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Abstract: This paper shows how to use the Kalman filter (Kalman 1960) to back out the shocks of a dynamic stochastic general equilibrium model. In particular, we use the smoothing algorithm as described in Hamilton (1994) to estimate the shocks of a sticky-prices and sticky-wages model using all the information up to the end of the sample.

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1. Introduction

This paper shows how to use the Kalman Filter (Kalman, 1960) to back out the shocks of a dynamic stochastic general equilibrium model. The idea is as follows: First, we write the model in what is called the *State-Space Representation*. Second, we use the Kalman Filter to write the likelihood function of the observed data and estimated the structural parameters of the model.[1] Third, using the estimate parameters, we estimated the values of the model perturbations during the sample period conditional on all the observed data.[2] This procedure is very useful because it allows us to use a general equilibrium model to make inference about which shocks the economy was facing during any period in the sample based on the full set of collected data. In the first part of the paper we describe how to implement the Kalman filter to estimate structural parameters and smooth the shocks to an abstract linear system. In the second part, we use a sticky price model to show how to use the procedure in practice.

2. The State-Space Representation

The first step is to write the model in State-Space form. Let η_t be a $(nx1)$ vector the observed variables at date t and let ξ_t be a $(rx1)$ vector of unobserved variables at date t (this vector

[1] By structural parameters, we mean those that define technology, preferences and the stochastic processes.
[2] This procedure is called "smoothing" of the shocks.

is also called the *state vector*). The *State-Space Representation* of the system is[3]

$$\xi_{t+1} = F\xi_t + v_{t+1} \tag{1}$$

$$\eta_t = H'\xi_t + w_t \tag{2}$$

where F and H' are matrices of the needed dimensions. Equation (1) is called *State Equation*, and (2) is the *Observed Equation*. v_t and w_t are uncorrelated normally distributed white noise vectors, therefore:

$$E\left(v_t v'_\tau\right) = \begin{cases} Q \text{ for } \tau = t \\ 0 \text{ otherwise} \end{cases}$$

$$E\left(w_t w'_\tau\right) = \begin{cases} R \text{ for } \tau = t \\ 0 \text{ otherwise} \end{cases}$$

and

$$E\left(w_t v'_\tau\right) = 0 \text{ for all } t, \tau$$

3. Using the Kalman Filter to Write the Likelihood Function of the Model

Once the model has be written in State-Space form, the second step is to estimate the structural parameters that define the model. In order to do that, we will need to write the likelihood function of the observed data, $\eta^T = \{\eta_1, \eta_2, ..., \eta_t\}$. We will use the Kalman Filter to write the likelihood function $\ell\left(\eta^T | F, H', Q, R\right) = \prod_{t=1}^{T} \ell\left(\eta_t | \eta_{t-1} F, H', Q, R\right)$.

[3] To see examples of State-Space representations of linear models, see Hamilton (1994) Chapter 13.

First, we need to introduce some notation. Let

$$\xi_{t+1|t} = E\left(\xi_{t+1}|\eta^t\right),$$

be the linear projection of ξ_{t+1} on η^t and a constant, and let

$$\eta_{t+1|t} = E\left(\eta_{t+1}|\eta^t\right) = H'\xi_{t+1|t}$$

be the linear projection of η_{t+1} on η^t and a constant. Also let

$$P_{t+1|t} = E\left(\xi_{t+1} - \xi_{t+1|t}\right)\left(\xi_{t+1} - \xi_{t+1|t}\right)',$$

be the mean squared forecasting error when projecting ξ_{t+1}, and let

$$\Sigma_{t+1|t} = E\left(\eta_{t+1} - \eta_{t+1|t}\right)\left(\eta_{t+1} - \eta_{t+1|t}\right)' =$$

$$= E\left(H'\xi_{t+1} + w_{t+1} - H'\xi_{t+1|t}\right)\left(H'\xi_{t+1} + w_{t+1} - H'\xi_{t+1|t}\right)' =$$

$$H'P_{t+1|t}H + R$$

be the mean squared forecasting error when projecting η_{t+1}. Finally notice that

$$E\left(\eta_{t+1} - \eta_{t+1|t}\right)\left(\xi_{t+1} - \xi_{t+1|t}\right)' =$$

$$= E\left(H'\xi_{t+1} + w_{t+1} - H'\xi_{t+1|t}\right)\left(\xi_{t+1} - \xi_{t+1|t}\right)' =$$

$$H'P_{t+1|t}$$

Since v_t and w_t are normally distributed and the system is linear $\eta_{t|t-1}$ and $\xi_{t|t-1}$ will be normally distributed and can write

$$\begin{bmatrix} \xi_t \\ \eta_t \end{bmatrix} |\eta^{t-1} \sim N\left(\begin{bmatrix} \xi_{t|t-1} \\ H'\xi_{t|t-1} \end{bmatrix}, \begin{bmatrix} P_{t|t-1} & P_{t|t-1}H \\ H'P_{t|t-1} & H'P_{t|t-1}H + R \end{bmatrix} \right)$$

which implies that[4]

$$\xi_t|\eta^t \sim N(\xi_{t|t}, P_{t|t})$$

where the expectation of ξ_t conditional on η^t is:

$$\xi_{t|t} = \xi_{t|t-1} + H'P_{t|t-1}\left[H'P_{t|t-1}H + R\right]^{-1}\left(\eta_t - \eta_{t|t-1}\right),$$

and its MSE is:

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}H\left[H'P_{t|t-1}H + R\right]^{-1}H'P_{t|t-1}.$$

3.1. The algorithm

Therefore, given $\xi_{t|t-1}$, $P_{t|t-1}$ and observation η_t, the Kalman filter algorithm is:

$$\eta_{t|t-1} = H'\xi_{t|t-1},$$

[4] This result is due to the following feature of the normal distribution. If X and Y conditional on w are jointly normal

$$[X'|w\ Y'|w]' \sim N\left(\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}, \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix} \right)$$

then $X'|y, w$ is also normally distributed with the following distribution:

$$X|y, w \sim N\left(\overline{x} + \Sigma_{xy}\Sigma_{yy}^{-1}\left(y - \overline{y}\right), \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}\right)$$

$$\Sigma_{t|t-1} = H' P_{t|t-1} H + R,$$

$$\xi_{t|t} = \xi_{t|t-1} + H' P_{t|t-1} \left[H' P_{t|t-1} H + R \right]^{-1} \left(\eta_t - \eta_{t|t-1} \right),$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1} H \left[H' P_{t|t-1} H + R \right]^{-1} H' P_{t|t-1},$$

$$\xi_{t+1|t} = F \xi_{t|t},$$

and

$$P_{t+1|t} = F P_{t|t} F' + Q.$$

3.2. The Likelihood Function

Since $\left[\begin{array}{cc} \xi_t' & \eta_t' \end{array} \right]' | \eta^{t-1}$ is normally distributed, $\eta_t | \eta^{t-1}$ is also normally distributed. In addition, since the considered model has only one lag, we can write $\eta_t | \eta^{t-1} = \eta_t | \eta_{t-1}$. This implies that

$$\log \ell \left(\eta^T | F, H', Q, R \right) = \sum_{t=1}^{T} \log \ell \left(\eta_t | \eta_{t-1}, F, H', Q, R \right) =$$
$$- \sum_{t=1}^{T} \left[\frac{n}{2} \log 2\pi + \frac{1}{2} \log \left| \Sigma_{t|t-1} \right| + \frac{1}{2} \sum_{t=1}^{T} \left(\eta_t - \eta_{t|t-1} \right)' \Sigma_{t|t-1}^{-1} \left(\eta_t - \eta_{t|t-1} \right) \right]$$

Once the likelihood function of η^T has been obtained, we can either maximize it or combine it with some prior distribution about F, H', Q, and R to get the MLE or the posterior distribution of the structural parameters.

4. Smoothing

At this point we have used the Kalman Filter to estimate the structural parameters of a linear model. In some settings, the vector ξ_t has a structural interpretation.[5] An important goal is to estimate $\xi^T = \{\xi_t\}_{t=1}^T$ conditional on the full set of data η^T. Therefore, the objective of this section is to get the smoothed estimated of ξ_t denoted by:

$$\xi_{t|T} = E\left(\xi_t|\eta^T\right).$$

Assume that, given η^t, we are told the true value of ξ_{t+1}. Then the linear projection of ξ_t on ξ_{t+1}, η^t and a constant is[6]

$$E\left(\xi_t|\xi_{t+1},\eta^t\right) = \xi_{t|t} + \left(E\left(\xi_t - \xi_{t|t}\right)\left(\xi_{t+1} - \xi_{t+1|t}\right)'\right) P_{t+1|t}^{-1}\left[\xi_{t+1} - \xi_{t+1|t}\right].$$

But

$$E\left(\xi_t - \xi_{t|t}\right)\left(\xi_{t+1} - \xi_{t+1|t}\right)' = P_{t|t}F'$$

therefore

$$E\left(\xi_t|\xi_{t+1},\eta^t\right) = \xi_{t|t} + J_t\left[\xi_{t+1} - \xi_{t+1|t}\right]$$

[5] In a Dynamic Stochastic General Equilibrium Model ξ_t can be interpreted as the structural shocks that the economy faces at time t.

[6] The result is based on the formula for updating the linear projection

$$P\left(Y_3|Y_2,Y_1\right) = P\left(Y_3|Y_1\right) + H_{32}H_{22}^{-1}\left[Y_2 - P\left(Y_2|Y_1\right)\right]$$

where

$$H_{32} = E\left(Y_3 - P\left(Y_3|Y_1\right)\right)\left(Y_2 - P\left(Y_2|Y_1\right)\right)'$$
$$H_{22} = E\left(Y_2 - P\left(Y_2|Y_1\right)\right)\left(Y_2 - P\left(Y_2|Y_1\right)\right)'$$

where

$$J_t = P_{t|t} F' P_{t+1|t}^{-1} \tag{3}$$

Finally, notice that[7]

$$E\left(\xi_t | \xi_{t+1}, \eta^t\right) = E\left(\xi_t | \xi_{t+1}, \eta^{t+j}\right) \ \forall j > 0$$

that implies that

$$E\left(\xi_t | \xi_{t+1}, \eta^t\right) = E\left(\xi_t | \xi_{t+1}, \eta^T\right) = \xi_{t|t} + J_t\left[\xi_{t+1} - \xi_{t+1|t}\right].$$

The final step is to get

$$E\left(\xi_t | \eta^T\right) = E\left(E\left(\xi_t | \xi_{t+1}, \eta^T\right) | \eta^T\right) = \xi_{t|t} + J_t\left[E\left(\xi_{t+1} | \eta^T\right) - \xi_{t+1|t}\right] = \xi_{t|t} + J_t\left[\xi_{t+1|T} - \xi_{t+1|t}\right] \tag{4}$$

4.1. Smoothing Algorithm

The sequence of smoothed estimates $\left\{\xi_{t|T}\right\}_{t=1}^{T}$ is calculated as follows: first, using the Kalman Filter, we calculate $\left\{\xi_{t|t}\right\}_{t=1}^{T}$, $\left\{\xi_{t+1|t}\right\}_{t=0}^{T-1}$, $\left\{P_{t|t}\right\}_{t=1}^{T}$, and $\left\{P_{t+1|t}\right\}_{t=0}^{T-1}$. Note that $\xi_{T|T}$ is the last entry of $\left\{\xi_{t|t}\right\}_{t=1}^{T}$ and that we can use (3) to calculate $\left\{J_t\right\}_{t=1}^{T}$ and (4) to determine $\left\{\xi_{t|T}\right\}_{t=1}^{T}$.

[7] This is true because we can write η_{t+j} as

$$\eta_{t+j} = H'\left(F^{j-1}\xi_{t+1} + F^{j-2}v_{t+2} + ... + v_{t+j}\right) + w_{t+j}$$

for any $j > 0$, and

$$u_t = \xi_t - E\left(\xi_t | \xi_{t+1}, \eta^t\right)$$

is uncorrelated with $\xi_{t+1}, v_{t+2}, ..., v_{t+j}$, and w_{t+j}.

5. An Application: The Sticky Price and Sticky Wage Model

In this section we are going to describe how to use the Smoothing Algorithm just described to estimate the perturbations that the economy faced at any historical date t.

We are not going to fully describe the model here. For a full description see Rabanal and Rubio-Ramírez (2003).

The equations are obtained by taking a log-linear approximation around the symmetric steady-state equilibrium with zero price and wage inflation rates. In what follows, the lower-case variables denote log-deviations from the steady-state value.[8]

First, we have the Euler equation, which relates consumption, c_t, with the real rate of interest, r_t, inflation, Δp_{t+1}, and a preference-shifter (or demand shock) g_t in the following way:

$$c_t = E_t c_{t+1} - \sigma(r_t - E_t \Delta p_{t+1} + E_t g_{t+1} - g_t). \tag{5}$$

The production function relates output gap with a productivity shock, a_t, and hours worked, n_t:

$$y_t = a_t + (1 - \delta) n_t. \tag{6}$$

The marginal cost, mc_t, is related with wages, w_t, prices, p_t, hours worked and output:

$$mc_t = w_t - p_t + n_t - y_t. \tag{7}$$

The marginal rate of substitution, mrs_t, between consumption and hours worked takes the

[8] For every variable X_t, we define the log-linear approximation as $x_t = \log X_t - \log(X^{SS})$, where X^{SS} is the variable's steady state-value.

form

$$mrs_t = \frac{1}{\sigma} y_t + \gamma n_t - g_t. \tag{8}$$

We use the following specification for the Taylor rule

$$r_t = \rho_r r_{t-1} + (1 - \rho_r) \left[\gamma_\pi \Delta p_t + \gamma_y y_t \right] + ms_t, \tag{9}$$

where γ_π and γ_y are the long-run responses of the monetary authority to deviations of inflation and output from their steady-state values and ms_t is the monetary shock, to be defined below. We include an interest rate smoothing parameter, ρ_r, following recent empirical work (as in Clarida, Galí, and Gertler, 2000).

The pricing decision of the firm under the Calvo-type restriction delivers the following forward-looking equation for inflation:

$$\Delta p_t = \beta E_t \Delta p_{t+1} + \kappa_p (mc_t + \lambda_t), \tag{10}$$

where $\kappa_p = (1-\delta)(1-\theta_p\beta)(1-\theta_p)/(\theta_p(1+\delta(\bar{\varepsilon}-1)))$, $\bar{\varepsilon} = \bar{\lambda}/(\bar{\lambda}-1)$ is the steady-state value of ε, and λ_t is a mark-up shock.

The nominal wage growth equation (Δw_t) is:

$$\Delta w_t = \beta E_t \Delta w_{t+1} + \kappa_w (mrs_t - (w_t - p_t)), \tag{11}$$

where $\kappa_w = (1 - \theta_w)(1 - \beta\theta_w) / \left(\theta_w \left(1 + \phi\gamma\right)\right)$.

Real wages, $w_t - p_t$, relate to last period real wage, inflation and nominal growth in the following way:

$$w_t - p_t = w_{t-1} - p_{t-1} + \Delta w_t - \Delta p_t. \tag{12}$$

The market clearing condition is:

$$c_t = y_t. \tag{13}$$

We specify the shocks in the following way

$$a_t = \rho_a a_{t-1} + \varepsilon_t^a, \tag{14}$$

$$g_t = \rho_g g_{t-1} + \varepsilon_t^g, \tag{15}$$

$$ms_t = \varepsilon_t^m, \tag{16}$$

and

$$\lambda_t = \varepsilon_t^\lambda, \tag{17}$$

where each innovation ε_t^i follows a $N(0, \sigma_i^2)$ distribution. The innovations are uncorrelated with each other. Let $\varepsilon_t = \left(\varepsilon_t^a, \varepsilon_t^{ms}, \varepsilon_t^\lambda, \varepsilon_t^g\right)'$.

5.1. The State-Space Representation of the Model

To write the State-Space representation of the model, we solve it using the Uhlig (1999) algorithm. Let

$$x_t = (w_t - p_t, r_t, \Delta p_t, \Delta w_t, y_t)'$$

$$\mu_t = (n_t, mc_t, mrs_t, c_t)'$$

and

$$z_t = (a_t, ms_t, \lambda_t, g_t)'$$

Then, if we write:

$$0 = Ax_t + Bx_{t+1} + C\mu_t + Dz_t,$$

$$0 = E_t \left(Fx_{t+1} + Gx_t + Hx_{t-1} + J\mu_{t+1} + K\mu_t + Lz_{t+1} + Mz_t \right),$$

and

$$z_t = Nz_{t-1} + \varepsilon_t,$$

Uhlig (1999) shows how to get a solution of the form:

$$x_t = Px_{t-1} + Zz_t,$$

$$\mu_t = Rx_{t-1} + Sz_t,$$

and

$$z_t = Nz_{t-1} + \varepsilon_t.$$

In our case the observable vector is $\eta_t = (w_t - p_t, r_t, \Delta p_t, y_t)'$. Therefore, if we let $\xi_t = (x_t', z_t')'$, the the State Space representation of the model is:

$$\xi_t = \begin{bmatrix} P & ZN \\ N & 0 \end{bmatrix} \xi_{t-1} + \begin{bmatrix} Z \\ I \end{bmatrix} \varepsilon_t, \tag{18}$$

and

$$\eta_t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \xi_t. \tag{19}$$

If we want to make the mapping from (18) and (19) to (1) and (2), we need to consider the following relations:

$$F = \begin{bmatrix} P & ZN \\ N & 0 \end{bmatrix}$$

$$H' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$v_t = \begin{bmatrix} Z \\ I \end{bmatrix}$$

$$Q = \begin{bmatrix} Z \\ I \end{bmatrix} \begin{bmatrix} \sigma_a^2 & 0 & 0 & 0 \\ 0 & \sigma_{ms}^2 & 0 & 0 \\ 0 & 0 & \sigma_\lambda^2 & 0 \\ 0 & 0 & 0 & \sigma_g^2 \end{bmatrix} \begin{bmatrix} Z \\ I \end{bmatrix}'$$

and

$$w_t = 0$$

Now we are ready to apply the kalman filter to estimate the structural parameters of the model and smooth ξ_t. Notice that the $6th$, $7th$, $8th$ and $9th$ components of ξ_t are the structural shocks that affect the model, and as a consequence, when smoothing ξ_t we are also smoothing the structural perturbations.

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