 returns regardless of the ex ante equity premium value we consider. Inspection of the skewness and kurtosis rows in Table 3 confirms that higher moments of the simulated returns data are also consistent with historical S&P 500 returns for all values of the ex ante equity premium considered.

Another interesting quantity arising in our simulations is the ratio we call the discounted dividend growth rate (denoted $y$), defined in terms of the growth rate of dividends, $g_t \equiv (D_{t+1} - D_t)/D_t$, and the discount rate, $r_t$:

$$y_t = \frac{1 + g_t}{1 + r_t}.$$

As explained in Appendix 1, the present value price of a stock can be decomposed into a sum-product of expected dividend growth rates multiplied by the most recent dividend:

$$P_t = D_t \sum_{i=1}^{\infty} \mathcal{E}_t \left\{ \Pi_{k=1}^{i} y_{t+k-1} \right\}. \tag{3}$$

We calibrate our simulations to produce interest rates and dividend growth rates consistent with those we have seen in the US economy over the last 47 years; one would therefore anticipate that the discounted dividend growth rates we produce in our simulations should be consistent with the actual economy's data. Table 4 and Panel A of Figure 2 present evidence that the experiences of the US economy and the simulated economies are indeed similar. While the US realized discounted dividend dividend growth rate of 0.94 appears to be in the left tail of the plain line PDF (for the 2% ex ante equity premium case), the $5^{th}$ percentile from the appropriate row of Table 4 is 0.931. Thus, based on a 90% confidence bound implied by the simulated distribution for the 2% ex ante equity premium case, the US realization of the mean $y$ is not statistically unusual. Based on the percentiles in Table 4, we see that the US realized mean discounted dividend growth rate is within a 90% confidence interval for all values of the ex ante equity premium we consider. With regard to higher moments, the US realized standard deviation, skewness and kurtosis are nowhere statistically unusual according to the simulated percentiles. It is clear from that the sample moments of the discounted dividend growth rates observed over recent history in the US are consistent with a broad range of possible ex ante equity premia.

In Panel B of Figure 2 and in Table 5, we consider the distribution of the Sharpe ratio (or reward-to-risk ratio, calculated as the average annual difference between the arithmetic return and the risk-free rate divided by the standard deviation of the annual differences). As with the ex post equity premium, each of our 2,000 simulated economies produces a single (47 year) estimate of the Sharpe ratio, hence Table 4 conveys the distribution of only the mean Sharpe ratio, and not higher moments. The observed Sharpe ratio of 0.501 for the S&P 500 is nowhere unusual at conventional levels of significance in each case shown in Table 4 and in each panel plotted in Panel B.

In Panel C of Figure 2, we consider distributions of the first order autocorrelation coefficient estimate on returns (obtained from the regression of returns on lagged returns) for various values of the ex ante equity premium. The distribution is very similar across simulations, and the actual autocorrelation coefficient of -.134, based on the last 47 years of S&P 500 return data, is not statistically unusual in any case. In Table 6, the first of each set of two rows provides the mean observed S&P 500 autocorrelation coefficient as well the mean and percentiles of the distribution of simulated return autocorrelation coefficients (corresponding to the PDFs shown in Panel C). The second row in each set reports the standard deviation of the autocorrelation coefficient estimated on S&P 500 returns and the mean and distribution percentiles for the standard deviation of the autocorrelation coefficients from the simulated economies. The distributions of the standard deviations are very similar across values of the ex ante equity premium, and in no case is the observed S&P 500 standard deviation statistically unusual.[8]

The final set of univariate distributions we consider is shown in Panel D of Figure 2 for the first order autoregressive conditional heteroskedasticity (ARCH) coefficient, obtained from the regression of squared residuals on lagged squared residuals. Once again, the PDFs are very similar across values of the ex ante equity premium, and the actual S&P 500 value of 0.25 is not statistically unusual in any case. In Table 7, the first row among each set of two provides the value of the ARCH coefficient based on S&P 500 returns as well as the mean and percentiles of the ARCH coefficients based on the simulated data. The second row in each set of rows contains the standard deviation of the ARCH coefficient estimate based on the S&P 500 data as well as the mean and percentiles of the ARCH estimates' standard deviations based on simulated data. Again, the standard deviation distributions are similar across ex ante equity premia, and in no case do we find evidence that the S&P 500 estimates are statistically unusual.

Collectively, the panels of Figures 1 and 2 and Tables 1 through 7 suggest that the true ex ante equity premium consistent with the US economy is likely greater than 2% and less than 6%. In some cases we found that US observed values (such as the mean dividend yield,

[8]Coincidentally, the mean and median standard deviation of the simulated AR(1) coefficients is for all cases identical (to three decimals) to the standard deviation of the AR(1) coefficient estimated using the S&P 500 returns.

at a 1% level of significance) were inconsistent with ex ante equity premia as high as 6%. In other cases we found that US observed values were not likely given an ex ante equity premium as low as 2% (such as the mean return, at a 1% level of significance). We will see below that with additional examination of the statistics produced in our simulations, we can further and substantially narrow the range of plausible ex ante equity premium values.

## 2.2 Multivariate Conditional Distributions

Having considered the plausible range of ex ante equity premia implied by univariate distributions of various financial statistics produced in our simulations, we now consider the range implied by the joint distributions. Because our simulations produce returns, ex post equity premia, Sharpe ratios, dividend yields, *etc.* based on a series of simulated dividends and interest rates calibrated to US data, we can consider the joint distributions of these quantities that arise in our simulations (once again conditional on various values of the ex ante equity premium). In Section 3 we document the robustness of our results to changes in the basic parameters upon which our simulations are based.

In Panel A of Table 8, we present $\chi^2$ statistics for testing the hypothesis that simulated distributions of various financial quantities are jointly consistent with values observed in the US. We present test statistics for simulations based on values of the ex ante equity premium ranging from 2% to 6%. The column labeled Case 1 reports statistics pertaining to the joint distribution of the mean return, return standard deviation, mean dividend yield, ex post equity premium, AR(1) coefficient estimate for returns, and ARCH(1) coefficient estimate for returns.[9] For our joint tests, we do not consider variables which can be derived directly from these variables (such as the Sharpe ratio which is a function of mean returns, interest rates, and the return standard deviation). Of course, we also do not consider the financial

[9]More precisely, the $\chi^2$ tests are based on joint normality of (in some cases, joint normality of simple *transformations* of) sample estimates of moments of the simulated data, which follow an asymptotic normal distribution based on a law of large numbers (see White [1984] for details). For instance, we consider the mean *logarithm* of gross returns and the mean *logarithm* of the dividend yield, both of which more closely follow an asymptotic normal than the mean return or mean dividend yield. Similarly, we consider the *cube root* of the return variance which is approximately normally distributed (see Kendall and Stuart [1976, page 399] for further details).

variables to which we calibrate our simulations (interest rates and dividend growth rates), as the simulated mean, variance, and covariance of these variables are, by construction, identical to the corresponding moments of the actual data to which we calibrate. The column labeled Case 2 reports statistics pertaining to joint distributions of the same set of variables as Case 1 with the exception of the AR(1) and ARCH(1) coefficients. We saw that the simulated AR(1) and ARCH(1) distributions are consistent with the observed S&P 500 data in every case we consider, and hence one might expect that excluding these variables in our test might increase its power.

A significant $\chi^2$ test statistic, in this context, suggests that the combination of financial statistics observed for the US economy is significantly different from the joint collection of simulated data. One, two, and three asterisks indicate significance at the 10%, 5%, and 1% level respectively. Among the test statistics shown in Panel A (results in other panels are discussed in Section 3), we only observe insignificant values for the 4% ex ante equity premium (both Cases 1 and 2). For all other values of the ex ante equity premium, 2%, 2.5%, 3%, 3.5%, 4.5%, 5%, and 6%, we observe very strong rejections (significant at the 1% level in each case) of the hypothesis that the US realizations of mean dividend yield, ex post equity premium, *etc.* are consistent with the simulated economies. That is, only when the ex ante equity premium is 4% in our simulations are we able to match the joint realization of returns, return volatility, dividend yield, ex post equity premium, return autocorrelation, and degree of ARCH. Thus we conclude that the true value of the ex ante equity lies in the close vicinity of 4%, but not as high as 4.5% or as low as 3.5%.

To motivate the intuition behind the results of the joint tests, we provide bivariate plots of the simulated data, conditional on various values of the ex ante equity premium. Obviously we cannot construct a four-or-higher-dimensional plot of the financial variables that interest us. Three-dimensional plots, while feasible to construct, are too dense to be understandable. Thus we consider successive bivariate combinations. Because the univariate plots discussed in the previous section established that the plausible range of ex ante equity premia lies somewhere between 2% and 6%, and because the joint tests shown in Panel A of Figure 8 suggest values very close to 4%, we show bivariate plots based on ex ante equity premia of 3.5%, 4%, 4.5% and 5%.

In every case, the pair of statistics we plot are dependent on each other in some way, allowing us to make interesting conditional statements. Among the bivariate distributions we consider, we will see some that serve primarily to confirm the ability of our simulations to produce the character and diversity of results observed in US markets. Some sets of figures will rule out ex ante equity premia as low as 3.5% while others will rule out ex ante equity premia as high as 4.5% or 6%. Viewed collectively, the figures serve to confirm that the range of ex ante equity premia consistent with US market data is in the close vicinity of 4%, just as we saw with our $\chi^2$ tests.

Consider first Figure 3 which reports joint distributions of mean returns and return standard deviations arising in our simulations based on four particular values of the ex ante equity premium (3.5% in Panel A, 4% in Panel B, 4.5% in Panel C, and 5% in Panel D). Each panel contains a scatterplot of two thousand points, each point representing a pair of statistics (mean return versus return standard deviation) arising in one of the simulated 47-year economies. The combination based on the US realization over the 47-year period 1952-1998 is shown in each plot with crosshairs (solid straight lines with the intersection marked by a solid dot). The set of simulated pairs in each panel is surrounded by an ellipse which represents a 99% bivariate confidence bound, based on the asymptotic normality (or log-normality, where appropriate) of the variables plotted.[10] The confidence ellipses for the 3.5% case is marked with stars, the 4% case with asterisks, the 4.5% case with circles and

[10]The 99% confidence ellipsoids are asymptotic approximations based on joint normality of the sample estimates of the moments of the simulated data. Consistent with our construction of the $\chi^2$ tests reported above, for some of the variables (return mean, return standard deviation, dividend yield, and interest rate mean) the moment used to construct the ellipsoid was the mean of a transformation of the data: the logarithm of the gross return, the logarithm of the interest rate, the logarithm of the dividend yield and the cube root of the variance of the return. (As stated previously, the cube root of a $\chi^2$ random variable is approximately normally distributed. See Kendall and Stuart [1976, page 399] for details.) All of the sample moment estimates we consider are asymptotically normally distributed, as can be seen by appealing to the appropriate law of large numbers. See White [1984] for further details.

the 5% case with triangles. Notice that the intersection of the crosshairs is within a 99% confidence ellipse in all cases except that of the 3.5% ex ante equity premium. That is, our simulations produce mean returns and return volatility that roughly match the US observed moments of returns (without our having calibrated to returns), but based on this set of plots, we can conclude that ex ante equity premia less than or equal to 3.5% are inconsistent with the observed mean return and return volatility of S&P 500 returns.[11]

We can easily condense the information contained in these four individual plots into one plot, as shown in Panel A of Figure 4. The scatterplot of points representing individual simulations are omitted in the condensed plot, but the confidence ellipses themselves (and the symbols used to distinguish between them) are retained, with the 4% ellipse now indicated in bold. As with Figure 3, we can tell by comparing the confidence ellipses with the crosshairs representing the S&P 500 return and standard deviation combination that only the 3.5% ex ante equity premium case is rejected at the 1% significance level. In presenting results for additional bivariate combinations, we follow the same practice, omitting the points that represent individual simulations, using the same set of symbols to distinguish between confidence ellipses based on ex ante equity premia of 3.5%, 4%, 4.5%, and 5%, and indicating the 4% ellipse in bold.[12]

In Panel B of Figure 4 we consider the four sets of confidence ellipses for mean return and mean dividend yield combinations. Notice that as we increase the ex ante equity premium, the confidence ellipses shift upward and to the right. Notice also that with higher values of the ex ante equity premium we tend to have more variable dividend yields. That is, the confidence ellipse covers a larger range of dividend yields when the value of the ex ante equity premium is larger. The observed combination of S&P 500 mean return and mean dividend

[11]Consistent with the $\chi^2$ tests presented above, plots of the bivariate distribution of the mean and standard deviation of returns for ex ante equity premia below 3.5% show the distributions shifting down and flattening out, resulting in increasing deviations of the bivariate distribution from the S&P 500 crosshairs.

[12]A detailed supplement available at www.markkamstra.com provides plots that are based on *all* the values of the ex ante equity premium we consider in this paper: 2%, 2.5%, 3%, 3.5%, 4%, 4.5%, 5%, and 6%. The plots in that supplement show the scatterplot of points that represent individual simulations as well as the 99% confidence ellipses.

yield, represented by the intersecting crosshairs, lies within the confidence ellipse for all the cases shown, though it is close to the edge for some cases. That is, at the 1% significance level, we cannot reject any of these ex ante equity premium values on the basis of only these two variables.

Panel C of Figure 4 plots confidence ellipses for mean interest rates versus mean ex post equity premia. The intersection of the crosshairs is within all four of the confidence ellipses shown. Notice as well that the confidence ellipses are all negatively sloped: we see high interest rates with low equity premia and low interest rates with high equity premia. Many researchers, including Weil [1989], have commented that the flip side of the high equity premium puzzle is the low risk-free rate puzzle. Here we confirm that the dual "puzzle" arises in our simulated economies as well.

Panel D of Figure 4 contains the confidence ellipses for paired mean interest rates and dividend yields, showing a strong linear relationship between the variables. Note that the range of interest rates and dividend yields in this figure is truncated, going no higher than 6% for dividend yields and no higher than 15% for interest rates. We truncate the range of this figure to focus attention on the crosshairs as the ellipses are very narrow for this combination of variables. Considering the ellipses around the joint distribution of interest rates and dividend yields, only when the ex ante equity premium is 4% (for the ellipse marked in bold with asterisks) do we ever see combinations like that realized in the US market data. The intersection of the crosshairs falls far outside the 99% confidence ellipses for all the other cases shown, and we can deduce that the distribution would flip from one side of the market pair to the other when the ex ante equity premium is just above 4%. This is our starkest result among the bivariate plots, sharply narrowing the range of plausible ex ante equity premia to lie around 4%.[13]

[13]The strength of this relationship, and its linearity, bear some comment. Consider the Gordon [1962] growth model which expresses share price, $P$, as a simple function of dividends ($D$), discount rates ($r$) and the dividend growth rate ($g$) when $r$ and $g$ are constant: $P = D/(r-g)$. This expression can be re-arranged as $D/P = r - g$, revealing a positive linear relationship between dividend yield and interest rates, just as we see in Panel D of Figure 4. This suggests that the Gordon model holds approximately, on average, even when discount rates $r$ and dividend growth rates $g$ are not constant. (Recall that we model both $r$

In Panel A of Figure 5, the Sharpe ratio (or reward-to-risk ratio, calculated as the average annual difference between the arithmetic return and the risk-free rate divided by the standard deviation of the annual differences) is plotted against the mean dividend yield. As the ex ante equity premium is increased from 3.5%, the confidence ellipses shift from the left of the intersected crosshairs to the right. The crosshairs intersection, at a Sharpe ratio of 0.5 and a mean dividend yield of about 4%, is well outside the 99% confidence ellipse for the 3.5% ex ante equity premium case, suggesting a 3.5% ex ante equity premium is inconsistent with the jointly observed S&P 500 Sharpe ratio and mean dividend yield. Indeed Fama and French [2002] and Jagannathan, McGrattan, and Scherbina [2001] make reference to dividend yields to argue that the equity premium may be much smaller than 6%; our analysis gives us a glimpse of just how much smaller it might be.

In Panel B of Figure 5, we consider Sharpe ratio and mean interest rate pairs. The crosshair intersection is within a 99% confidence ellipse in all cases shown, though it is right on the edge for the 3.5% ex ante equity premium case.

In Panel C, mean dividend yields are plotted against mean ex post equity premia. We notice that as the ex ante equity premium rises from 3.5% to 5%, the confidence ellipses shift upward and to the right. Only for the 4%, 4.5%, and 5% cases do we find the intersected crosshairs lie within the 99% confidence ellipses.

In Panel D we consider the joint distribution of first order autoregressive coefficients (from regressing returns on lagged returns) versus first order ARCH coefficients (from regressing squared regression errors on lagged squared errors). Note that our simulations are able to replicate the autoregressive and ARCH properties of observed S&P 500 returns without having calibrated to returns. For all four values of the ex ante equity premium considered here, the observed AR and ARCH coefficients on S&P 500 returns are well within a 99% confidence ellipse.

Among the panels of Figures 4 and 5, there are some cases that serve primarily to confirm

and $g$ as autocorrelated.) That is, the Gordon model can be interpreted as an unconditional version of our present-value expression, Equation (3).

that our simulations produce sensible outcomes for returns, dividend yields, *etc.*, which we hope inspires confidence in the method and calibration of our exercise. Many of scatterplots are more illuminating in that they imply boundaries on the plausible set of ex ante equity premia; that is, ex ante equity premia which imply returns, dividend yields, *etc.* consistent with what has been observed in the US economy over the past 47 years. The joint realization of key characteristics of the US market data suggests that the true ex ante equity premium is no lower than 3.5%, no higher than 4.5%, and is most likely near or very slightly above 4%. These findings are consistent with the $\chi^2$ tests presented earlier, which showed that only the joint distributions of simulated mean dividend yields, ex post equity premia, *etc.* based on an ex ante equity premium of 4% are consistent with the observed US data.

# 3 Sensitivity Analysis

A reasonable question is whether our results are sensitive to the values of fundamental parameters or the assumptions upon which our simulations are based. Thus, we conducted extensive robustness checks. The three types of sensitivity checks we performed (described in detail below) are (1) changing the degree of autocorrelation in the fundamental variables upon which our simulations are based, interest rates and dividend growth rates, (2) restricting the processes driving interest rates and dividend growth rates to be independent and identically distributed instead of calibrating to the time series properties observed in practice, and (3) allowing for time-varying equity premia. We find that small changes in parameters leads to small changes in our findings, while large changes (like modeling interest rates and dividend growth rates as iid rather than autocorrelated and cross-correlated) leads to large changes in our results.

## 3.1 Sensitivity to Degree of Autocorrelation in $r_t$ and $g_t$

As described in Appendix 1, the entire set of variables we consider in this study are generated by specifying processes for only the interest rate and the dividend growth rate. Each of these two quantities is calibrated to follow the degree of autoregression observed for these series in the US economy. We study the time-series properties of historic annual dividend growth rates from the S&P 500 and annual US T-bill rates, and we determine that dividend growth rates follow a first order moving average process, MA(1), while interest rates follow a first order autoregressive process, AR(1). The coefficient in each of these processes is set to match the mean of the observed process. For our first sensitivity check, we consider the impact of both increasing and decreasing the magnitude of the MA(1) and AR(1) coefficients by as much as one standard deviation of the estimated coefficients.

Figures 6 and 7 contain confidence ellipses (around joint distributions of the same financial variables considered in Figures 4 and 5) for simulations based on various changes in the degree of autoregression in the interest rate and dividend growth rate processes. We base all the sensitivity tests on a 4% ex ante equity premia since the analysis in Section 2 suggests the true ex ante equity premium lies very near that value.[14] In each panel we plot four confidence ellipses. Confidence ellipses marked with a star correspond to simulations in which the MA(1) coefficient for dividend growth rates has been reduced by one standard deviation. Confidence ellipses marked with an asterisk are for the case where the MA(1) coefficient on the dividend growth rate process has been increased by one standard deviation. The confidence ellipses indicated with circles are based on reducing both the MA(1) coefficient for dividend growth rates and the AR(1) coefficient for interest rates by one standard deviation. For the case marked by a triangle, both coefficients have been increased (the MA(1) coefficient is increased by one standard deviation, but the AR(1) coefficient is increased by only half a standard deviation, as it borders on a unit root process).

Examination of the panels in Figures 6 and 7 reveals that the simulation results are fairly robust to the degree of autocorrelation in the dividend growth rates. That is, the actual US market data indicated by the crosshairs remain broadly consistent with the joint distributions with a few exceptions. In Panels A, B, and D of Figure 6, we see that the crosshairs are not

[14]Sensitivity results based on a 3.5% or 4.5% ex ante equity premium are very similar, though obviously less supportive of those respective premia as plausible values for the true ex ante equity premium.

encompassed by the 99% confidence ellipses with circles, corresponding to reducing both the MA(1) and AR(1) coefficients at the same time. In Panel D of Figure 6, increasing both the MA(1) and AR(1) coefficients at the same time places the realized mean US interest rate and mean S&P 500 dividend yield outside the 99% confidence ellipse (denoted by triangles). In the remainder of the panels of Figures 6 and 7, (a total of 28 of the 32 cases plotted in Figures 6 and 7) we still find a 4% ex ante equity premium is consistent with the realized US outcomes, suggesting our findings are fairly robust to changes in the time series processes for interest rates and dividend growth rates underlying our simulations.

Joint tests reported in Panel B of Table 8 allow us to make more formal statements about the consistency of these joint distributions with observed US data. Recall that the column labeled Case 1 reports $\chi^2$ test statistics pertaining to the joint distribution of the mean return, return standard deviation, mean dividend yield, ex post equity premium, AR(1) coefficient estimate for returns, and ARCH(1) coefficient estimate for returns, while the column labeled Case 2 reports statistics pertaining to joint distributions of the same set of variables with the exception of the AR(1) and ARCH(1) coefficients. The test statistics are mostly insignificant, consistent with our previous statement that the results are fairly robust to changes in the degree of autocorrelation used for our simulations. We only find significant $\chi^2$ statistics for the simulations where we either reduced or increased the degree of autocorrelation in *both* the dividend growth rate and the interest rate.

## 3.2 Sensitivity to iid $r_t$ and $g_t$

Next, we explore the impact of failing to calibrate the dividend growth rate and interest rate processes to observed US data. We do so by modeling $g_t$ and $r_t$ as independently and identically distributed (iid) processes (*i.e.* not autocorrelated and not cross-correlated with each other).[15] As we show below, under such restrictive conditions the simulated distributions are almost always at odds with data from the US economy. Figures 8 and 9

[15]Sensitivity checks based instead on *constant* dividend growth rates and *constant* interest rates yield scatterplots that are very similar to the iid cases depicted in Figures 8 and 9.

contain scatterplots (and confidence ellipses in cases where they are not degenerate) for simulations based on iid dividend growth rates and interest rates. Once again the ex ante equity premium is set to 4%. Scanning through the set of figures, not only is it clear that the US market data are typically at odds with the joint distributions, but it is also obvious that the simulated data lose the richness apparent in the simulations that capture the degree of autocorrelation and cross-correlation found in actual interest rates and dividend growth rates. Dividend yields become virtually constant (see Panels B and D of Figure 8 and Panels A and C of Figure 9) whereas in reality they vary substantially over time and across exchanges. In Panel C of Figure 8 we notice that the joint distribution of interest rates and ex post equity premia becomes more condensed, and in fact the duality of low interest rates and high equity premia that we observed in our previous figures (and in market data) is no longer as striking in these simulations as they were previously, though we still see a negative relationship between ex post equity premia and interest rates. More formally, the joint test statistics reported in Panel C of Table 8 are strongly significant (for all values of the ex ante equity premia we consider),[16] confirming the notion that modeling $g_t$ and $r_t$ as iid processes is inconsistent with the US data.

### 3.3 Sensitivity to Time-Varying Equity Premia

Finally, we consider the implications of allowing equity premia to vary over time. A recent strand of the equity premium puzzle literature has been to observe that equity premium estimates have been falling in recent years. (See Claus and Thomas [2001], Jagannathan, McGrattan, and Scherbina [2001] and Fama and French [2002], among others.) We should note that stock market returns are sufficiently volatile that estimates of the equity premium over short time periods are extremely unreliable. However, to the extent that the equity premium *does* time-vary, we seek to verify the robustness of our results.

[16] We conducted simulations based on iid interest rate and dividend growth rates only for the values of the ex ante equity premia shown in Panel C of Table 8: 2%, 3%, 4%, 4.5%, and 6%. It is reasonable to assume that findings would be similar for intermediary values of the ex ante equity premia considered elsewhere in the paper, including 2.5%, 3.5%, and 5%.

We consider four different model types for producing annual ex ante equity premia that vary over time. All four types are somehow linked to annual equity premia estimated over 1952 to 1998 based on a CAPM model which is fully explained in Appendix 2. In all four models, by working with logs, the time varying equity premia we generate have a lower bound of 0%. Further, the four models we adopt allow the ex ante equity premium to evolve as a time series process which covaries with the interest rate and dividend growth rate processes we use in this paper. For the first model, Type 1, we estimate a time series process (which turns out to be AR(1)) for the log of the estimated equity premia that emerge from the CAPM model explained in Appendix 2. We use this time series model to simulate annual time-varying ex ante equity premia. We set the mean ex ante equity premium to 4%, consistent with our finding that the most plausible value for the ex ante equity premium is around 4%.

The second time-varying equity premium model we consider, Type 2, is also based on the model of log equity premia used for Type 1, but the Type 2 time-varying equity premia also incorporate a downward time-trend. Although the Type 2 overall average ex ante equity premium is still set to 4%, the mean premium is set to 6% at the beginning of the 47-year simulation period and trends downward linearly to a mean of 2% at the end of the 47-year simulation period.[17]

The third time-varying equity premium model we consider, Type 3, models the annual ex ante equity premium as a log-normal random variable (instead of an AR(1) process) with a 4% mean and no time trend. The standard deviation for the log-normal process is based on the equity premium estimates that emerge from the CAPM model explained in Appendix 2. Type 4 also models the ex ante equity premium as a log normal random variable, with the additional feature of a downward time-trend. Although the Type 4 mean ex ante equity premium is still 4%, the mean premium trends down linearly from 6% to 2% over the course of the 47-year simulation periods.

[17]While the time trend in the ex ante equity premium in this case is modeled as deterministic, the realized ex post equity premium is of course still random.

Results based on all four types of time-varying equity premia are shown in Figures 10 and 11. Consider the 99% confidence ellipse marked with asterisks and the ellipse marked with stars, corresponding to the AR(1) model and the log-normal model respectively (both without time trend), *i.e.* corresponding to Types 1 and 3 described above. The crosshairs representing US market data are within these two sets of confidence ellipses almost without exception. In contrast, the confidence ellipses marked with circles and triangles corresponding to the two downward time-trend models (Types 2 and 4) frequently do not encompass the crosshairs. Thus, while our results are robust to some forms of time-varying equity premia, they are not robust to time-varying equity premia that incorporate a deterministic downward trend. One way to understand this finding is as follows. Our earlier collection of results based on non-time-varying equity premia suggested that ex ante equity premia in the range of 4% are best supported by the actual US data. With time-varying ex ante equity premia that follow a downward trend from 6% to 2%, the simulated data deviate from the 4% ex ante equity premium for much of the 47 year period we simulate, leading to confidence ellipses that deviate from the 4% cases seen earlier. This suggests a linear time trend in the equity premium is not consistent with the US data, though it does not necessarily rule out other sorts of trending equity premia.

Formal $\chi^2$ tests are shown in Panel D of Table 8. Simulations based on both types of time-varying equity premia that incorporate a downward trend (Types 2 and 4) yield significant test statistics, suggesting these scenarios are inconsistent with observed US data. Simulations based on time-varying equity premia Types 1 and 3, however, are more consistent with the observed US data, as three of the four $\chi^2$ statistics are insignificant for model Types 1 and 3.

Overall, the sensitivity checks considered in this section based on alternate specifications for interest rates and dividend growth rates and based on time-varying processes for the ex ante equity premium make it clear that accounting for autocorrelation doesn't merely make our scatterplots noisier or more variable, coincidentally allowing them to envelope realizations of the US economy. Rather, calibrating the interest rates and dividend growth rates produced in our simulations to the degree of autocorrelation observed in the market data leads to joint distributions of important financial statistics with characteristics consistent with what is observed empirically (like simultaneously low interest rates and high equity premia but not the reverse, for instance). Furthermore, the range of plausible ex ante equity premia implied by the simulations are remarkably robust to minor modifications in the parameter values upon which the simulations are based as well as to some forms of time-varying equity premia.

# 4 Conclusions

The equity premium of interest in theoretical models is the extra return investors anticipate when purchasing risky stock instead of risk-free debt. Unfortunately, we do not observe this *ex ante* premium in the data; we only observe the returns that investors actually receive *ex post*, after they purchase the stock and hold it over some period of time during which random economic shocks impact prices. US stocks have historically returned roughly 6% more than risk-free debt, which is higher than warranted by standard economic theory; hence the equity premium puzzle.

In this paper we have devised a method to simulate the distribution from which ex post equity premia are drawn, conditional on various values for investors' ex ante equity premium and calibrated to fundamentals of the US economy. Even though ex post estimates provided by recent papers suggest the US equity premium may be falling in recent years, these estimates are imprecise and do not rule out puzzlingly high ex ante equity premia. We have therefore sought to determine whether realized financial statistics from the US economy are consistent with various settings of the ex ante equity premium. That is, if investors demand (ex ante) a particular equity premium, could we still observe combinations of Sharpe ratios, dividend yields, interest rates, and ex post equity premia observed in practice? Not only do we find that our choice of models and parameters are consistent with the range of financial statistics observed in market data, but our results are surprisingly robust to changes in the parameter values underlying our study. On the basis of our fundamentals-based analysis, we conclude that the most plausible range of ex ante equity premia is in a very narrow vicinity of 4%, in contrast to previous empirical work which could not reliably rule out premia as low as 0% or as high as 9%.

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# Appendices

## Appendix 1: Technical Details on the Simulations

### A1.1 Fundamentals

In creating distributions of financial variables modeled on the US economy, we start by generating the fundamental factors that drive asset prices: dividends and discount rates. We define $P_t$ as a stock's beginning-of-period-$t$ price and $\mathcal{E}_t$ as the expectations operator conditional on information available up to but not including the beginning of period $t$. The discount rate, $r_t$ is the rate investors use to discount payments received during period $t$ (*i.e.* from the beginning of period $t$ to the beginning of period $t+1$). Investor rationality requires that the time $t$ market price of a stock, which will pay a dividend $D_{t+1}$ one period later and then sell for $P_{t+1}$, satisfy Equation (4):

$$P_t = \mathcal{E}_t \left\{ \frac{P_{t+1} + D_{t+1}}{1 + r_t} \right\}. \tag{4}$$

Then $R_t \equiv \{(P_{t+1} + D_{t+1})/P_t\} - 1$ is the return on stock; i.e. the equity return.

Invoking the standard transversality condition that the expected present value of the stock price $P_{t+i}$ falls to zero as $i$ goes to infinity, and defining the growth rate of dividends during period $t$ as $g_t \equiv (D_{t+1} - D_t)/D_t$ , allows us rewrite Equation (4) as:

$$P_t = D_t \mathcal{E}_t \left\{ \sum_{i=1}^{\infty} \left( \Pi_{k=1}^{i} \left[ \frac{1 + g_{t+k-1}}{1 + r_{t+k-1}} \right] \right) \right\}. \tag{5}$$

One attractive feature of expressing the present value stock price as in Equation (5), in terms of dividend growth rates and discount rates, is that this form highlights the irrelevance of inflation, at least to the extent that expected and actual inflation are the same. Notice that working with nominal growth rates and discount rates, as we do, is equivalent to working with deflated nominal rates (*i.e.* real rates). That is, $\frac{1+([g_t-I_t]/[1+I_t])}{1+([r_t-I_t]/[1+I_t])} = \frac{(1+g_t)}{(1+r_t)}$, where $I_t$ is inflation. Working with nominal values in our simulations removes a potential source of measurement error associated with attempts to estimate inflation.

## A1.2 Calibration

The first step in obtaining stock prices from Equation (5) is to estimate time series models for dividend growth and interest rates so that our Monte Carlo simulations will generate dividends and discount rates that share key features with observed S&P 500 dividends and US discount rates. The discount rate is defined to be the risk-free interest rate plus a constant premium of $X\%$, where $X$ is chosen to produce a target equity premium as explained in Section A1.3. Economic theory admits a wide range of possible processes for the risk-free interest rate, from constant to autoregressive and highly non-linear heteroskedastic forms. The AR(1) model of the logarithm of interest rates, as described in Hull [1993, page 408] will be used here as it fits our data well and restricts nominal interest rates to be positive. Standard specification tests for normality, autocorrelation, and ARCH on the error term from an AR(1) model of the logarithm of interest rates do not reject the null of no misspecification. The 1-year T-bill rates on our annual data have mean 0.059 and standard deviation 0.03 over 1952-1998, the time period we study.[18] The AR(1) coefficient estimate in the regression of log interest rates on lagged log interest rates equals 0.83.

Since dividend growth rates have a minimum value of -100% and no theoretical maximum, a natural choice for their distribution is the log-normal. The logarithm of 1 plus the annual dividend growth rate has mean 0.0531 and standard deviation 0.035 for the S&P 500 over 1952 to 1998. We estimated simple ARMA time series models for the logarithm of 1 plus the dividend growth rate and found the best model by the Bayesian Information Criterion to be an MA(1) model with the MA(1) coefficient equal to 0.60. Standard tests for normality of this error term (and hence conditional log-normality of dividend growth rates) do not reject

[18] The starting year of 1952 was motivated by the U.S. Federal Reserve Board's adoption of a modern monetary policy regime in 1951.

the null of normality, and standard tests for autocorrelation and ARCH fail to reject the null of homoskedasticity and no serial correlation. Finally, the error terms from the MA(1) model of log dividend growth rates and log interest rates are correlated, with a correlation coefficient of 0.21.

Properties of prices and returns produced by Equation (5) depend in important ways on the modeling of the dynamics of the dividend growth and interest rate processes. For instance, the stock price will equal a constant multiple of the dividend level and returns will be very smooth over time if dividend growth and interest rates are set equal to constants plus independent innovations. However, using models that capture the serial dependence of dividend growth rates and interest rates observed in the data, as we do, will typically lead to time-varying price-dividend ratios and variable returns of the sort we observe in the S&P 500 stock market data.

## A1.3 The Equity Premium versus the Discount Premium

When discounting a stream of uncertain cashflows to determine the present value of a share of stock, one typically uses a discount rate that exceeds the risk-free interest rate. In this case the appropriate risk adjustment, which we refer to as the "discount premium," is the amount one needs to add to the risk-free rate in order to produce a risk-adjusted discount rate that correctly prices the stock: In the finance literature and in common practice, the equity premium and discount premium are typically used interchangeably, with estimates of one substituted for the other in various applications.[19] However, as we now show, in general they are not equivalent.

The ex ante equity premium, $\pi_e$, is defined as the difference between the expected equity return and the expected risk-free rate:

$$\pi_e \equiv \mathcal{E}\left\{R\right\} - \mathcal{E}\left\{r_f\right\}. \tag{1}$$

[19] See, for instance, Claus and Thomas [2001] who comment on the equity premium but then calculate the discount premium instead, and Lee, Myers, and Swaminathan [1999] who calculate the equity premium but then use it as a discount premium for the purpose of discounting future cash flows.

The discount premium $\pi_d$, however, is the premium added to the risk-free rate, $r_f$, for the purpose of *discounting future risky cash flows,* as in determining the present value of a share of stock. As stated at the beginning of the appendix, the price of a company's stock which will pay a dividend $D_{t+1}$ one period from now and then sell for price $P_{t+1}$ is:

$$P_t = \mathcal{E}_t \left\{ \frac{P_{t+1} + D_{t+1}}{1 + r_{f,t} + \pi_d} \right\}. \tag{4}$$

Notice that by dividing both sides of Equation (4) by $P_t$, defining the stock return $R_t$ as $(P_{t+1} + D_{t+1})/P_t - 1$, and applying the law of iterated expectations we obtain the following:

$$1 = \mathcal{E} \left\{ \frac{1 + R_t}{1 + r_{f,t} + \pi_d} \right\}. \tag{6}$$

By Jensen's Inequality, the value of $\pi_d$ that sets the expectation in Equation (6) equal to 1 will not necessarily be the equity premium $\pi_e = \mathcal{E}\{R\} - \mathcal{E}\{r_f\}$ defined above. Only in the special case where the denominator of Equation (6) is not random (*e.g.* if $r_f$ and $\pi_d$ are both constant) can we multiply both sides of Equation (6) by $\mathcal{E}\{1 + r_f + \pi_d\}$ and derive the equality of $\pi_d$ and $\pi_e$. Therefore, simply adding $\pi_d = X\%$ to the risk-free rate will not in general yield an equity premium of $\pi_e = X\%$. In most cases, $\pi_d \neq \pi_e$.

In this study, we determine the appropriate discount premium by finding the value of $\pi_d$ that satisfies the expectation in Equation (6). In our application, $\pi_d$ exceeds $\pi_e$ by roughly 20 basis points.[20] In other words, if a 2% equity premium is desired, then we must add a 2.2% discount premium to the risk-free interest rate. The difference of 20 basis points is a relatively important adjustment when one considers the power of compounding in present

[20] Donaldson, Kamstra, and Kramer [2002] highlight the difference between $\pi_d$ and $\pi_e$ which arises due to Jensen's inequality. They also provide a simple method of moments estimator that can be used for estimating $\pi_d$ without bias.

value calculations. Donaldson, Kamstra, and Kramer [2002] call the difference between the discount premium and the equity premium the "risk gap." They document that the risk gap can be as large as 40 basis points among stocks listed on the S&P 500, and that failure to account for the risk gap when conducting asset valuation can lead to pricing errors as large as 50%.

One way to verify that our estimator delivers the correct discount premium for the desired equity premium, and that we have made the equity-premium-to-discount-premium bias adjustment correctly, is to observe that our simulations replicate the equity premium assumed in each case. As discussed in the main text, when we assume a 2% ex ante equity premium, we obtain the correct mean of 2% for the ex post equity premium, having added the adjusted discount premium of 2.2% to the risk-free rate in our simulations.

## A1.4 Numerical Simulation

We now detail the numerical simulation by which our figures and tables are produced. That is, we detail for the $n^{th}$ economy the formation of the prices ($P_t^n$), returns ($R_t^n$), ex post equity premia ($\hat{\pi}_e^n$), *etc.*, (where $n = 1, \cdots, N$ and $t = 1, \cdots, T$), given dividends, dividend growth rates, risk-free interest rates and the equity premium of the $n^{th}$ economy: $D_t^n$, $g_{t-1}^n$, and $r_{t-1}^n = r_{f,t-1}^n + \pi_d$.[21]

In terms of timing and information, recall that $P_t^n$ is the stock's beginning-of-period-$t$ price, $r_t^n$ is the rate used to discount payments received during period $t$ and is known at the beginning of period $t$, $D_t^n$ is paid at the beginning of period $t$, $g_t^n$ is defined as $(D_{t+1}^n - D_t^n)/D_t^n$ and is not known at the beginning of period $t$ since it depends on $D_{t+1}^n$, and $\mathcal{E}_t\{\cdot\}$ is the conditional expectation operator, with the conditioning information being the set of information available to investors up to but not including the beginning of period $t$. Finally, recall Equation (5), rewritten to correspond to the $n^{th}$ economy:

[21] We set the number of economies, $N$, at 2000. This is a sufficiently large number of replications to produce results with very small simulation error, as discussed below.

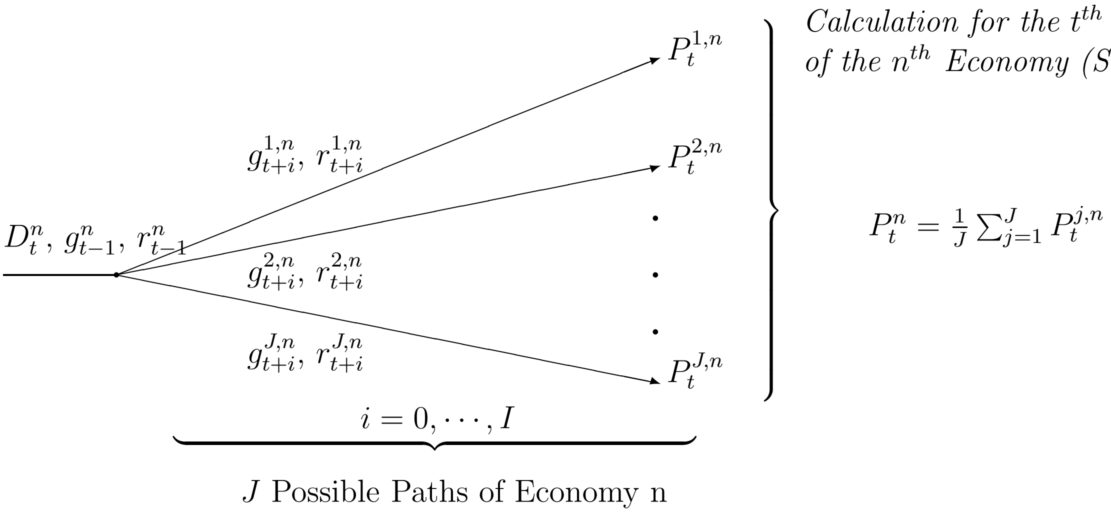
$$P_t^n = D_t^n \mathcal{E}_t \left\{ \sum_{i=1}^{\infty} \left( \Pi_{k=1}^{i} \left[ \frac{1 + g_{t+k-1}^n}{1 + r_{t+k-1}^n} \right] \right) \right\}. \tag{7}$$

Returns are constructed as $R_t^n = (P_{t+1}^n + D_{t+1}^n - P_t^n)/P_t^n$, and $\hat{\pi}_e^n = \overline{R}^n - \overline{r}_f^n$ where $\overline{R}^n = \frac{1}{T}\sum_{t=1}^{T} R_t^n$ and $\overline{r}_f^n = \frac{1}{T}\sum_{t=1}^{T} r_{f,t}^n$.

Based on Equation (7), we generate prices by generating a multitude of possible streams of dividends and discount rates, present-value discounting the dividends with the discount rates, and averaging the results; *i.e.* by conducting a Monte Carlo integration. Hence we produce prices ($P_t^n$), returns ($R_t^n$), ex post equity premia ($\hat{\pi}_e^n$), and a myriad of other financial quantities, utilizing only dividend growth rates and discount rates. The exact procedure is described below and summarized in Exhibit 1.[22]

**Exhibit 1**

*Diagram of a Simple Market Price Calculation for the $t^{th}$ Observation of the $n^{th}$ Economy (Steps 1 and 2)*




**Step 1:** When forming $P_t^n$, the most recent fundamental information available to a market trader would be $g_{t-1}^n$, $D_t^n$, and $r_{t-1}^n$. The quantities $g_{t-1}^n$, $D_t^n$, and $r_{t-1}^n$ must therefore

[22] According to Equation (7), the stream of dividends and discount rates should be infinitely long, however truncating the stream at a sufficiently distant point in time denoted $I$ leads to a very small approximation error. We discuss this point more fully below.

be generated directly in our simulations, whereas $P_t^n$ is calculated based on these $g$, $r$ and $D$. The objective of Steps 1(a)-(c) outlined below is to produce dividend growth rates and interest rates that replicate the real world dividend growth and interest rate data. That is, the simulated dividend growth rates and interest rates must have the same mean, variance, covariance, and autocorrelation structure as the observed S&P 500 dividend growth rates and US interest rates.

*Step 1(a):* Note that since, as described above, the logarithm of one plus the dividend growth rate is modeled as an MA(1) process, $log(1 + g_t^n)$ is a function of only innovations, labeled $\epsilon_g^n$. Note also that since the logarithm of the interest rate is modeled as an AR(1) process, $log(r_{f,t}^n)$ is a function of $log(r_{f,t-1}^n)$ and an innovation labeled $\epsilon_r^n$. Set the initial dividend, $D_1^n$, equal to the total S&P 500 dividend value for 1951 (observed at the end of 1951), and the lagged innovation of the logarithm of the dividend growth rates $\epsilon_{g,0}^n$ to 0. To match the real-world interest rate data, set $log(r_{f,0}^n) = -3.05$ (the mean value of log interest rates required to produce interest rates matching the mean and variance of observed T-bill rates). Then generate two independent standard normal random numbers, $\eta_1^n$ and $\nu_1^n$ (note that the subscript on these random numbers indicates time, $t$), and form two correlated random variables, $\epsilon_{r,1}^n = 0.242(0.21\eta_1^n + (1 - .21^2)^{.5}\nu_1^n)$ and $\epsilon_{g,1}^n = 0.0305\eta_1^n$. These are the simulated innovations to the interest rate and dividend growth rate processes, formed to have standard deviations of 0.242 and 0.0305 respectively to match the data, and to be correlated with correlation coefficient 0.21 as we find in the S&P 500 return and T-bill rate data. Next, form $log(1+g_1^n) = 0.0531+0.60\epsilon_{g,0}^n+\epsilon_{g,1}^n$ and $log(r_{f,1}^n) = -0.18+0.94log(r_{f,0}^n)+\epsilon_{r,1}^n$.[23] Also form $D_2^n = D_1^n(1 + g_1^n)$.

*Step 1(b):* Produce two correlated normal random variables, $\epsilon_{r,2}^n$ and $\epsilon_{g,2}^n$ as in Step 1(a) above, and conditioning on $\epsilon_{g,1}^n$ and $log(r_{f,1}^n)$ from Step 1(a) produce $log(1 + g_2^n) =$

[23]Notice that the AR(1) parameter for the log interest rate process is estimated to be 0.83 but we have set it to 0.94 in the simulations. It is well known that the coefficient estimate in an AR(1) OLS regression is biased downwards; see for instance Kennedy [1992, page 147]. Numeric simulations were employed to determine the appropriate correction for our data, as in Orcutt and Winokur [1969], and this led to the setting of 0.94. The intercept term had to be adjusted as well to reflect this new setting.

$0.0531 + 0.60\epsilon^n_{g,1} + \epsilon^n_{g,2}$, $log(r^n_{f,2}) = -0.18 + 0.94 log(r^n_{f,1}) + \epsilon^n_{r,2}$ and $D^n_3 = D^n_2(1 + g^n_2)$.

*Step 1(c):* Repeat Step 1(b) to form $log(1 + g^n_t)$, $log(r^n_{f,t})$ and $D^n_t$ for $t = 3, 4, 5, \cdots, T$ and for each economy $n = 1, 2, 3, \cdots, N$. Then calculate the dividend growth rate $g^n_t$ and the discount rate $r^n_t$ (which equals $r^n_{f,t}$ plus the bias-corrected discount premium needed to obtain the desired equity premium).

**Step 2:** For each time period $t = 1, 2, 3, \cdots, T$ and economy $n = 1, 2, 3, \cdots, N$ we next calculate prices, $P^n_t$. In order to do this we must solve for the expectation of the infinite sum of discounted future dividends conditional on time $t - 1$ information for economy $n$. That is, we must produce a set of possible paths of dividends and interest rates that might be observed in periods $t, t + 1, t + 2, \cdots$ given what is known at period $t - 1$ and use these to solve the expectation of Equation (7). We use the superscript $j$ to index the possible paths of future economies that could possibly evolve from the current state of the economy.

*Step 2(a):* Set $\epsilon^{j,n}_{g,t-1} = \epsilon^n_{g,t-1}$ and $log(r^{j,n}_{f,t-1}) = log(r^n_{f,t-1})$ for $j = 1, 2, 3, \cdots, J$.[24] Generate two independent standard normal random numbers, $\eta^{j,n}_t$ and $\nu^{j,n}_t$ and form two correlated random variables $\epsilon^{j,n}_{r,t} = 0.242(0.21\eta^{j,n}_t + (1 - .21^2)^{.5}\nu^{j,n}_t)$ and $\epsilon^{j,n}_{g,t} = 0.0305\eta^{j,n}_t$ for $j = 1, 2, 3, \cdots, J$.[25] These are the simulated innovations to the interest rate and dividend growth rate processes, respectively. Form $log(1 + g^{j,n}_t) = 0.0531 + 0.60\epsilon^{j,n}_{g,t-1} + \epsilon^{j,n}_{g,t}$ and $log(r^{j,n}_{f,t}) = -0.18 + 0.94 log(r^{j,n}_{f,t-1}) + \epsilon^{j,n}_{r,t}$.

*Step 2(b):* Produce two correlated normal random variables $\epsilon^{j,n}_{r,t+1}$ and $\epsilon^{j,n}_{g,t+1}$ as in Step 2(a) above, and conditioning on $\epsilon^{j,n}_{g,t}$ and $log(r^{j,n}_{f,t})$ from Step 2(a) produce $log(1 + g^{j,n}_{t+1}) =$

[24] We choose $J$ to equal 2,000, in order to ensure the simulation error in calculating prices and returns was controlled to be very small. To determine the simulation error, we conducted a simulation of the simulations. Unlike some Monte Carlo experiments (such as those estimating the size of a test statistic under the null) the standard error of the simulation error for most of our estimates (returns, prices, *etc.*) are themselves analytically intractable, and must be simulated. In order to estimate the standard error of the simulation error in estimating market prices, we estimated a single market price 2,000 times, each time independent of the other, and from this set of prices computed the mean and variance of the price estimate. If the experiment had no simulation error, each of the price estimates would be identical. With the number of possible paths, $J$, equal to 2,000 we find that the standard deviation of the simulation error is less than 0.20% of the price, which is sufficiently small as not to be a source of concern for our study.

[25] For our random number generation we made use of a variance reduction technique, stratified sampling. This technique has us drawing pseudo-random numbers ensuring that $q$% of these draws come from the $q^{th}$ percentile, so that our sampling does not weight any grouping of random draws too heavily.

$0.0531 + 0.60\epsilon_{g,t}^{j,n} + \epsilon_{g,t+1}^{j,n}$, and $log(r_{f,t+1}^{j,n}) = -0.18 + 0.94 log(r_{f,t}^{j,n}) + \epsilon_{r,t+1}^{j,n}$ for $j = 1, 2, 3, \cdots, J$.

*Step 2(c):* Repeat Step 2(b) to form $log(1 + g_{t+i}^{j,n})$ and $log(r_{t+i}^{j,n})$ for $i = 2, 3, 4, \cdots, I$, $j = 1, 2, 3, \cdots, J$, and economies $n = 1, 2, 3, \cdots, N$. Solve for the dividend growth rate $g_{t+i}^{j,n}$, the dividends $D_{t+i}^{j,n}$, and the discount rate $r_{t+i}^{j,n}$ (which equals $r_{f,t+i}^{j,n}$ plus the bias-corrected discount premium needed to obtain the desired equity premium) for $i = 0, 1, 2, \cdots, I$.

*Step 2(d):* The present discounted value of each of the individual $J$ streams of dividends is now taken in accordance with Equation (7), with the $j^{th}$ present value price noted as $P_t^{j,n}$. Finally, the price for the $n^{th}$ economy in period $t$ is formed: $P_t^n = \frac{1}{J} \sum_{j=1}^{J} P_t^{j,n}$.

In considering these prices, note that according to Equation (7) the stream of discount and dividend growth rates should be infinitely long, while in our simulations we extend the stream only a finite number of periods, $I$. Since the ratio of gross dividend growth rates to gross discount rates are less than unity in steady state, the individual product elements in the infinite sum in Equation (7) eventually converge to zero as $I$ increases. (Indeed, this convergence to zero is exactly what is required for the standard transversality condition that the expected present value of the stock price $P_{t+i}$ falls to zero as $i$ goes to infinity.) We therefore set $I$ large enough in our simulations so that the truncation does not materially effect our results. We find that setting $I = 1000$ years is sufficient in all cases we studied. That is, the discounted present value of a dividend payment received 1000 years in the future is essentially zero. Also note that the steps above are required to produce $P_t^n$, $D_t^n$, $g_t^n$, and $r_t^n$ for $n = 1, \cdots, N$ and $t = 1, \cdots, T$; the intermediate terms superscripted with a $j$ are required only to perform the numerical integration that yields $P_t^N$. Note that the length of the time series $T$ is chosen to be 47 to imitate the 47 years of annual data we have available from the S&P 500 from 1952 to 1998.

**Step 3:** After performing Steps 1(a)-(c) and 2(a)-(d) for $t = 1, \cdots, T$, rolling out $N$ independent economies for $T$ periods, we construct the market returns for each economy, $R_t^n = (P_{t+1}^n + D_{t+1}^n - P_t^n)/P_t^n$, and the equity premium that agents in the $n^{th}$ economy would observe, $\hat{\pi}_e^n$, estimated from Equation (1) as the mean difference in market returns and the risk-free rate. These simulated equity premia form the basis for the analysis in the rest of this paper.

# Appendix 2: Model Underlying our Four Types of Time-Varying Equity Premia

We detail below four types of time-varying ex ante equity premia. Briefly, the first type is based on an AR(1) model of equity premia that emerges from Merton's [1980] conditional CAPM, the second is based on Merton's CAPM and a downward trend, the third models the time-varying equity premia as iid, and the fourth incorporates a downward trend to the iid time-varying equity premia.

Merton's conditional CAPM is expressed in terms of returns in excess of the risk-free rate, or, in other words, the period by period equity premium. For the $i^{th}$ asset,

$$E_{t-1}(r_{i,t}) = \lambda \; cov_{t-1}(r_{i,t} r_{m,t}), \tag{8}$$

where $r_{i,t}$ are excess returns on the asset, $r_{m,t}$ are excess returns on the market portfolio, and $cov_{t-1}$ is the time-varying conditional covariance between excess returns on the asset and on the market portfolio. For the expected excess market return, (8) becomes

$$E_{t-1}(r_{m,t}) = \lambda \; var_{t-1}(r_{m,t}) \tag{9}$$

where $var_{t-1}$ is the market time-varying conditional variance. Merton [1980] argues that $\lambda$ in (9) is the weighted sum of the reciprocal of each investor's coefficient of relative risk aversion, with the weight being related to the distribution of wealth among individuals.

Equation (9) defines a time-varying equity premium, but this model has the equity premium varying only as a function of time-varying conditional variance. Following Bekaert and

Harvey [1995], it is possible to allow $\lambda$ in (9) to vary over time by making it a parametric function of conditioning variables. We set $\lambda_{t-1} = exp\left(\lambda_0 + \lambda_1 \frac{D_{t-1}}{P_{t-1}}\right)$ since the dividend yield is a well known predictor of future returns.

We make use of a simple ARCH specification to model $var_{t-1}(r_{m,t})$. Once again we calibrate to the S&P 500 over 1952 to 1998, estimating the following model:

$$r_{m,t} = \lambda_{t-1}\ var_{t-1}(r_{m,t}) + e_{m,t} \tag{10}$$

$$var_{t-1}(r_{m,t}) = h_t = \omega + \alpha e^2_{m,t-1} \tag{11}$$

$$\lambda_{t-1} = exp\left(\lambda_0 + \lambda_1 \frac{D_{t-1}}{P_{t-1}}\right) \tag{12}$$

The values of estimated parameters are $\lambda_0 = -.90$, $\lambda_1 = 0.53$, $\omega = 0.021$, and $\alpha = 0.066$.

For our simulations, we model the time series process of the ex ante time varying equity premium (denoted $\pi_{e,t}$) by using the excess return as a proxy for the equity premium:

$$\hat{\pi}_{e,t} = \hat{\lambda}_{t-1} v\hat{a}r_{t-1}(r_{m,t}) \tag{13}$$

where $\hat{\lambda}_{t-1} = exp\left(-0.9 + 0.53\frac{D_{t-1}}{P_{t-1}}\right)$, $v\hat{a}r_{t-1}(r_{m,t}) = 0.021 + 0.066\hat{e}^2_{m,t-1}$, and $\hat{e}_{m,t-1} = r_{m,t-1} - \hat{\pi}_{t-1}$. The time-varying equity premium we estimate here, $\hat{\pi}_{e,t}$, follows a strong AR(1) time series process, similar to that of the risk-free interest rate.[26]

To generate the first time varying equity premium model we consider, labeled Type 1 (with autocorrelated ex ante equity premia), we start by producing three sets of iid normal random variables. We calibrate two of these random variables to residuals from the time series models we estimated for S&P 500 dividend growth rates and US interest rates, as

[26] The mean of the estimated equity premium from this model is 8% and the standard deviation is 6.2%. An AR(1) model of the natural logarithm of the equity premium has a coefficient of 0.72 on the lagged equity premium, with a standard error of 0.089 and an $R^2$ of 0.599. The error from this regression is insignificantly correlated with the innovation from the interest rate model but is significantly positively correlated with the innovation from the MA(1) model of dividend growth rates, with a correlation coefficient of 0.216.

detailed in Appendix 1. (Recall from Appendix 1 that interest rates follow an AR(1) process and dividend growth rates follow an MA(1) process.) We calibrate the third set of iid normal random variables to the residuals from the AR(1) model of the equity premia that emerge from estimating the Merton CAPM described above using S&P 500 returns. The interest rates, dividend growth rates, and equity premia we generate not only follow the time series processes to which they are calibrated, they also mimic the covariance structure between the residuals from the time series models of interest rates, dividend growth rates, and equity premia as estimated using US data. In evolving a 47-year simulated time series of the (natural logarithm of the) equity premium, the 'year 1' simulated value and the intercept term are set to the appropriate values necessary to generate a 4% mean equity premium.

In producing model Type 2 (autocorrelated and downward trending ex ante equity premia), we model the same AR(1) structure used for Type 1 equity premia (and the same time series properties for interest rates and dividend growth rates). We then incorporate a downward trend in the premium by setting the mean equity premium to be 6% at the start of the 47-year simulation period, 2% at the end, and 4% overall.

Model Types 3 and 4 do not use the AR(1) process used for time-varying equity premia in model Types 1 and 2, though they are still linked to the CAPM model described above (and the interest rate and dividend growth rate processes are still calibrated as described above). The Type 3 time-varying equity premium is a log-normal random variable which has a mean of 4%, a standard deviation calibrated to the variability of the equity premium estimates that emerge from the CAPM model, and no downward trend. Finally, for Type 4 we model the ex ante equity premium as a log-normal random variable with a mean premium of 6% at the start of the 47-year simulation period, 2% at the end, and 4% overall.

**Table 1**
**Statistics on Ex Post Equity Premium Estimates ($\hat{\pi}_e$) for the Simulated Market Economies Based on Various Values of the Ex Ante Equity Premium**

This table presents means and percentiles of the ex post equity premium estimates ($\hat{\pi}_e$) arising in our simulated economies. The ex post equity premium is estimated as the difference between the mean return and the mean interest rate over 47 years of simulated data. The results reported in each set of rows correspond to simulations in which the ex ante equity premium was set to a value ranging from 2% through 6%.

| Ex Ante Equity Premium | Mean of Simulated $\hat{\pi}_e$ | Percentiles of Simulated $\hat{\pi}_e$ | | | | |
|---|---|---|---|---|---|---|
| | | 1% | 5% | 50% | 95% | 99% |
| 2 % | 2.032 | -7.008 | -2.977 | 2.452 | 5.316 | 6.244 |
| 2.5 % | 2.516 | -5.230 | -1.773 | 2.907 | 5.626 | 6.847 |
| 3 % | 2.954 | -5.198 | -1.518 | 3.345 | 6.076 | 7.107 |
| 3.5 % | 3.498 | -3.911 | -0.625 | 3.872 | 6.494 | 7.278 |
| 4 % | 3.980 | -3.319 | 0.027 | 4.344 | 6.825 | 7.618 |
| 4.5 % | 4.532 | -2.268 | 0.705 | 4.934 | 7.317 | 8.200 |
| 5 % | 5.024 | -1.610 | 1.334 | 5.397 | 7.662 | 8.522 |
| 6 % | 6.040 | -0.099 | 2.362 | 6.304 | 8.561 | 9.248 |

**Table 2**
**Statistics on Dividend Yields ($D/P$) for the Simulated Market Economies Based on Various Values of the Ex Ante Equity Premium**

This table presents the first four moments of annual S&P 500 dividend yield (dividend divided by price, estimated each year during 1952-1998) along with means and percentiles of the first four moments of the annual dividend yields arising in our simulated economies. The results reported in each set of rows correspond to simulations in which the ex ante equity premium was set to a value ranging from 2% through 6%.

| Ex Ante Equity Premium | Moment | S&P 500 $D/P$ | Mean of Simulated $D/P$ | Percentiles of Simulated $D/P$: 1% | 5% | 50% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|
| 2% | Mean | 3.789 | 1.439 | 0.824 | 0.912 | 1.328 | 2.284 | 3.299 |
| | $\sigma$ | 1.060 | 0.465 | 0.108 | 0.145 | 0.356 | 1.062 | 1.914 |
| | Skewness | 0.843 | 0.873 | -0.192 | 0.099 | 0.803 | 1.907 | 2.483 |
| | Kurtosis | 3.144 | 3.503 | 1.629 | 1.900 | 2.988 | 6.993 | 9.934 |
| 2.5% | Mean | 3.789 | 2.031 | 1.179 | 1.315 | 1.893 | 3.214 | 4.180 |
| | $\sigma$ | 1.060 | 0.616 | 0.134 | 0.197 | 0.481 | 1.535 | 2.281 |
| | Skewness | 0.843 | 0.844 | -0.191 | 0.030 | 0.766 | 1.842 | 2.513 |
| | Kurtosis | 3.144 | 3.456 | 1.618 | 1.843 | 2.942 | 6.795 | 10.228 |
| 3% | Mean | 3.789 | 2.618 | 1.541 | 1.733 | 2.434 | 4.116 | 5.341 |
| | $\sigma$ | 1.060 | 0.768 | 0.161 | 0.241 | 0.594 | 1.884 | 3.077 |
| | Skewness | 0.843 | 0.823 | -0.235 | 0.027 | 0.783 | 1.756 | 2.355 |
| | Kurtosis | 3.144 | 3.368 | 1.624 | 1.870 | 2.936 | 6.190 | 9.390 |
| 3.5% | Mean | 3.789 | 3.159 | 1.935 | 2.134 | 2.970 | 4.837 | 6.108 |
| | $\sigma$ | 1.060 | 0.856 | 0.212 | 0.284 | 0.707 | 1.948 | 3.150 |
| | Skewness | 0.843 | 0.791 | -0.339 | 0.001 | 0.747 | 1.742 | 2.295 |
| | Kurtosis | 3.144 | 3.305 | 1.615 | 1.818 | 2.876 | 5.985 | 9.145 |
| 4% | Mean | 3.789 | 3.720 | 2.296 | 2.533 | 3.485 | 5.562 | 7.532 |
| | $\sigma$ | 1.060 | 0.978 | 0.236 | 0.328 | 0.779 | 2.246 | 4.014 |
| | Skewness | 0.843 | 0.777 | -0.281 | -0.006 | 0.731 | 1.730 | 2.277 |
| | Kurtosis | 3.144 | 3.253 | 1.613 | 1.832 | 2.821 | 6.067 | 9.021 |
| 4.5% | Mean | 3.789 | 4.231 | 2.764 | 3.009 | 4.001 | 6.262 | 7.761 |
| | $\sigma$ | 1.060 | 1.053 | 0.264 | 0.365 | 0.834 | 2.527 | 4.126 |
| | Skewness | 0.843 | 0.783 | -0.286 | -0.010 | 0.733 | 1.745 | 2.315 |
| | Kurtosis | 3.144 | 3.278 | 1.616 | 1.851 | 2.866 | 6.038 | 9.314 |
| 5 % | Mean | 3.789 | 4.762 | 3.115 | 3.414 | 4.532 | 6.791 | 8.773 |
| | $\sigma$ | 1.060 | 1.118 | 0.293 | 0.395 | 0.928 | 2.489 | 4.145 |
| | Skewness | 0.843 | 0.759 | -0.293 | -0.017 | 0.693 | 1.714 | 2.232 |
| | Kurtosis | 3.144 | 3.211 | 1.610 | 1.852 | 2.797 | 6.066 | 8.513 |
| 6% | Mean | 3.789 | 5.792 | 3.936 | 4.257 | 5.534 | 8.233 | 9.835 |
| | $\sigma$ | 1.060 | 1.245 | 0.320 | 0.429 | 1.050 | 2.822 | 4.145 |
| | Skewness | 0.843 | 0.745 | -0.316 | -0.053 | 0.694 | 1.725 | 2.183 |
| | Kurtosis | 3.144 | 3.216 | 1.632 | 1.811 | 2.790 | 5.959 | 8.768 |

**Table 3**
**Statistics on Returns for**
**the Simulated Market Economies Based on Various**
**Values of the Ex Ante Equity Premium**

This table presents the first four moments of annual S&P 500 returns (estimated each year during 1952-1998) along with means and percentiles of the first four moments of the annual returns arising in our simulated economies. The results reported in each set of rows correspond to simulations in which the ex ante equity premium was set to a value ranging from 2% through 6%.

| Ex Ante Equity Premium | Moment | S&P 500 | Mean of Simulated Data | Percentiles of Simulated Data 1% | 5% | 50% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|
| 2 % | Mean | 13.439 | 8.074 | 4.587 | 5.514 | 7.907 | 11.182 | 13.605 |
| | $\sigma$ | 15.008 | 15.476 | 9.294 | 10.526 | 14.960 | 22.000 | 27.310 |
| | Skewness | -0.106 | 0.462 | -0.409 | -0.188 | 0.410 | 1.268 | 1.788 |
| | Kurtosis | 2.468 | 3.388 | 1.958 | 2.147 | 2.979 | 5.936 | 8.467 |
| 2.5 % | Mean | 13.439 | 8.578 | 5.009 | 5.958 | 8.411 | 11.738 | 14.369 |
| | $\sigma$ | 15.008 | 14.475 | 8.388 | 9.617 | 14.067 | 21.031 | 24.825 |
| | Skewness | -0.106 | 0.447 | -0.428 | -0.211 | 0.413 | 1.281 | 1.870 |
| | Kurtosis | 2.468 | 3.379 | 1.947 | 2.141 | 2.989 | 5.689 | 9.099 |
| 3 % | Mean | 13.439 | 9.087 | 5.644 | 6.510 | 8.844 | 12.642 | 15.041 |
| | $\sigma$ | 15.008 | 13.741 | 7.924 | 9.057 | 13.267 | 20.248 | 23.179 |
| | Skewness | -0.106 | 0.427 | -0.493 | -0.177 | 0.382 | 1.220 | 1.731 |
| | Kurtosis | 2.468 | 3.321 | 1.914 | 2.107 | 2.983 | 5.704 | 8.172 |
| 3.5 % | Mean | 13.439 | 9.579 | 6.002 | 6.979 | 9.369 | 12.783 | 15.499 |
| | $\sigma$ | 15.008 | 13.169 | 7.619 | 8.785 | 12.710 | 19.010 | 23.221 |
| | Skewness | -0.106 | 0.433 | -0.423 | -0.181 | 0.381 | 1.223 | 1.731 |
| | Kurtosis | 2.468 | 3.347 | 1.928 | 2.152 | 2.990 | 5.669 | 8.584 |
| 4 % | Mean | 13.439 | 10.111 | 6.474 | 7.430 | 9.803 | 13.833 | 16.641 |
| | $\sigma$ | 15.008 | 12.636 | 7.368 | 8.432 | 12.089 | 18.522 | 22.605 |
| | Skewness | -0.106 | 0.411 | -0.498 | -0.247 | 0.356 | 1.229 | 1.799 |
| | Kurtosis | 2.468 | 3.311 | 1.865 | 2.102 | 2.963 | 5.629 | 8.566 |
| 4.5 % | Mean | 13.439 | 10.595 | 7.186 | 7.984 | 10.359 | 14.034 | 16.692 |
| | $\sigma$ | 15.008 | 12.236 | 7.273 | 8.164 | 11.763 | 17.992 | 21.921 |
| | Skewness | -0.106 | 0.426 | -0.461 | -0.206 | 0.373 | 1.207 | 1.781 |
| | Kurtosis | 2.468 | 3.331 | 1.861 | 2.124 | 2.976 | 5.723 | 9.131 |
| 5 % | Mean | 13.439 | 11.108 | 7.410 | 8.318 | 10.862 | 14.688 | 17.297 |
| | $\sigma$ | 15.008 | 11.834 | 6.870 | 7.791 | 11.376 | 17.506 | 21.028 |
| | Skewness | -0.106 | 0.415 | -0.442 | -0.214 | 0.372 | 1.194 | 1.745 |
| | Kurtosis | 2.468 | 3.301 | 1.937 | 2.162 | 2.972 | 5.485 | 8.713 |
| 6 % | Mean | 13.439 | 12.113 | 8.524 | 9.390 | 11.864 | 15.694 | 18.412 |
| | $\sigma$ | 15.008 | 11.135 | 6.452 | 7.390 | 10.743 | 16.359 | 20.501 |
| | Skewness | -0.106 | 0.417 | -0.472 | -0.204 | 0.365 | 1.187 | 1.888 |
| | Kurtosis | 2.468 | 3.289 | 1.957 | 2.149 | 2.943 | 5.576 | 8.508 |

**Table 4**

**Statistics on Discounted Dividend Growth Rates ($y$) for the Simulated Market Economies Based on Various Values of the Ex Ante Equity Premium**

This table presents the first four moments of S&P 500 discounted dividend growth rates (based on annual dividend growth rates estimated over 1952-1998, and reported in decimal form) as well as means and percentiles of the first four moments of discounted dividend growth rates arising in our simulated economies. The discounted dividend growth rate, $y_t$, is defined as $(1+g_t)/(1+r_t)$ where $g_t$ is the growth rate of dividends and $r_t$ is the discount rate. The results reported in each set of rows correspond to simulations in which the ex ante equity premium was set to a value ranging from 2% through 6%.

| Ex Ante Equity Premium | Moment | S&P 500 $y$ | Mean of Simulated $y$ | Percentiles of Simulated $y$ 1% | 5% | 50% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|
| 2 % | Mean | 0.942 | 0.978 | 0.889 | 0.931 | 0.983 | 1.011 | 1.019 |
| | $\sigma$ | 0.037 | 0.042 | 0.027 | 0.030 | 0.040 | 0.062 | 0.083 |
| | Skewness | 0.023 | -0.049 | -1.122 | -0.689 | -0.036 | 0.574 | 0.829 |
| | Kurtosis | 2.631 | 2.719 | 1.803 | 1.995 | 2.578 | 3.882 | 4.858 |
| 2.5 % | Mean | 0.942 | 0.974 | 0.892 | 0.925 | 0.978 | 1.007 | 1.016 |
| | $\sigma$ | 0.037 | 0.042 | 0.028 | 0.030 | 0.040 | 0.061 | 0.078 |
| | Skewness | 0.023 | -0.059 | -1.056 | -0.738 | -0.047 | 0.556 | 0.836 |
| | Kurtosis | 2.631 | 2.691 | 1.772 | 1.985 | 2.581 | 3.727 | 4.603 |
| 3 % | Mean | 0.942 | 0.969 | 0.883 | 0.916 | 0.974 | 1.001 | 1.011 |
| | $\sigma$ | 0.037 | 0.042 | 0.027 | 0.030 | 0.040 | 0.064 | 0.085 |
| | Skewness | 0.023 | -0.062 | -1.123 | -0.722 | -0.048 | 0.538 | 0.846 |
| | Kurtosis | 2.631 | 2.701 | 1.801 | 1.960 | 2.596 | 3.823 | 4.625 |
| 3.5 % | Mean | 0.942 | 0.965 | 0.884 | 0.917 | 0.969 | 0.997 | 1.005 |
| | $\sigma$ | 0.037 | 0.042 | 0.027 | 0.030 | 0.039 | 0.061 | 0.082 |
| | Skewness | 0.023 | -0.046 | -1.087 | -0.722 | -0.038 | 0.597 | 0.820 |
| | Kurtosis | 2.631 | 2.704 | 1.787 | 1.963 | 2.586 | 3.827 | 4.791 |
| 4 % | Mean | 0.942 | 0.960 | 0.873 | 0.913 | 0.964 | 0.994 | 1.001 |
| | $\sigma$ | 0.037 | 0.042 | 0.027 | 0.030 | 0.039 | 0.062 | 0.086 |
| | Skewness | 0.023 | -0.048 | -1.080 | -0.727 | -0.038 | 0.580 | 0.840 |
| | Kurtosis | 2.631 | 2.693 | 1.809 | 1.972 | 2.577 | 3.840 | 4.747 |
| 4.5 % | Mean | 0.942 | 0.956 | 0.875 | 0.909 | 0.960 | 0.987 | 0.995 |
| | $\sigma$ | 0.037 | 0.041 | 0.027 | 0.030 | 0.039 | 0.062 | 0.083 |
| | Skewness | 0.023 | -0.056 | -1.146 | -0.763 | -0.042 | 0.595 | 0.884 |
| | Kurtosis | 2.631 | 2.718 | 1.794 | 1.962 | 2.585 | 3.831 | 4.867 |
| 5 % | Mean | 0.942 | 0.951 | 0.870 | 0.905 | 0.956 | 0.983 | 0.990 |
| | $\sigma$ | 0.037 | 0.041 | 0.027 | 0.030 | 0.039 | 0.059 | 0.081 |
| | Skewness | 0.023 | -0.051 | -1.074 | -0.771 | -0.038 | 0.594 | 0.884 |
| | Kurtosis | 2.631 | 2.733 | 1.805 | 1.968 | 2.607 | 3.934 | 4.939 |
| 6 % | Mean | 0.942 | 0.943 | 0.867 | 0.895 | 0.947 | 0.974 | 0.982 |
| | $\sigma$ | 0.037 | 0.040 | 0.026 | 0.029 | 0.038 | 0.059 | 0.076 |
| | Skewness | 0.023 | -0.045 | -1.055 | -0.732 | -0.020 | 0.553 | 0.792 |
| | Kurtosis | 2.631 | 2.715 | 1.800 | 1.985 | 2.606 | 3.802 | 4.572 |

**Table 5**
**Statistics on Sharpe Ratios for**
**the Simulated Market Economies Based on Various**
**Values of the Ex Ante Equity Premium**

This table presents the mean Sharpe ratio for the S&P 500 over 1952-1998 as well as means and percentiles of the Sharpe ratio arising in our simulated economies. The Sharpe ratio is defined as excess return divided by the standard deviation of the excess return. The results reported in each row correspond to simulations in which the ex ante equity premium was set to a value ranging from 2% through 6%.

| Ex Ante Equity Premium | S&P 500 | Mean of Simulated Data | Percentiles of Simulated Data | | | | |
|---|---|---|---|---|---|---|---|
| | | | 1% | 5% | 50% | 95% | 99% |
| 2 % | 0.501 | 0.154 | -0.284 | -0.139 | 0.157 | 0.425 | 0.522 |
| 2.5 % | 0.501 | 0.199 | -0.230 | -0.091 | 0.196 | 0.505 | 0.612 |
| 3 % | 0.501 | 0.243 | -0.238 | -0.078 | 0.241 | 0.564 | 0.725 |
| 3.5 % | 0.501 | 0.293 | -0.156 | -0.037 | 0.289 | 0.646 | 0.799 |
| 4 % | 0.501 | 0.343 | -0.157 | 0.002 | 0.339 | 0.699 | 0.867 |
| 4.5 % | 0.501 | 0.396 | -0.107 | 0.039 | 0.395 | 0.752 | 0.930 |
| 5 % | 0.501 | 0.453 | -0.081 | 0.078 | 0.441 | 0.847 | 1.021 |
| 6 % | 0.501 | 0.571 | -0.005 | 0.140 | 0.559 | 1.012 | 1.181 |

**Table 6**
**Statistics on Market Return Autoregressive Coefficients for the Simulated Market Economies Based on Various Values of the Ex Ante Equity Premium**

The first line in each cell of this table contains the first order autoregressive coefficient, AR(1), for S&P 500 returns along with means and percentiles of the AR(1) coefficients arising in our simulated economies. The second line in each cell contains the same statistics based on the standard deviation of the AR(1) estimate for the S&P 500 and the simulated economies. S&P 500 AR(1) coefficients are estimated over 1952-1998. The results reported in each set of rows correspond to simulations in which the ex ante equity premium was set to a value ranging from 2% through 6%.

| Ex Ante Equity Premium | Moment | S&P 500 | Mean of Simulated Data | Percentiles of Simulated Data 1% | 5% | 50% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|
| 2 % | AR(1) | -0.134 | -0.058 | -0.398 | -0.305 | -0.058 | 0.182 | 0.306 |
| | $\sigma$ of estimate | 0.151 | 0.151 | 0.134 | 0.142 | 0.151 | 0.157 | 0.165 |
| 2.5 % | AR(1) | -0.134 | -0.039 | -0.371 | -0.281 | -0.043 | 0.209 | 0.307 |
| | $\sigma$ of estimate | 0.151 | 0.151 | 0.133 | 0.142 | 0.151 | 0.157 | 0.167 |
| 3 % | AR(1) | -0.134 | -0.029 | -0.375 | -0.282 | -0.033 | 0.233 | 0.344 |
| | $\sigma$ of estimate | 0.151 | 0.151 | 0.132 | 0.142 | 0.151 | 0.158 | 0.165 |
| 3.5 % | AR(1) | -0.134 | -0.033 | -0.393 | -0.287 | -0.032 | 0.227 | 0.309 |
| | $\sigma$ of estimate | 0.151 | 0.151 | 0.136 | 0.142 | 0.151 | 0.158 | 0.167 |
| 4 % | AR(1) | -0.134 | -0.027 | -0.360 | -0.267 | -0.030 | 0.225 | 0.337 |
| | $\sigma$ of estimate | 0.151 | 0.151 | 0.136 | 0.143 | 0.151 | 0.157 | 0.166 |
| 4.5 % | AR(1) | -0.134 | -0.018 | -0.371 | -0.267 | -0.022 | 0.238 | 0.336 |
| | $\sigma$ of estimate | 0.151 | 0.151 | 0.134 | 0.143 | 0.151 | 0.157 | 0.165 |
| 5 % | AR(1) | -0.134 | -0.026 | -0.380 | -0.268 | -0.031 | 0.231 | 0.334 |
| | $\sigma$ of estimate | 0.151 | 0.151 | 0.137 | 0.143 | 0.151 | 0.157 | 0.165 |
| 6 % | AR(1) | -0.134 | -0.019 | -0.348 | -0.263 | -0.023 | 0.235 | 0.364 |
| | $\sigma$ of estimate | 0.151 | 0.151 | 0.136 | 0.142 | 0.151 | 0.157 | 0.167 |

**Table 7**
**Statistics on ARCH Coefficients for the Simulated Market Economies Based on Various Values of the Ex Ante Equity Premium**

The first line in each cell of this table contains the first order autoregressive conditional heteroskedasticity coefficient, ARCH(1), for S&P 500 returns along with means and percentiles of the ARCH(1) coefficients arising in our simulated economies. The second line in each cell contains the same statistics based on the standard deviation of the ARCH(1) estimate for the S&P 500 index and the simulated economies. S&P 500 ARCH(1) coefficients are estimated over 1952-1998. The results reported in each set of rows correspond to simulations in which the ex ante equity premium was set to a value ranging from 2% through 6%.

| Ex Ante Equity Premium | Moment | S&P 500 | Mean of Simulated Data | Percentiles of Simulated Data 1% | 5% | 50% | 95% | 99% |
|---|---|---|---|---|---|---|---|---|
| 2 % | ARCH | 0.250 | -0.002 | -0.265 | -0.209 | -0.025 | 0.273 | 0.434 |
| | $\sigma$ of estimate | 0.147 | 0.153 | 0.108 | 0.143 | 0.153 | 0.160 | 0.206 |
| 2.5 % | ARCH | 0.250 | 0.003 | -0.269 | -0.203 | -0.018 | 0.274 | 0.410 |
| | $\sigma$ of estimate | 0.147 | 0.154 | 0.107 | 0.142 | 0.153 | 0.159 | 0.218 |
| 3 % | ARCH | 0.250 | 0.004 | -0.285 | -0.202 | -0.020 | 0.271 | 0.417 |
| | $\sigma$ of estimate | 0.147 | 0.153 | 0.117 | 0.143 | 0.153 | 0.162 | 0.194 |
| 3.5 % | ARCH | 0.250 | 0.006 | -0.275 | -0.197 | -0.019 | 0.293 | 0.439 |
| | $\sigma$ of estimate | 0.147 | 0.154 | 0.110 | 0.141 | 0.154 | 0.162 | 0.226 |
| 4 % | ARCH | 0.250 | 0.001 | -0.285 | -0.206 | -0.024 | 0.279 | 0.407 |
| | $\sigma$ of estimate | 0.147 | 0.153 | 0.115 | 0.141 | 0.153 | 0.162 | 0.204 |
| 4.5 % | ARCH | 0.250 | -0.002 | -0.279 | -0.207 | -0.022 | 0.272 | 0.433 |
| | $\sigma$ of estimate | 0.147 | 0.153 | 0.115 | 0.142 | 0.154 | 0.160 | 0.202 |
| 5 % | ARCH | 0.250 | 0.006 | -0.269 | -0.209 | -0.013 | 0.291 | 0.423 |
| | $\sigma$ of estimate | 0.147 | 0.153 | 0.109 | 0.142 | 0.153 | 0.161 | 0.202 |
| 6 % | ARCH | 0.250 | -0.002 | -0.280 | -0.199 | -0.022 | 0.271 | 0.419 |
| | $\sigma$ of estimate | 0.147 | 0.154 | 0.114 | 0.143 | 0.153 | 0.161 | 0.218 |

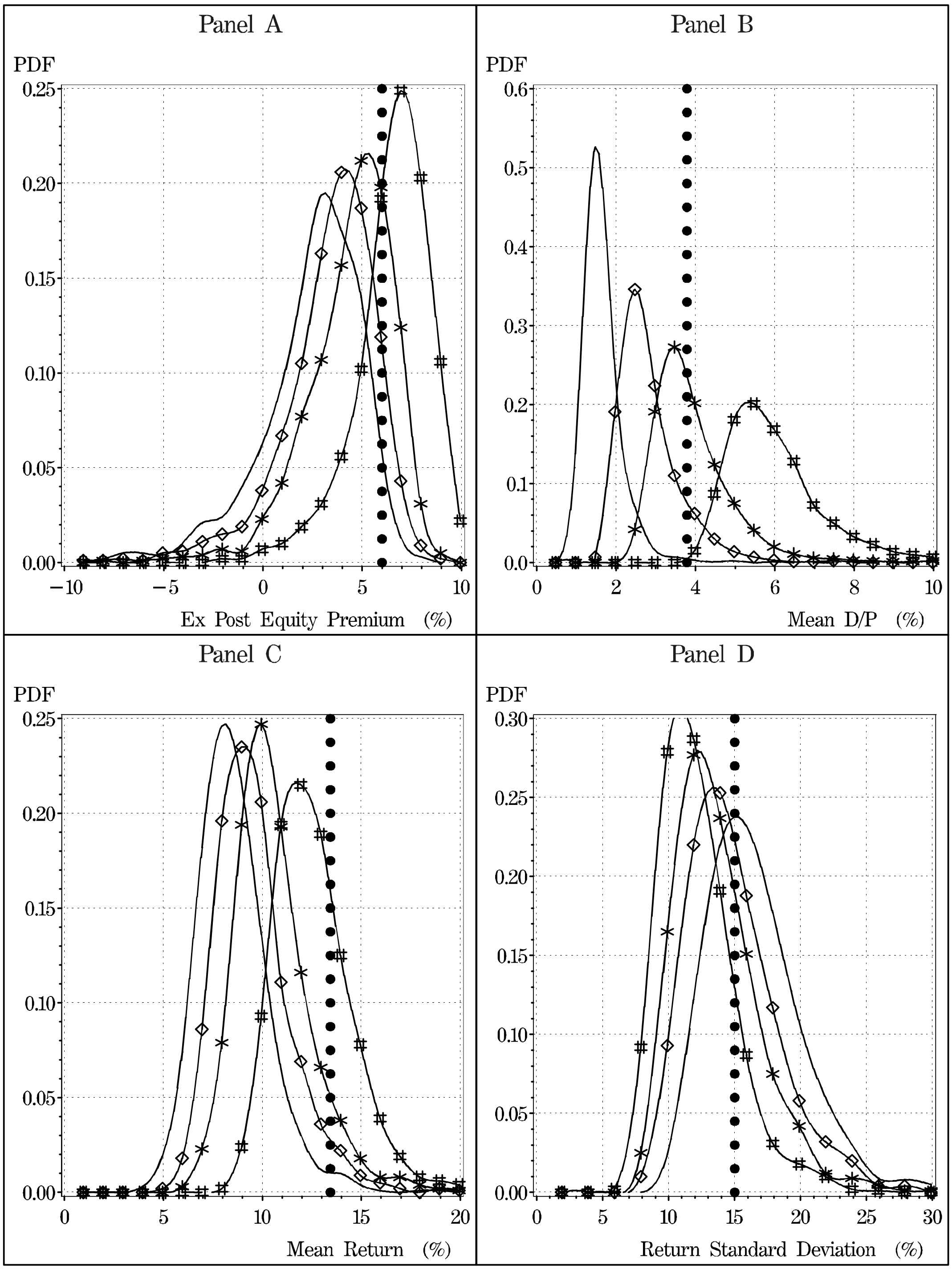
**Table 8**
**Joint Tests: Are Joint Distributions of the Simulated Data Consistent with Observed Combinations of Financial Statistics?**

We present $\chi^2$ statistics for testing whether various joint collections of data simulated over 47-year periods are consistent with US financial statistics observed over 1952-1998. For Case 1, we consider whether the joint simulated distributions of the mean return, return standard deviation, mean dividend yield, ex post equity premium, AR(1) coefficient estimate for returns, and ARCH(1) coefficient estimate for returns are consistent with the values of these quantities observed in practice. For Case 2, we consider joint simulated distributions of the same financial statistics with the exception of the AR(1) and ARCH(1) coefficients. The Case 1 and 2 $\chi^2$ statistics have 6 and 4 degrees of freedom respectively. One, two, and three asterisks indicate significance at the 10%, 5%, and 1% level of significance respectively.

| | Case 1 $\chi^2$ Statistic | Case 2 $\chi^2$ Statistic |
|---|---|---|
| **Panel A: Various Values of the Ex Ante Equity Premium** | | |
| Ex Ante Equity Premium of 2% | 293.90*** | 281.89*** |
| Ex Ante Equity Premium of 2.5% | 153.71*** | 146.86*** |
| Ex Ante Equity Premium of 3% | 62.51*** | 56.09*** |
| Ex Ante Equity Premium of 3.5% | 22.31*** | 16.77*** |
| Ex Ante Equity Premium of 4% | 8.16 | 4.19 |
| Ex Ante Equity Premium of 4.5% | 20.07*** | 16.32*** |
| Ex Ante Equity Premium of 5% | 51.46*** | 47.31*** |
| Ex Ante Equity Premium of 6% | 195.16*** | 186.62*** |
| **Panel B: Sensitivity to Parameter Settings with a 4% Ex Ante Equity Premium** | | |
| Reduced Auto in Dividend Growth | 8.31 | 4.22 |
| Increased Auto in Dividend Growth | 8.50 | 4.08 |
| Reduced Auto in Dividend Growth & Interest Rates | 119.41*** | 114.83*** |
| Increased Auto in Dividend Growth & Interest Rates | 12.52* | 7.30 |
| **Panel C: Sensitivity to Various Values of the Ex Ante Equity Premium with iid Dividend Growth Rates and Interest Rates** | | |
| Ex Ante Equity Premium of 2% | $9.86\times10^{6}$*** | $9.87\times10^{6}$*** |
| Ex Ante Equity Premium of 3% | $8.44\times10^{5}$*** | $8.42\times10^{5}$*** |
| Ex Ante Equity Premium of 4% | $4.54\times10^{6}$*** | $4.54\times10^{6}$*** |
| Ex Ante Equity Premium of 4.5% | $1.01\times10^{7}$*** | $1.01\times10^{7}$*** |
| Ex Ante Equity Premium of 6% | $1.05\times10^{8}$*** | $1.05\times10^{8}$*** |
| **Panel D: Sensitivity to Time Varying Equity Premia with a 4% Average Ex Ante Equity Premium** | | |
| Type 1: Autocorrelated Equity Premium | 10.99* | 5.59 |
| Type 2: Autocorrelated & Downward Trending Equity Premium | 146.79*** | 140.53*** |
| Type 3: iid Equity Premium | 9.14 | 4.65 |
| Type 4: Downward Trending Equity Premium | 162.54*** | 156.61*** |

## Figure 1: Probability Distribution Function of Simulated Ex Post Equity Premia, Dividend Yields, Mean Returns, and Return Standard Deviations

This figure contains PDFs for various financial statistics generated in 2,000 simulated economies. Each panel contains a PDF for each of four different assumed values of the ex ante equity premium: 2%, 3%, 4%, and 6% (marked with a plain line, $\diamond$, $*$, and # respectively). Panel A shows distributions of the ex post equity premium (mean return minus mean interest rate), Panel B shows the mean dividend yield distributions (dividend divided by price), Panel C shows mean return distributions, and Panel D shows distributions of the standard deviation of returns. In each panel, the vertical column of dots indicates the value of the ex post equity premium, the mean dividend yield, the mean return, or the return standard deviation estimated using actual US data over 1952-1998. The simulated statistics are estimated over 47 years of generated data for each economy.



## Figure 2: Probability Distribution Function of Simulated Mean Discounted Dividend Growth Rates, Sharpe Ratios, AR Coefficients, and ARCH Coefficients

This figure contains probability distribution functions (PDFs) for various financial statistics generated in 2,000 simulated economies. Each panel contains a PDF for each of four different assumed values of the ex ante equity premium: 2%, 3%, 4%, and 6% (marked with a plain line, $\diamond$, $*$, and # respectively). Panel A shows distributions of the mean discounted dividend growth rate ($y_t = (1+g_t)/(1+r_t)$ where $g_t$ is the growth rate of dividends and $r_t$ is the discount rate), Panel B shows the Sharpe ratio distributions (excess return divided by the standard deviation of the excess return), Panel C shows the return autocorrelation coefficient (the OLS parameter estimate from regressing returns on lagged returns), and Panel D shows distributions of the ARCH coefficient (the OLS parameter estimate from regressing squared residuals on lagged squared residuals). In each panel, the vertical column of dots indicates the value of the discounted dividend growth rate, the Sharpe ratio, the return AR coefficient, or the ARCH coefficient estimated using actual US data over 1952-1998. The simulated statistics are estimated over 47 years of generated data for each economy.

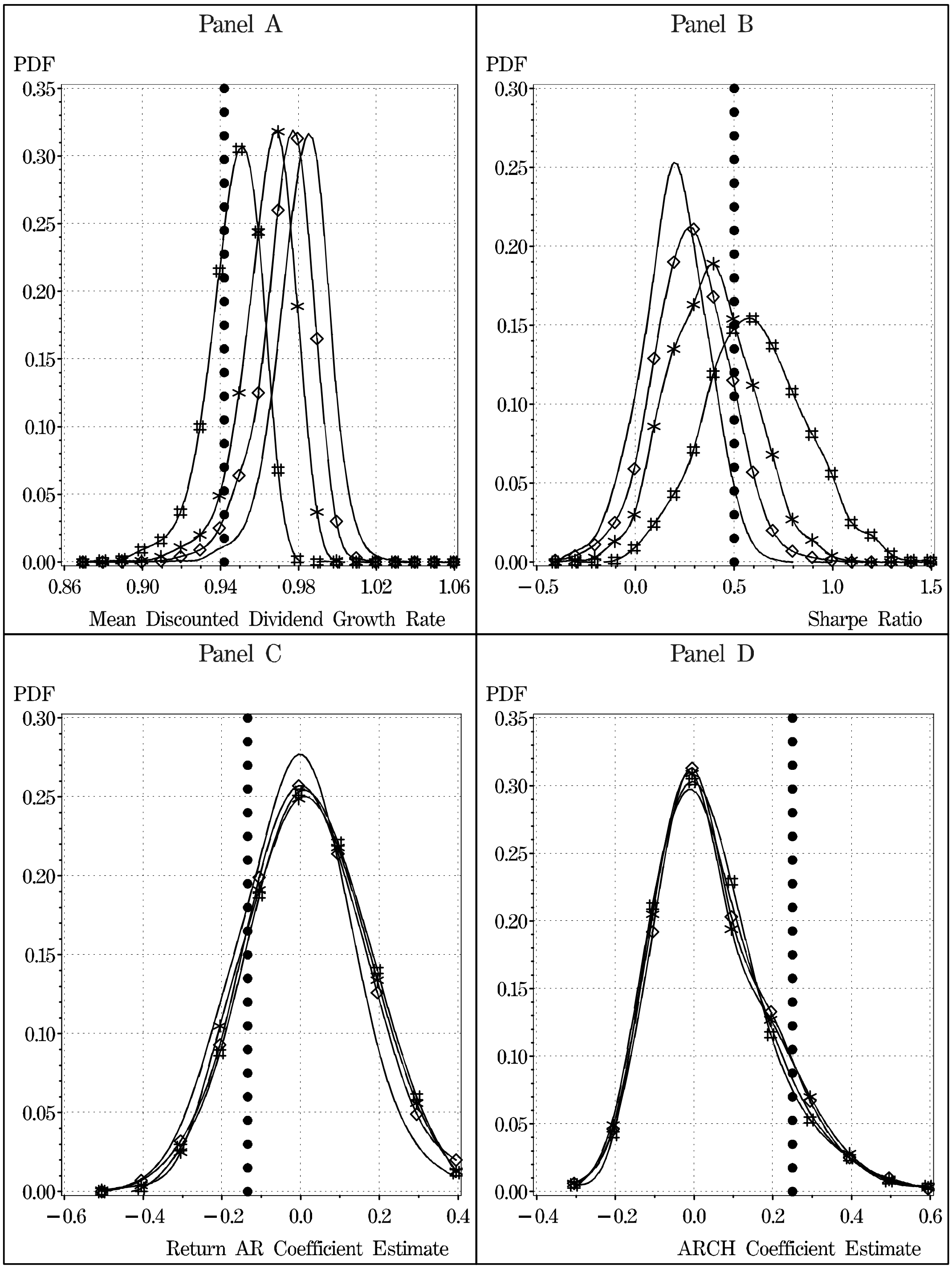
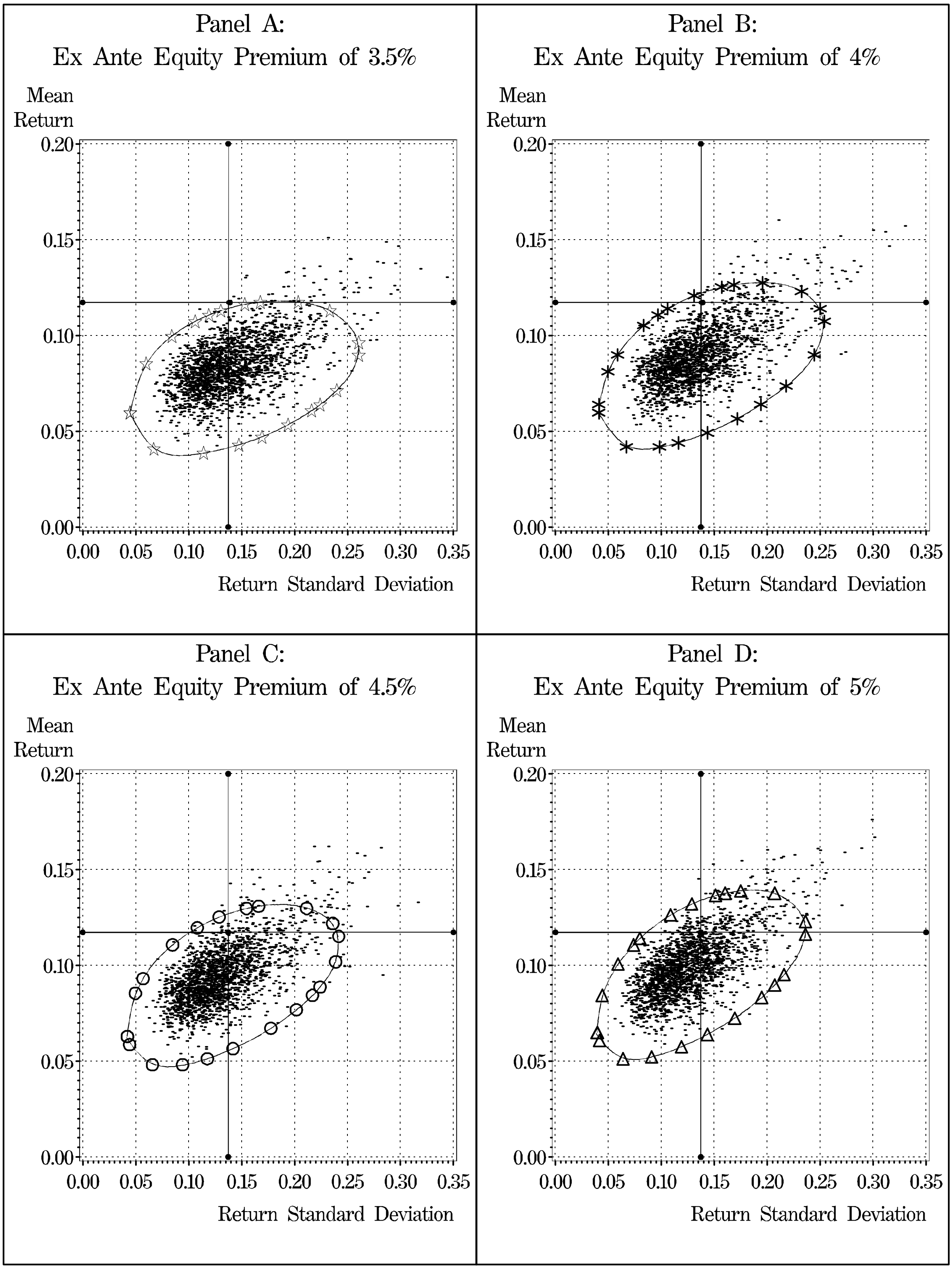


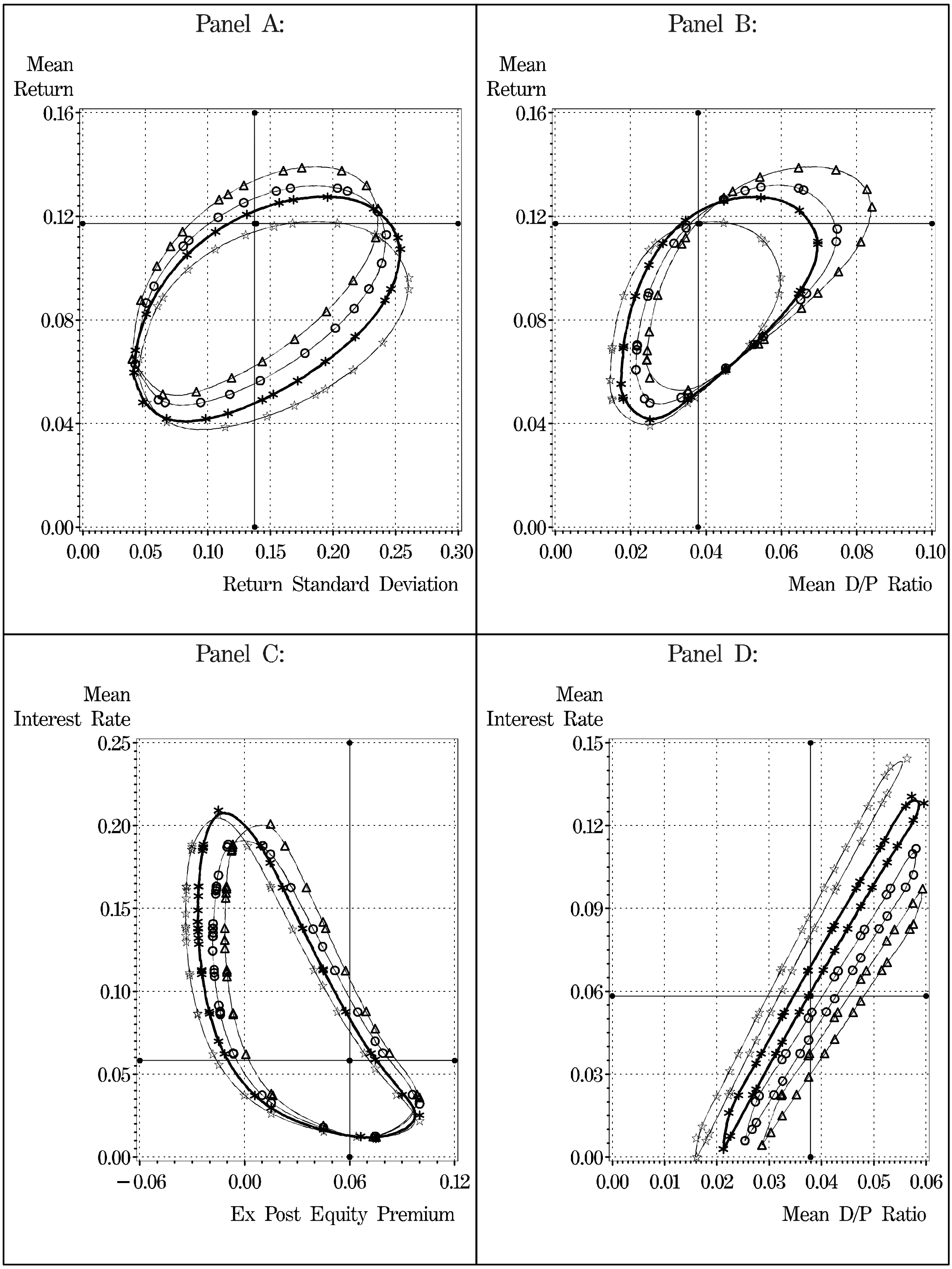
**Figure 3: Bivariate Distributions for Mean Returns Versus Return Standard Deviations Based on Several Values of the Ex Ante Equity Premium**

Bivariate distributions of mean returns versus return standard deviations are shown in this figure. Panels A, B, C, and D are based on ex ante equity premia of 3.5%, 4%, 4.5% and 5% respectively. The 2,000 simulated pairs, indicated with points, are based on data calibrated to dividend growth rates from the S&P 500 and 1-year US T-bill rates (1952-1998). Simulated points are surrounded by a thin line representing a 99% asymptotic confidence ellipse. The US realization (the non-simulated pair) is indicated by the intersection of the cross-hairs. The simulated statistics are estimated over 47 years of generated data.



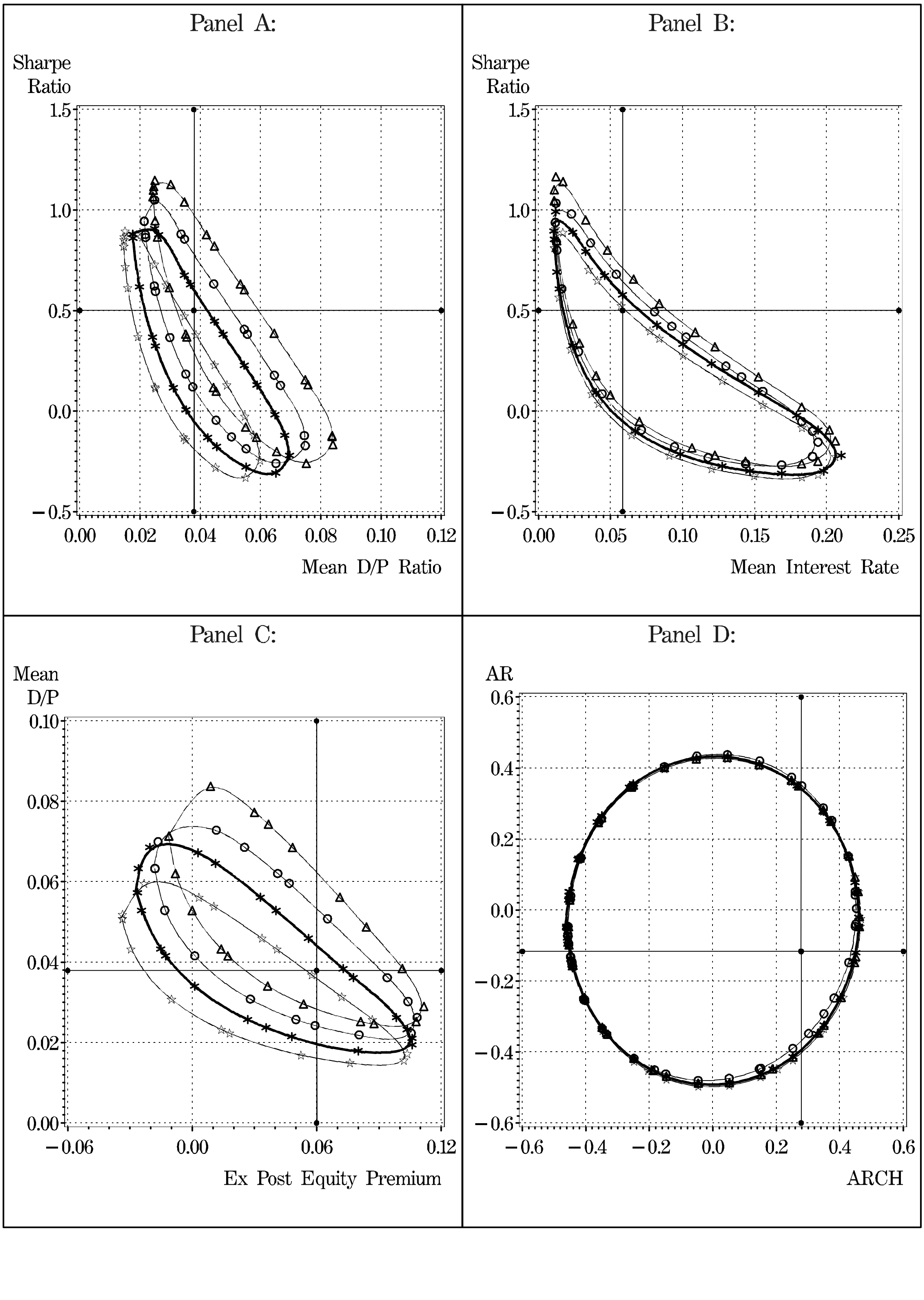
## Figure 4: Bivariate Distributions for Various Combinations of Variables Based on Several Values of the Ex Ante Equity Premium

Panels A, B, C, and D show bivariate distributions for mean returns versus return standard deviation, mean return versus mean dividend yield, mean interest rate versus ex post equity premium, and mean interest rate versus mean dividend yield respectively. 99% asymptotic confidence confidence ellipses are shown for the following cases: ex ante equity premia of 3.5% (marked by $\star$), 4% (marked by $*$), 4.5% (marked by $\circ$), and 5% (marked by $\triangle$). The 2,000 simulations underlying each confidence ellipse are based on data calibrated to dividend growth rates from the S&P 500 and 1-year US T-bill rates (1952-1998). The US realization (the non-simulated pair) is indicated by the intersection of the cross-hairs. The simulated statistics are estimated over 47 years of generated data.



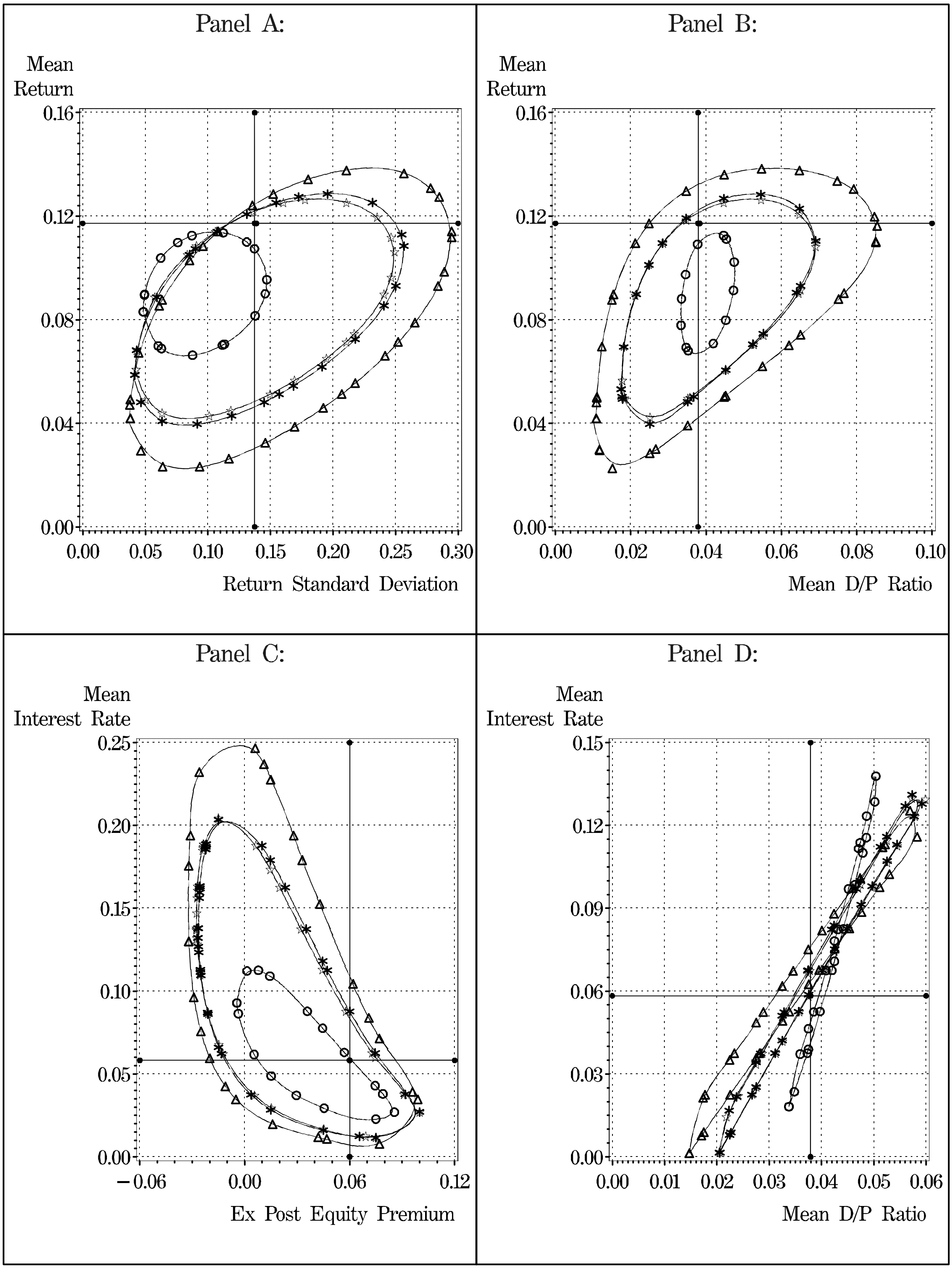
## Figure 5: Bivariate Distributions for Various Combinations of Variables Based on Several Values of the Ex Ante Equity Premium

Panels A, B, C, and D show bivariate distributions for Sharpe ratio versus mean dividend yield, Sharpe ratio versus mean interest rate, mean dividend yield versus ex post equity premium, and first order autoregressive coefficient versus first order ARCH coefficient respectively. 99% asymptotic confidence confidence ellipses are shown for the following cases: ex ante equity premia of 3.5% (marked by $\star$), 4% (marked by $*$), 4.5% (marked by $\circ$), and 5% (marked by $\triangle$). The 2,000 simulations underlying each confidence ellipse are based on data calibrated to dividend growth rates from the S&P 500 and 1-year US T-bill rates (1952-1998). The US realization (the non-simulated pair) is indicated by the intersection of the cross-hairs. The simulated statistics are estimated over 47 years of generated data.



# Figure 6: Sensitivity to Changes in Autocorrelation for Various Combinations of Variables Based on 4% Ex Ante Equity Premium

Panels A, B, C, and D show bivariate distributions for mean returns versus return standard deviation, mean return versus mean dividend yield, mean interest rate versus ex post equity premium, and mean interest rate versus mean dividend yield respectively. 99% asymptotic confidence confidence ellipses are shown for the following cases: reduced autocorrelation in dividend growth (marked by $\star$), increased autocorrelation in dividend growth (marked by $*$), reduced autocorrelation in dividend growth and interest rates (marked by $\circ$), and increased autocorrelation in dividend growth and interest rates (marked by $\triangle$). The 2,000 simulations underlying each confidence ellipse are based on data calibrated to dividend growth rates from the S&P 500 and 1-year US T-bill rates (1952-1998). The US realization (the non-simulated pair) is indicated by the intersection of the cross-hairs. The simulated statistics are estimated over 47 years of generated data. Plots in all four panels are based on an ex ante equity premium of 4%.



## Figure 7: Sensitivity to Changes in Autocorrelation for Various Combinations of Variables Based on 4% Ex Ante Equity Premium

Panels A, B, C, and D show bivariate distributions for Sharpe ratio versus mean dividend yield, Sharpe ratio versus mean interest rate, mean dividend yield versus ex post equity premium, and first order autoregressive coefficient versus first order ARCH coefficient respectively. 99% asymptotic confidence confidence ellipses are shown for the following cases: reduced autocorrelation in dividend growth (marked by $\star$), increased autocorrelation in dividend growth (marked by $*$), reduced autocorrelation in dividend growth and interest rates (marked by $\circ$), and increased autocorrelation in dividend growth and interest rates (marked by $\triangle$). The 2,000 simulations underlying each confidence ellipse are based on data calibrated to dividend growth rates from the S&P 500 and 1-year US T-bill rates (1952-1998). The US realization (the non-simulated pair) is indicated by the intersection of the cross-hairs. The simulated statistics are estimated over 47 years of generated data. Plots in all four panels are based on an ex ante equity premium of 4%.

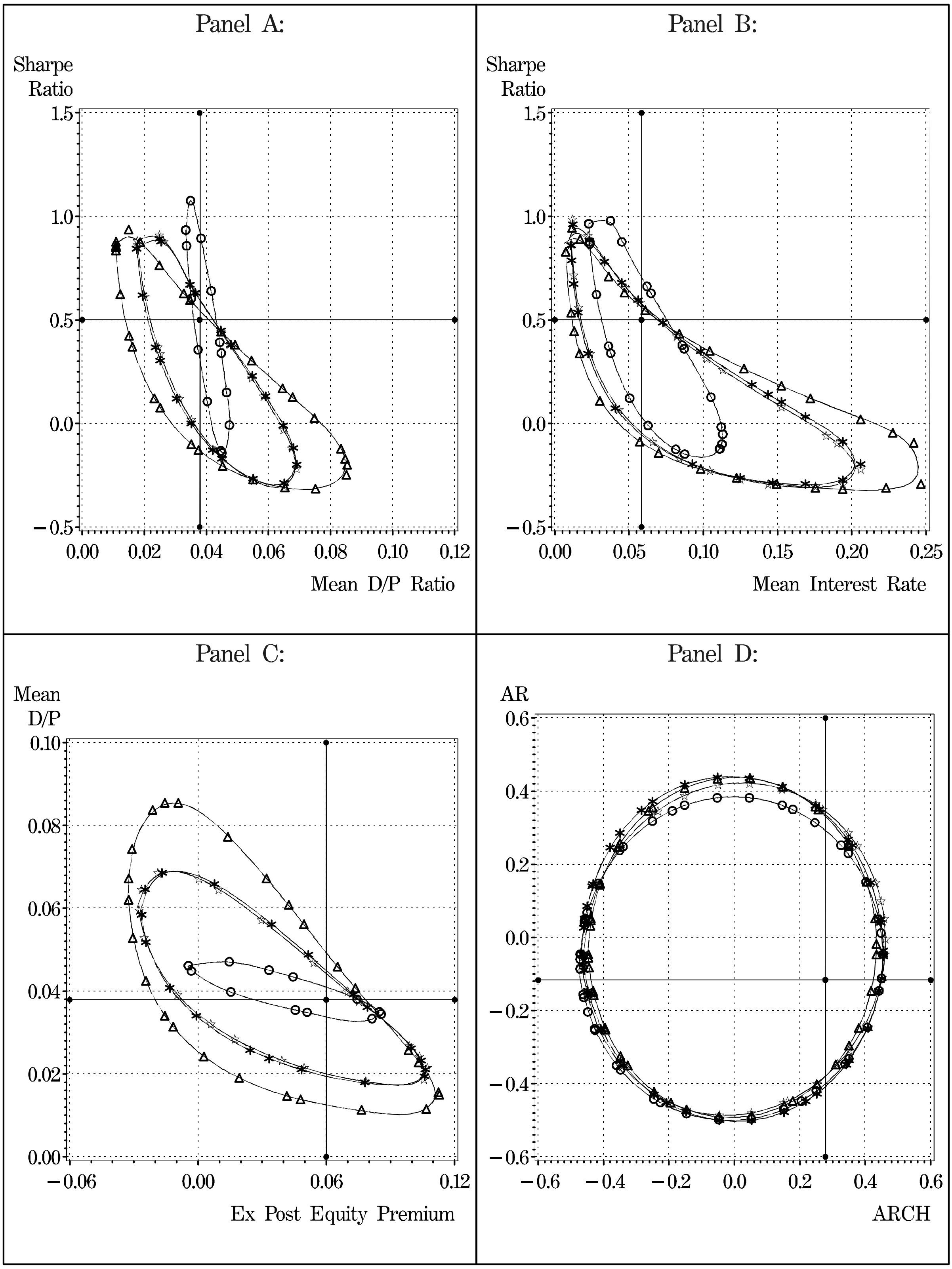
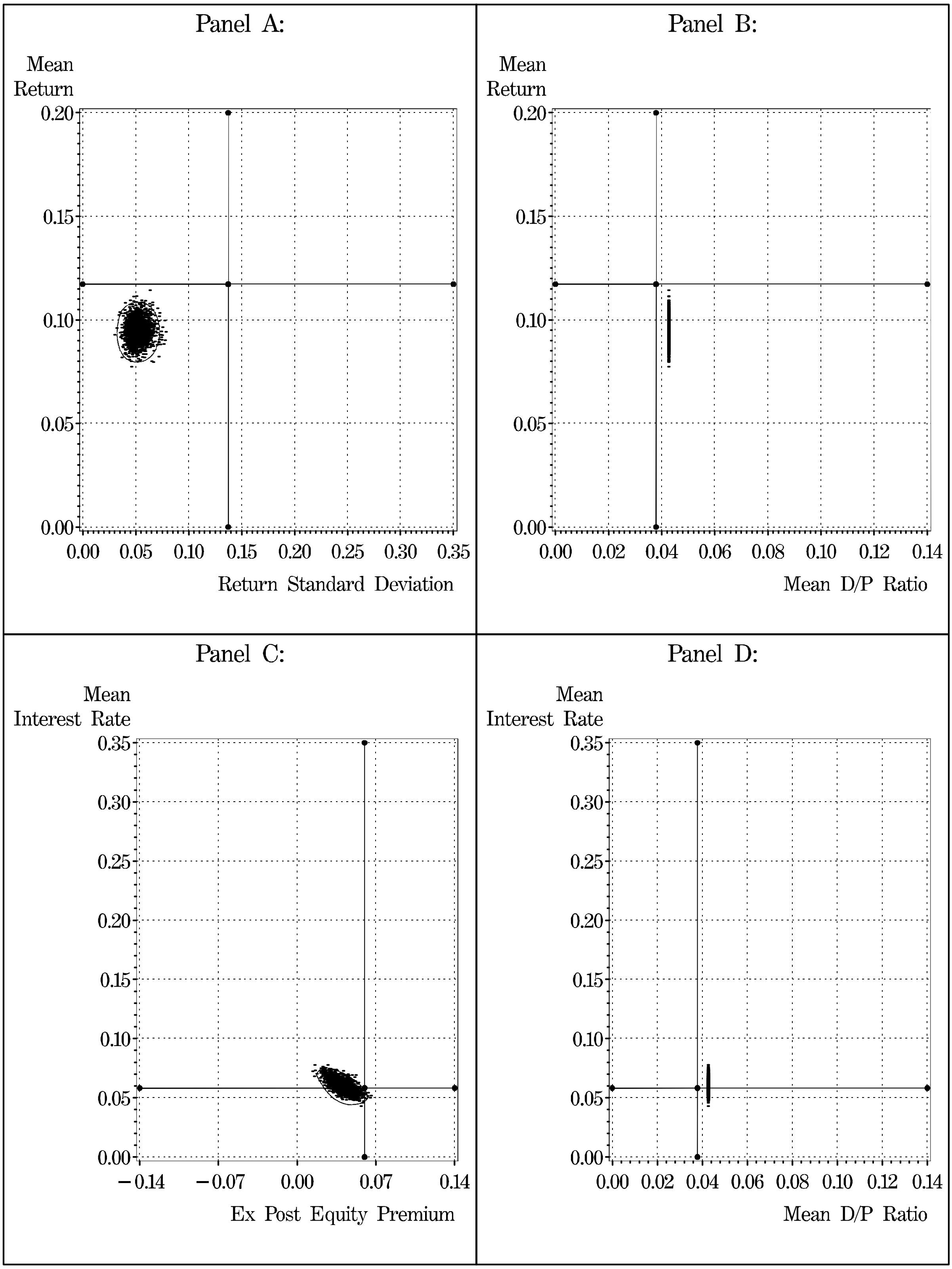


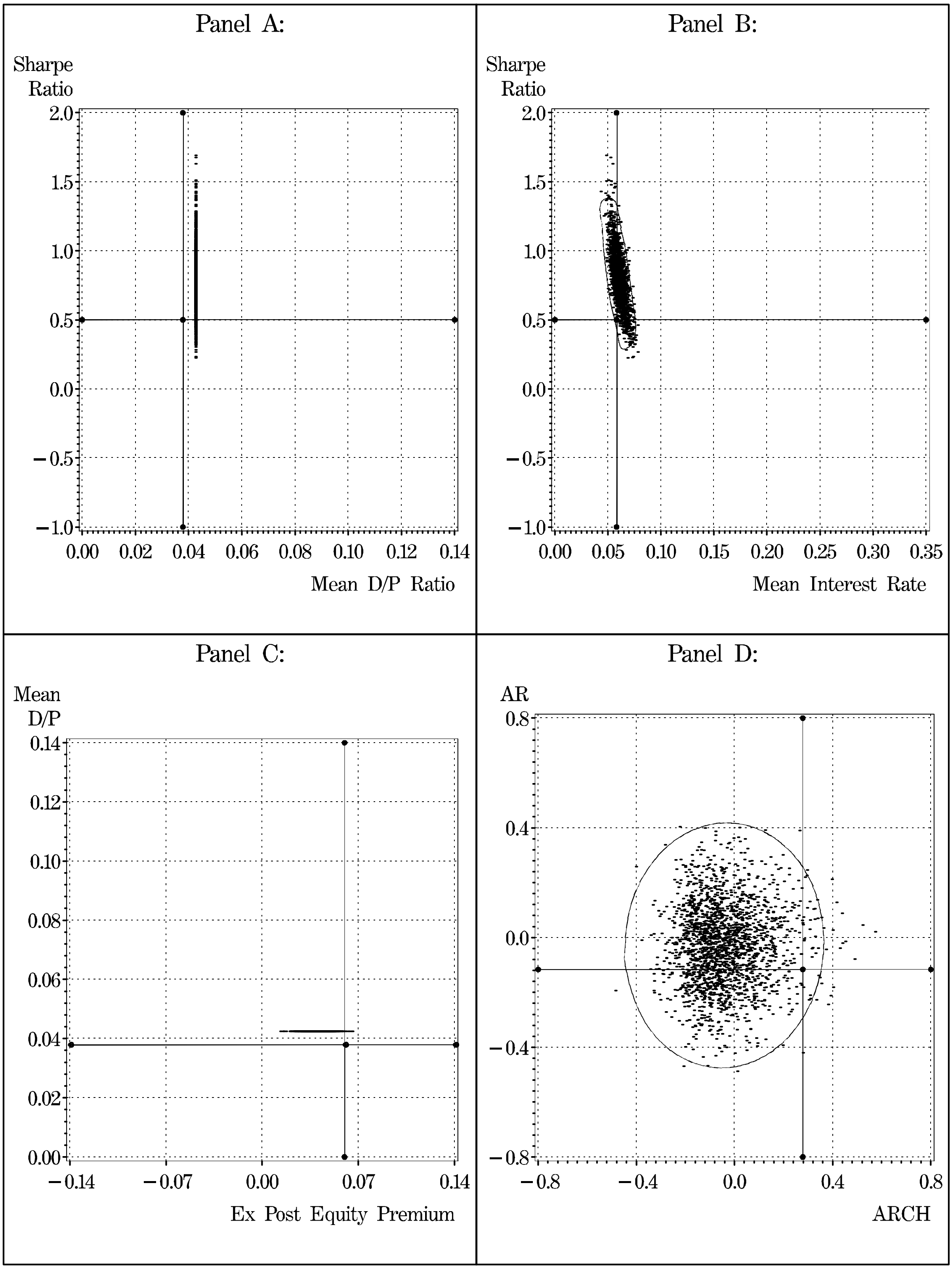
**Figure 8: Sensitivity to iid Dividend Growth Rates and Mean Interest Rates for Various Combinations of Variables Based on 4% Ex Ante Equity Premium**

Panels A, B, C, and D show bivariate distributions for Sharpe ratio versus mean dividend yield, Sharpe ratio versus mean interest rate, mean dividend yield versus ex post equity premium, and first order autoregressive coefficient versus first order ARCH coefficient respectively. The 2,000 simulated pairs, indicated with points, are based on iid dividend growth rates and interest rates. 99% asymptotic confidence confidence ellipses are shown for Panels A and C, though they are partially obscured by the scatterplots themselves. Confidence ellipses not defined for Panels B and D in which one of the two plotted variables is constant or near-constant. The US realization (the non-simulated pair) is indicated by the intersection of the cross-hairs. All statistics are estimated over 47 years of actual US (1952-1998) or simulated data. Plots in all four panels are based on an ex ante equity premium of 4%.



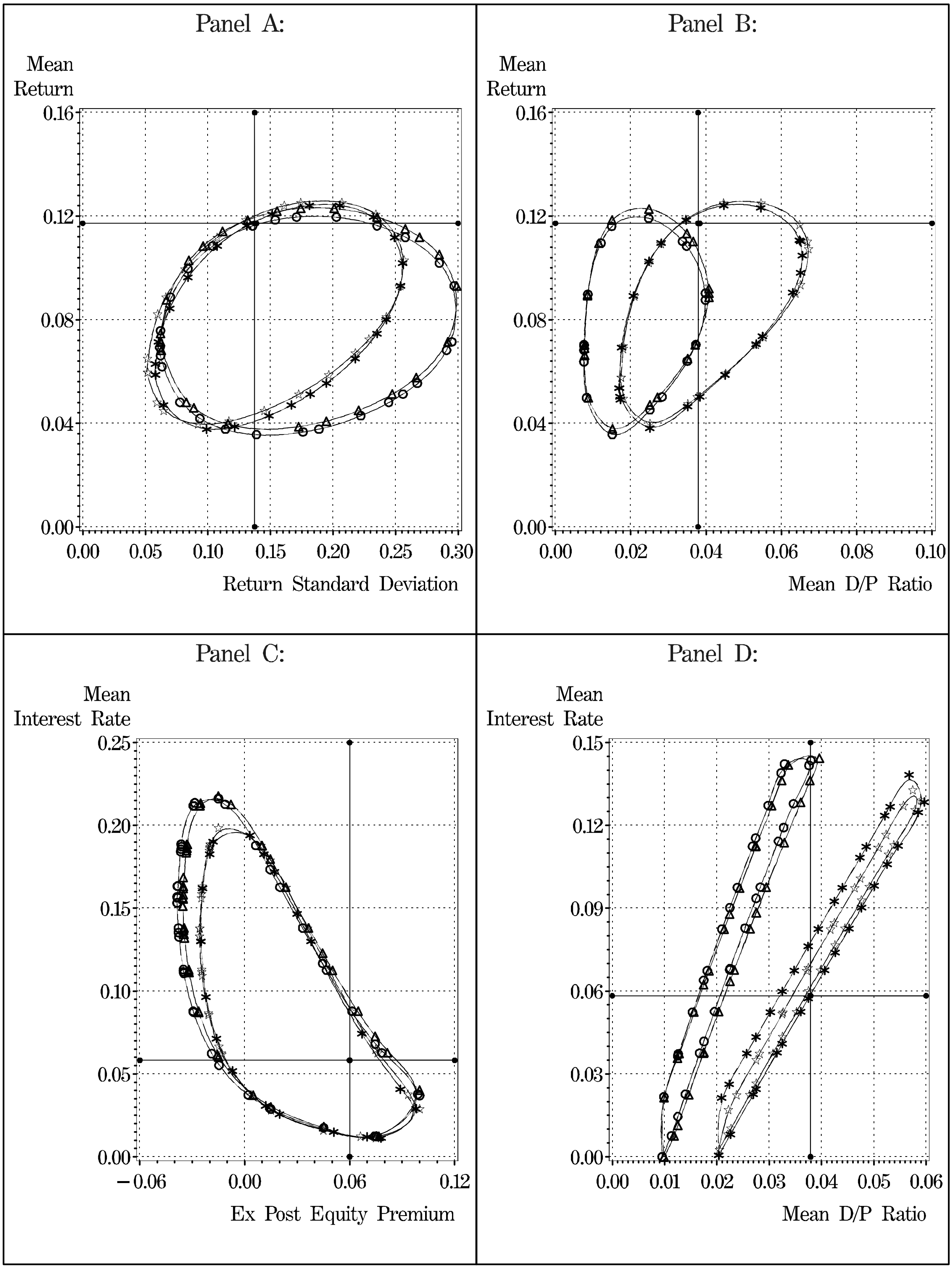
## Figure 9: Sensitivity to iid Dividend Growth Rates and Mean Interest Rates for Various Combinations of Variables Based on 4% Ex Ante Equity Premium

Panels A, B, C, and D show bivariate distributions for Sharpe ratio versus mean dividend yield, Sharpe ratio versus mean interest rate, mean dividend yield versus ex post equity premium, and first order autoregressive coefficient versus first order ARCH coefficient respectively. The 2,000 simulated pairs, indicated with points, are based on iid dividend growth rates and interest rates. 99% asymptotic confidence confidence ellipses are shown for Panels B and D, though they are partially obscured by the scatterplots themselves. Confidence ellipses not defined for Panels A and C in which one of the two plotted variables is constant or near-constant. The US realization (the non-simulated pair) is indicated by the intersection of the cross-hairs. All statistics are estimated over 47 years of actual US (1952-1998) or simulated data. Plots in all four panels are based on an ex ante equity premium of 4%.



## Figure 10: Sensitivity to Time-Varying Equity Premium for Various Combinations of Variables

Panels A, B, C, and D show bivariate distributions for mean returns versus return standard deviation, mean return versus mean dividend yield, mean interest rate versus ex post equity premium, and mean interest rate versus mean dividend yield respectively. 99% asymptotic confidence confidence ellipses are shown for the following cases: autocorrelated equity premium (Type 1, marked by $*$); autocorrelated and downward trending equity premium (Type 2, marked by $\triangle$); iid equity premium (Type 3, marked by $\star$); and downward trending equity premium (Type 4, marked by $\circ$) respectively. The 2,000 simulations underlying each confidence ellipse are based on data calibrated to dividend growth rates from the S&P 500 and 1-year US T-bill rates (1952-1998). The US realization (the non-simulated pair) is indicated by the intersection of the cross-hairs. All statistics are estimated over 47 years of actual US (1952-1998) or simulated data. The average ex ante equity premium equals roughly 4% for all four panels.



## Figure 11: Sensitivity to Time-Varying Equity Premium for Various Combinations of Variables

Panels A, B, C, and D show bivariate distributions for mean returns versus return standard deviation, mean return versus mean dividend yield, mean interest rate versus ex post equity premium, and mean interest rate versus mean dividend yield respectively. 99% asymptotic confidence confidence ellipses are shown for the following cases: autocorrelated equity premium (Type 1, marked by $*$); autocorrelated and downward trending equity premium (Type 2, marked by $\triangle$); iid equity premium (Type 3, marked by $\star$); and downward trending equity premium (Type 4, marked by $\circ$) respectively. The 2,000 simulations underlying each confidence ellipse are based on data calibrated to dividend growth rates from the S&P 500 and 1-year US T-bill rates (1952-1998). The US realization (the non-simulated pair) is indicated by the intersection of the cross-hairs. All statistics are estimated over 47 years of actual US (1952-1998) or simulated data. The average ex ante equity premium equals roughly 4% for all four panels.